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FURTHER RESULTS ON THE QUALITATIVE PROPERTIES OF SOLUTIONS OF A CERTAIN CLASS OF FIFTH ORDER DIFFERENTIAL EQUATIONS

BY

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Abstract. In this paper, we extend to more general equations of the form (1.1) and (1.2), the results in [2] and [3] on existence of a unique globally exponentially stable solution which is bounded, periodic (or almost periodic).

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1. Introduction. We shall be concerned herewith the equations:

$$x^{(v)} + ax^{(iv)} + bx''' + f(x'') + g(x') + h(x) = p(t) \tag{1.1}$$

and

$$x^{(v)} + ax^{(iv)} + bx''' + f(x'') + g_1(x)x' + h(x) = p(t) \tag{1.2}$$

in which a and b are positive constants, the functions f g, g_1, h and p are continuous and depend only on the arguments displayed explicitly. It will be assumed that

$$f(0) = g(0) = g_1(0) = h(0) = 0 \tag{1.3}$$

Our purpose is to use the frequency-domain method to obtain sufficient conditions on equations (1.1) and (1.2) that guarantee
(P_α) existence of a unique solution $x(t)$, which together with its first four

derivatives $x'(t), x''(t), x'''(t)$ and $x^{(iv)}(t)$ is bounded in $\mathbb{R}$;
(P_β) existence of a solution which is globally exponentially stable, together with its first four derivatives;
(P_γ)existence of a solution which is periodic (or almost periodic) together with its first four derivatives.

We shall also show that the equation (1.2) is a dual (in the sense of Barbălat and Halanay [8]) to the equation (1.1) under a nonsingular matrix transformation.

Equations of the form (1.1) and (1.2) have recieved considerable attention using Lyapunov's second method (see for instance [1], [11] and [17]). For more expository results on the frequecy-domain techniques and its applications see e.g. ([4]-[10], [13], [14], [16] and [18]-[21]). The results obtained in this paper, generalize our earlier results in [2] and [3] to fifth-order non-linear differential equations with three non-linear terms. Our work shall utilize substantially, the generalized theorem of Yacobovich [8] which we shall now state without proof:
Consider the system:

$$X' = AX - B\varphi(\sigma) + P(t),\ \sigma = C^\star X \tag{1.4}$$

where A is an $n \times n$ real matrix, B and C are $n \times m$ real matrices with $C^\star$ as the transpose of C, $\varphi(\sigma) = col\varphi_j(\sigma_j), (j = 1, 2, ..., m)$ and $P(t)$ is an n-vector.

Suppose that in (1.3), the following assumptions are true:
(i) A is a stable matrix;
(ii) $P(t)$ is bounded for all t in $\mathbb{R}$
(iii) for some constants $\hat{\mu_j} \geq 0, (j = 1, 2, ..., m)$

$$0 \leq \frac{\varphi_j(\sigma_j) - \varphi_j(\hat{\sigma_j})}{\sigma_j - \hat{\sigma_j}} \leq \hat{\mu}_j, (\sigma_j \neq \hat{\sigma}_j) \tag{1.5}$$

(iv) there exists a diagonal matrix $D > 0$, such that the frequency-domain inequality

$$\pi(\omega) = MD + ReDG(i\omega) > 0 \tag{1.6}$$

holds for all ω in $\mathbb{R}$, where $G(i\omega) = C^\star(i\omega I - A)^{-1}B$ is the transfer function and $M = diag(\frac{1}{\hat{\mu}_j}), (j = 1, 2, ..., m)$. Then, the system (1.3) has properties P_α, P_β and P_γ.

Let us remark that the frequency-domain conditions obtained for equations (1.1) and (1.2) are necessary conditions for the existence of a positive definite Lyapunov function of the Lur'e-Postnikov form with a negative sign derivative. The main objective of this paper is to prove the following:

Theorem 1. *Consider the equation (1.1) under the assumptions (1.3). Let $p(t)$ be bounded in $\mathbb{R}$. Suppose that there exist positive parameters c, d, e, μ_1, μ_2 and μ_3 such that the functions h, g and f satisfy respectively the following inequalities*

$$e \leq \frac{h(z)-h(\bar{z})}{z-\bar{z}} \leq e+\mu_1, \ (z \neq \bar{z}) \tag{1.7}$$

$$d \leq \frac{g(z)-g(\bar{z})}{z-\bar{z}} \leq d+\mu_2, \ (z \neq \bar{z}) \tag{1.8}$$

$$c \leq \frac{f(z)-f(\bar{z})}{z-\bar{z}} \leq d+\mu_3, \ (z \neq \bar{z}) \tag{1.9}$$

