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ON SOME MAPS OF CONJUGACY CLOSED LOOPS

BY

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Abstract. It is shown that certain maps of a conjugacy closed loop are elements of the Bryant-Schneider group of the loop. It is also proved that conjugacy closed loops are embeddable into their Bryant-Schneider groups. The index of the automorphism group of a conjugacy closed loop in the Bryant-Schneider group is shown to be twice the cardinality of the loop if the index of nucleus of the loop is 2 in the loop.

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1. Introduction. A considerable amount of work has been done on conjugacy closed loops (otherwise known as CC-loops). The notion of CC-loop is itself due to GOODAIRE and ROBINSON [7,8], thence several other works has been done. This include that of CHIBOKA and SOLARIN [5,6], CHIBOKA [4], KUNNEN [9] the present author and SOLARIN [1]. The present work considered the middle inner mappings $T(x) = R_x L_{x^{-1}}$ of a CC-loop. It is proved that these mappings are elements of the Bryant-Schneider group provided the is flexible. It is also proved that a CC-loop is embeddable in its Bryant-Schneider group. The cardinality of the Bryant-Schneider group of a CC-loop whose nucleus is of index 2 in the loop is also considered.

Definition 1.1. A loop is conjugacy closed if and only if there are functions $f, g : Q \times Q \longrightarrow Q$ such that for all x, y, z:

$$RCC : x.yz = f(x, y).xz \ LCC : zy.x = zx.g(x, y)$$

As usual, the left and right multiplication is defined by $xy = xR_y = yL_x$, so that R_y and L_x are permutations of the set Q. With these, CC-loop can be defined in terms of conjugations:

Definition 1.2. A loop is conjugacy closed (CC-loop) if $L_xL_yL_x^{-1} \in \Pi_L$ and $R_xR_yR_x^{-1} \in \Pi_R$ where $\Pi_R = \{R_x \mid x \in Q\}$ and $\Pi_L = \{L_x \mid x \in Q\}$

Definition 1.3. Let $(Q, .)$ be a loop and BS$(Q, .)$ be the set of all permutations of Q such that $< \theta R_{g^{-1}}, \theta L_{f^{-1}}, \theta >$ is an autotopism of $(Q, .)$ for some $f, g \in Q$. Then, BS$(Q, .)$ is called the Bryant-Schneider group of the loop $(Q, .)$.

Lemma 1.1. [6] *Let $(Q, .)$ be a CC-loop. The the following statements are equivalent:*

1. *$(Q, .)$ is flexible*
2. *$(Q, .)$ has the inverse property*
3. *$(Q, .)$ is an M-loop*

Corollary 1.1. *In a flexible conjugacy closed loop $R_x^{-1} = R_{x^{-1}}$ and $L_x^{-1} = L_{x^{-1}}$*

Definition 1.4. Let $M(Q, .)$ be the multiplication group of a loop $(Q, .)$. The inner group $I(Q, .)$ is the subgroup of $M(Q, .)$ generated by the mappings $R(x, y)$, $L(x, y)$ and $T(x)$ for all $x, y \in Q$, where $R(x, y) = R_xR_yR_{xy}^{-1}$, $L(x, y) = L_xL_yL_{xy}^{-1}$, and $T(x) = R_xL_x^{-1}$ called the right inner mappings, the left inner mappings and the middle inner mappings respectively.

For the definition of a loop and concepts in theory of loops, readers may consult either Bruck [2] or Pflugfelder [10] or both.

2. Main Results

Theorem 2.1. *Let $(Q, .)$ be a flexible CC-loop. For each fixed element $u \in Q$, define the principal isotope $(Q, \star)$ of $(Q, .)$ by $x \star y = xR_u.yL_{u^{-1}}$ for all $x, y \in Q$. If θ_u is the mapping defined by $x\theta_u = xR_uL_{u^{-1}}$ for al $x \in Q$. Then θ_u is an element of the Bryant-Schneider group BS$(Q, .)$ of $(Q, .)$.*

Proof. If $xy\theta_u = xyR_uL_{u^{-1}}$ for all $x, y \in Q$ then

$$\begin{aligned} x\theta_u \star y\theta_u &= x\theta_uR_u.y\theta_uL_{u^{-1}} = xR_uL_{u^{-1}}R_u.yR_uL_{u^{-1}}L_{u^{-1}} = \\ &= (u^{-1}.xu)u.u^{-1}(u^{-1}.yu). \end{aligned}$$