Suppose further that μ_1, μ_2 and μ_3 satisfy the following inequalities

$$\frac{\mu_1\mu_2}{4(d\mu_2-c\mu_1)} < \frac{\tau_1}{\tau_2} < \frac{4e}{mu_2} \tag{1.10}$$

$$(\mu_2\mu_3)^2 \leq 16(d\mu_2-e\mu_3)(c\mu_3-b\mu_2) \tag{1.11}$$

$$(\mu_1\mu_3)^2 \leq 16e^2(a\mu_1+c\mu_3) \tag{1.12}$$

such that

$$\begin{aligned} &\frac{e}{\mu_2\mu_3}\left[e(\lambda_1\mu_2+\lambda_2\mu_3)+\lambda_3\mu_2\mu_3-\frac{4e^4}{\mu_1}\right] > \\ &> \left[\frac{b(32d-3b^2)}{128}(3b+(9b^2-32d)^{\frac{1}{2}}+d^2\right] \end{aligned} \tag{1.13}$$

where λ_1, λ_2 and λ_3 are positive parameters. Then the equation (1.1) has the properties P_α, P_β and P_γ.

Theorem 2. *Suppose that in the equation (1.2), there exist positive parameters c, d, e, μ_1, μ_2 and μ_3 such that h and f satisfy respectively the inequalities (1.7) and (1.9) with p(t) bounded in* $I\!R$. *Suppose further that the function g_1 satisfy the inequality*

$$d \leq \frac{1}{x}\int_0^x g_1(s)ds \leq d + \mu_2, \;\; x \neq 0 \tag{1.14}$$

with inequalities (2.10)-(2.13) satisfied. Then, equation (1.2) has the properties P_α, P_β and P_γ.

Remark 1. If $f(x^{''}) = ax^{''}$, $a > 0$ in the equations (1.1) and (1.2), then the Theorems 1 and 2 reduce respectively to Theorems 3.1 and 3.6 of [3].

Remark 2. If $f(x^{'}) = ax^{''}$, $a > 0$ and $g(x^{'}) = dx^{'}$, $d > 0$ in equation (1.1), then we obtain an earlier result (Theorem 3.5) of [3].

Remark 3. If $f(x^{'}) = ax^{''}$, $a > 0$ and $h(x) = ex$, $e > 0$ in the equations (1.1) and (1.2), then the Theorems 1 and 2 reduce respectively to Theorems 3.3 and 3.8 of [3].

Remark 4. Suppose that $g(x^{'}) = dx^{'}$, $d > 0$ and $h(x) = ex$ $e > 0$ in the equation (1.1), then the Theorem 1 reduces to Lemma 1 of [2].

Remark 5. Suppose that in the equations (1.1) and (1.2), $h(x) = ex$, $e > 0$, then the Theorems 1 and 2 reduce respectively to Theorems 2.1 and 2.2 of [2].

Remark 6. Suppose that in the equation (1.1), $g(x^{'}) = dx^{'}$, $d > 0$, then we have the following independent result.

Theorem 3. *Consider the equation:*

$$x^{(v)} + ax^{(iv)} + bx''' + f(x^{''}) + d(x') + h(x) = p(t) \tag{1.15}$$

where functions f, h and p are continuous, p(t) bounded in $I\!R$ *and a, b, and e are positive constants with $h(0) = f(0) = 0$. Suppose that there exist positive parameters c, e, μ_1 and μ_3 such that the functions h and f satisfy respectively the inequalities (1.7) and (2.8). Then the equation (1.15) has properties P_α, P_β and P_γ if the inequality (1.12) holds.*

2. Some preliminaries. The proof of all three theorems are based on the generalized theorem of Yacubovich [8]. Let us introduce the following:

The Routh-Hurwitz conditions for stability of solutions of the linear homogeneous equation of (1.1) and (1.2) are:

$$
\begin{aligned}
&a > 0, (ab - c) > 0, (ab - c)c - (ad - e)a > 0, \\
&(ab - c)(cd - be) - (ad - e)^2 > 0, e > 0
\end{aligned} \tag{2.1}
$$

and the consequences of these conditions are:

$$
\begin{aligned}
&b > 0, c > 0, \\
&cd - be > 0, \\
&ad - e > 0
\end{aligned} \tag{2.2}
$$

Thus, equations $v^2 a - vc + e = 0$ and $v^2 - vb + d = 0$ have two real positive roots given by v_1, v_2 and v_3, v_4 respectively, where

$$v_1 = \frac{1}{2a}[c - (c^2 - 4ae)^{\frac{1}{2}}] \tag{2.3}$$

$$v_2 = \frac{1}{2a}[c + (c^2 - 4ae)^{\frac{1}{2}}] \tag{2.4}$$

$$v_3 = \frac{1}{2}[b - (b^2 - 4d)^{\frac{1}{2}}] \tag{2.5}$$

$$v_4 = \frac{1}{2}[b + (b^2 - 4d)^{\frac{1}{2}}] \tag{2.6}$$

such that $0 < v_1 < v_3 < v_2 < v_4$.

3. The function $\pi(\omega)$. The main tool in the proof of our theorems is the function $\pi(\omega)$ defined by the inequality (1.6). For us to determine the function $\pi(\omega)$, we shall put our equations in the form of system (1.4). By setting $x_1 = x$, equation (1.1) is reduced to the system (1.4), with

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}; \ A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -e & -d & -c & -b & -a \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}; \ C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \ P(t) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ p(t) \end{pmatrix} \tag{3.1}$$

$$\sigma = C^\star X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \ \varphi(\sigma) = \begin{pmatrix} \hat{h}(x_1) \\ \hat{g}(x_2) \\ \hat{f}(x_3) \end{pmatrix}$$

Let

$$\hat{h}(z) = h(z) - ez \tag{3.2}$$

$$\hat{g}(z) = g(z) - dz \tag{3.3}$$

$$\hat{f}(z) = f(z) - cx \tag{3.4}$$

Then, the transfer function $G(i\omega) = C^\star(i\omega I - A)^{-1}B$ for the system (3.1) becomes

$$G(i\omega) = \frac{1}{\Delta} \begin{pmatrix} 1 & 1 & 1 \\ i\omega & i\omega & i\omega \\ -\omega^2 & -\omega^2 & -\omega^2 \end{pmatrix} \tag{3.5}$$

where

$$\Delta = (\omega^4 a - \omega^2 c + e) + i\omega(\omega^4 - b\omega^2 + d)$$

In order for us to get the function $\pi(\omega)$, we shall make use of the generalized theorem of Yacubovich as given in the introduction and this requires the existence of strictly positive numbers τ_1, τ_2 and τ_3 such that $D = diag(\tau_j)$ and $M = diag(\frac{1}{\mu_j})$ $(j = 1, 2, 3)$. After some calculations, we obtain

$$\pi(\omega) = \begin{pmatrix} \pi_{11} & \pi_{12} & \pi_{13} \\ \pi_{21} & \pi_{22} & \pi_{23} \\ \pi_{31} & \pi_{32} & \pi_{33} \end{pmatrix} > 0 \tag{3.6}$$

where

$$\pi_{11} = \tau_1 \left(\mu_1^{-1} + \frac{(\omega^4 a - \omega^2 c + e)}{|\Delta|^2} \right) \tag{3.7}$$

$$\pi_{22} = \tau_2 \left(\mu_2^{-1} + \omega^2 \frac{(\omega^4 - \omega^2 b + d)}{|\Delta|^2} \right) \tag{3.8}$$

$$\pi_{33} = \tau_3 \left(\mu_3^{-1} - \omega^2 \frac{(\omega^4 a - \omega^2 c + e)}{|\Delta|^2} \right) \tag{3.9}$$

$$\begin{aligned} \pi_{12} &= \frac{1}{2|\Delta|^2} \{ \big(\tau_1(\omega^4 a - \omega^2 c + e) + \omega^2 \tau_2 (\omega^4 - \omega^2 b + d) \big) \\ &\quad - i\omega \left(\tau_1(\omega^4 - \omega^2 b + d) + \tau_2(\omega^4 a - \omega^2 c + e) \right) \} \\ &= \bar{\pi}_{21} \end{aligned} \tag{3.10}$$

$$\begin{aligned} \pi_{13} &= \frac{1}{2|\Delta|^2} \{ (\tau_1 - \omega^2 \tau_3)(\omega^4 a - \omega^2 c + e) \\ &\quad - i\omega(\tau_1 + \omega^2 \tau_3)(\omega^4 - \omega^2 b + d) \} = \bar{\pi}_{31} \end{aligned} \tag{3.11}$$