Since a CC-loop is an M-loop [1] then $u^{-1}(u^{-1}.yu) = ((u^{-1})^2.y)$ for all $u, y \in Q$ and therefore we have

$$\begin{aligned} x\theta_u \star y\theta_u &= (u^{-1}.xu)u.((u^{-1})^2.y)u = ((u^{-1}.xu)u.(u^{-1})^2 y)u = \\ &= ((u^{-1}.xu)u^{-1}.y)u = (u^{-1}.xu)u^{-1}.yu = \\ &= u^{-1}x.yu = xL_{u^{-1}}.yR_u \end{aligned}$$

for all $x, y, u \in Q$

Now since Q is a CC-loop

$$T =< R_u, R_u L_{u^{-1}}, R_u >$$

is an autotopism for all $u \in Q$. This imply by [2] that

$$U =< JR_u J, R_u, R_u L_{u^{-1}} >=< L_{u^{-1}}, R_u, R_u L_{u^{-1}} >$$

is an autotopism for all $u \in Q$. Therefore,

$$xL_{u^{-1}}.yR_u = xyR_u L_{u^{-1}}$$

for all $x, y \in Q$ and

$$(x.y)\theta_u = x\theta_u \star y\theta_u$$

for all $x, y \in Q$ hence θ_n is an homomorphism from $(Q, .) \longrightarrow (Q, \star)$.

If $x\theta_u = y\theta_u$ then $xR_u L_{u^{-1}} = yR_u L_{u^{-1}}$ which imply that $xa = ya$ and hence $x = y$. Therefore, θ_u is injective.

For all $x \in Q$ where $x{=}u^{-1}[(ux.u^{-1})u]{=}(ux.u^{-1})R_u L_{u^{-1}}{=}(ux.u^{-1})\theta_u$ shows that θ_u is surjective.

Therefore, θ_u is an isomorphism and hence $\theta_u \in \mathrm{BS}(Q, .)$. □

Remark 2.1. Since the Bryant-Schneider group of a loop is an isotopic invariant [11] then θ_u is also a member of $BS(Q, \star)$.

Theorem 2.2. *Let $(Q, .)$ be a flexible CC-loop then the middle inner mappings $R_u L_u^{-1}$ is a automorphism for all $u \in Q$. Furthermore, this form of CC-loop is a group.*

Proof. For all $x, y \in Q$ we have

$$\begin{aligned} xR_u L_{u^{-1}}.yR_u L_{u^{-1}} &= (u^{-1}.xu).(u^{-1}.yu) = u^{-1}(xu.u^{-1}).yu = \\ &= u^{-1}(x.yu) = u^{-1}(xy.u) = xyR_u L_{u^{-1}} \end{aligned}$$

Therefore $R_uL_{u^{-1}}$ is an automorphism.

Since $R_uL_u^{-1}$ is an automorphism $< R_uL_u^{-1}, R_uL_u^{-1}, R_uL_u^{-1} >$ is an autotopism for all $u \in Q$. Also $< R_u, R_uL_u^{-1}, R_u >$ and $< L_uR_u^{-1}, L_u, L_u >$ are autotopisms of $(Q, .)$ for all $u \in Q$. So $< R_uL_u^{-1}, R_uL_u^{-1}, R_uL_u^{-1} >< L_uR_u^{-1}, L_u, L_u >=< I, R_u, R_u >$ for all $u \in Q$, hence for all $x, y \in Q$ $x.yu = xy.u$. Therefore $(Q, .)$ is a group. □

Theorem 2.3. *If $(Q, .)$ is a CC-loop then Q can be embedded in a subgroup of the Bryant-Schneider group $BS(Q, .)$ of $(Q, .)$.*

Proof. Let $I(Q, .) = \{\Psi_u \mid u \in Q\}$. Also, let $MGI(Q, .)$ be the group generated by $I(Q, .)$ for each $u \in Q$. From theorem 2.1 above θ_u defines an isomorphism from $(Q, .)$ to $(Q, .)$ and hence $MGI(Q, .) \leq BS(Q, .)$

For each $u \in Q$ the principal isotope given by

$$x * y = xR_u.yL_{u^{-1}}$$

for all $x, y, u \in Q$ hence $\Psi_u = R_uL_{u^{-1}}$.