$$\begin{aligned} \pi_{23} &= \frac{1}{2|\Delta|^2} \{ \big(\omega^2 \tau_2 (\omega^4 - \omega^2 b + d) - \tau_3 (\omega^4 a - \omega^2 c + e) \big) \\ &\quad + i\omega \left(\tau_2(\omega^4 a - \omega^2 c + e) - \tau_3 \omega^2 (\omega^4 - \omega^2 b + d) \right) \} \\ &= \bar{\pi}_{23} \end{aligned} \tag{3.12}$$

with $\bar{\pi_{ji}}$ $(j, i = 1, 2, 3)$ $j \neq i$ as the complex conjugate of π_{ij}. For inequality (3.6) to be true for all ω in $\mathbb{R}$, we shall use the Sylvester's criterion which requires that the determinant of the principal minors of $\pi(\omega)$ be positive for all ω in $\mathbb{R}$. We shall now give a series of Lemma which will be useful in the proof of the inequality (3.6).

Lemma 1. *Let*

$$S_1(v) = (-v^2a + vc - e) + \frac{v(v^2 - vb + d)}{(-v^2a + vc - e)}$$

where $\omega^2 = v$. *Then,* $\pi_{11}(\omega)$ *is positive for all* $v > 0$ *provided that*

$$\mu_1 > M_1(d, e) = S_1(v_0) = \max_{v_1 < v < v_2} S_1(v)$$

and

$$S_1(v_0) > S_1(v_3) = \frac{1}{2}(c - ab)\left(b - (b^2 - 4d)^{\frac{1}{2}}\right) + (ad - e)$$

where v_0 *is the unique real root of* $A_1(v) = 0$ *and* $M_1(d, e)$ *is the maximum value of* $S_1(v)$ *and attainable at say* $v = v_0$.

Proof. This has been proved in Lemma 4.1 of [3]. □

Lemma 2. *Let*

$$S_2(v) = (-v^2 + vb - d) + \frac{(v^2a - vc + e)^2}{v(-v^2 + vb - d)}$$

where $\omega^2 = v$. *Then,* $\pi_{22}(\omega)$ *is positive for all* $v > 0$, *provided that*

$$\mu_2 < M_2(d, e) = S_2(v_0) = \min_{v_3 < v < v_4} S_2(v)$$

and

$$S_2(v_0) = S_2(v_2) = \frac{1}{2a}\left(b - \frac{c}{a}\right)\left(c + (c^2 - 4ae)^{\frac{1}{2}}\right) - \left(d - \frac{e}{a}\right)$$

where v_0 *is the unique real root of* $A_2(v) = 0$ *with* $v_3 < v_0 < v_4$ *and* $M_2(d, e)$ *is the minimum of* $S_2(v)$ *and attainable at say* $v = v_0$. *Furthermore, if* $v_2 \neq \frac{b}{2}$, *then,* $S_2(v_2) > S_2(\frac{b}{2})$ *and if* $v = \frac{b}{2}$ *with* $e = \frac{2bc - b^2a}{4}$ *and* $\varepsilon < 2b(b^2 - 4d)(b - \frac{c}{a})$, $\varepsilon > 0$, *then,* $S_2(v_2) > S_2(\frac{b}{2})$.

Proof. See also the proof of Lemma 4.2 of [3]. □

Lemma 3. *Let*

$$S_3(v) = \frac{(v^2a - vc + e)}{v} + \frac{(v^2 - vb + d)^2}{(v^2a - vc + e)}$$

where $\omega^2 = v$. *Then,* $\pi_{33}(\omega)$ *is positive for all* $v > 0$ *provided that*

$$\mu_3 \leq M_3(c) = S_3(v_0) = \min_{v_1 < v < v_2} S_3(v)$$

and

$$S_3(v_0) = S_3(v_3) = ab - c - \frac{2(ad - e)}{(b - (b^2 - 4d)^{\frac{1}{2}}}$$

where v_0 *is the unique real root of* $A_3(v) = 0$ *with* $v_1 < v_0 < V_2$ *and* $M_3(c)$ *is the minimum value of* $S_3(v)$ *and attainable at say* $v = v_0$.