If a Binary operation $\circ$ is defined on $I(Q, .)$ by $\Psi_u \circ \Psi_v = \Psi_{uv}$ for all $u, v \in Q$ we obtain

$$x(\Psi_u \circ \Psi_v) = x\Psi_{uv}$$

Now to show that $I(Q, .)$ is a group.

From the definition above $\Psi_u \circ \Psi_v = \Psi_{uv} \in I(Q, .)$ since $u, v \in Q$ imply that $uv \in Q$. hence $I(Q, .)$ is closed.

If $e \in Q$ is the identity of $(Q, .)$ then $\Psi_e \circ \Psi_u = \Psi_{eu} = \Psi_{ue} = \Psi_u \circ \Psi_e = \Psi_u$. So Ψ_e is the identity element of $(I(Q, .), \circ)$.

$\Psi_u \circ \Psi_u^{-1} = \Psi_u \circ \Psi_{u^{-1}} = \Psi_{uu^{-1}} = \Psi_e$ implying that for all $\Psi_u \in (I(Q, .), \circ)$ there exist $\Psi_u^{-1} \in I(Q, .)$

If $(\Psi_u \circ \Psi_v) \circ \Psi_w = \Psi_u(\Psi_v \circ \Psi_w)$ for all $u, v, w \in Q$, then $\Psi_{uv.w} = \Psi_{u.vw}$ gives $uv.w = u.vw$ hence $(Q, .)$ is a group, hence $(I(Q, .), \circ)$ is a loop.

If a mapping $\Omega : Q \longrightarrow I(Q, .)$ is defined by $u\Omega = \Psi_u = R_uL_{u^{-1}}$ for all $u \in Q$. Ω is clearly a bijection. $(uv)\Omega = \Psi_{uv} = \Psi_u \circ \Psi_v = u\Omega \circ v\Omega$ therefore Ω is a homomorphism from $(Q, .)$ to $(I(Q, .), \circ)$. The group generated by $I(Q, .)$ $MGI(Q, .) \leq BS(Q, .)$ hence $(Q, .)$ is embeddable in $BS(Q, .)$. □

Corollary 2.1. *If $(R, .)$ is a sub-loop of a conjugacy closed loop $(Q, .)$ then $(R, .)$ is embeddable into a group generated by the set $\{R_aL_{a^{-1}} | a \in R\}$.*

Theorem 2.4. *Let* $(Q,.)$ *be a finite CC-loop then the order of the Bryant-Schneider group of* $(Q,.)$ *is given by* $2 \mid Q \mid\mid A(Q,.) \mid$ *if* $[Q : N(Q,.)] = 2$ *where* $\mid Q \mid$ *is the order of* $(Q,.)$, $\mid A(Q,.)|$ *is the order of the automorphism group and* $N(Q,.)$ *is the nucleus of* $(Q,.)$. *Furthermore,*
$[BS(Q,.) : A(Q,.)] = 2|Q|$.

Proof. For a CC-loop $(Q,.)$ with $[Q : N(Q,.)] = 2$ where $N(Q,.)$ is the nucleus of $(Q,.)$ we have $|Q| = 2|N(Q,.)|$.

Now by Bryant & Schneider [2]

$$|Q|^2|A(Q,.)| = |BS(Q,.)||N_\mu(Q,.)|$$

For a CC-loop the nuclei coincides so we have

$$|Q|^2|A(Q,.)| = |BS(Q,.)||N(Q,.)|$$

hence

$$|Q|^2|A(Q,.)| = \frac{1}{2}|BS(Q,.)||Q|$$

which imply that

$$2|Q||A(Q,.)| = |BS(Q,.)|$$

therefore

$$[BS(Q,.) : A(Q,.)] = 2|Q|$$

□

Corollary 2.2. *If* $(Q,.)$ *is a CC-loop of order* np *(where* n *is an even integer and* p *is any prime number) and* $[G : N] = 2$ *then the index of the automorphism group* $A(Q,.)$ *in the Bryant-Schneider group* $BS(Q,.)$ *is* $2np$.

Proof. Follows from theorem 2.4. □

Corollary 2.3. *If* $(Q,.)$ *is a CC-loop of order* $2pq$ *(where p&q are prime numbers) and* $[G : N] = 2$ *then the index of the automorphism group* $A(Q,.)$ *in the Bryant-Schneider group* $BS(Q,.)$ *is* $4pq$.

Proof. Follows from theorem 2.4. □

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