Proof. See the proof of Lemma 1 of [2]. □

Lemma 4. *For all* ω *in* $\mathbb{R}$ *and for* $1 \leq j, k < 3, (j \neq k)$

$$\pi_{jj}(\omega)\pi_{kk}(\omega) - |\pi_{jk}(\omega)|^2 > 0 \tag{3.13}$$

Proof. Let us first consider the case when $j = 1$ and $k = 2$ in the equation (3.12), then we have from the inequality (3.6) for $\omega^2 = v$,

(3.14)

$$\pi_{11}\pi_{22} - |\pi_{12}|^2 =$$

$$= \tau_1\tau_2 \left(\frac{1}{\mu_1\mu_2} + \frac{1}{|\Delta|^2}\left(\frac{v(v^2 - vb + d)}{\mu_1} + \frac{(v^2a - vc + e)}{\mu_2} - \frac{({\tau_1}^2 + v{\tau_2}^2)}{4\tau_1\tau_2}\right)\right)$$

This will be positive for all $v > 0$ in $\mathbb{R}$, if the inequality (1.10) is satisfied. The technique for the verification of this is the same as that used in the proof of Lemma 4.3 of [3] and we shall therefore skip details. Thus, $\pi_{11}\pi_{22} - |\pi_{12}|^2 > 0$ for all v in $\mathbb{R}$.

Next if $j = 2$ and $k = 3$, we have

(3.15)

$$\pi_{22}\pi_{33} - |\pi_{23}|^2 =$$

$$= \tau_2\tau_3 \left(\frac{1}{\mu_2\mu_3} + \frac{v}{|\Delta|^2}\left(\frac{(v^2 - vb + d)}{\mu_3} - \frac{(v^2a - vc + e)}{\mu_2} - \frac{({\tau_2}^2 + v{\tau_3}^2)}{4\tau_2\tau_3}\right)\right)$$

This will be positive for all $v > 0$ in $\mathbb{R}$, if the inequality (1.11) holds. This has already been proved in Lemma 3 of [3]. So we skip the details. Thus, $\pi_{22}\pi_{33} - |\pi_{23}|^2 > 0$ for all v in $\mathbb{R}$.

Finally, let $j = 3$ and $k = 1$. Again, we obtain

(3.16)
$$\pi_{33}\pi_{11} - |\pi_{31}|^2 = \\ = \tau_1\tau_3\left(\frac{1}{\mu_1\mu_3} + \frac{1}{|\Delta|^2}\left((v^2a - vc + e)\left(\frac{1}{\mu_3} - \frac{v}{\mu_1}\right) - \frac{({\tau_1}^2 + v^2{\tau_3}^2)}{4\tau_1\tau_3} - \frac{\omega^2}{2}\right)\right)$$

This will be positive for all $v > 0$ in $\mathbb{R}$, if

$$a^2 > 2b \tag{3.17}$$

$$b^2 > 2(ac + d) \tag{3.18}$$

$$c^2 < 2(ae - bd) \tag{3.19}$$

$$d^2 > 2ac \tag{3.20}$$

are satisfied. By our choice of $\mu_1\mu_3$ in the inequality (1.12), we have,

$$\frac{\mu_1\mu_3}{4(a\mu_1 + c\mu_3)} < \frac{\tau_1}{\tau_3} < \frac{4e^2}{\mu_1\mu_3} \tag{3.21}$$

This is possible since $(\mu_1\mu_3)^2 < 16e^2(a\mu_1 + c\mu_3)$.

Thus, $\pi_{33}\pi_{11} - |\pi_{13}|^2 > 0$ for all v in $\mathbb{R}$.

Lemma 5. *For all $v > 0$, $det\pi(\omega) > 0$ $(\omega^2 = v)$*

Proof. From the inequality (3.6),

$$\det\pi(\omega) = \pi_{11}\left(\pi_{22}\pi_{33} - |\pi_{23}|^2\right) + \pi_{22}\left(\pi_{11}\pi_{33} - |\pi_{31}|^2\right) + \\ + \pi_{33}\left(\pi_{11}\pi_{22} - |\pi_{12}|^2\right) - 2\pi_{11}\pi_{22}\pi_{33} + 2Re(\pi_{12}\pi_{23}\pi_{31}) > 0 \tag{3.22}$$

which becomes

(3.23)

$$\det\pi(\omega)=\tau_1\tau_2\tau_3\left[\frac{1}{\mu_1\mu_2\mu_3}+\frac{1}{|\Delta|^2}\left(\frac{v(v^2-vb+d)}{\mu_1\mu_3}-\right.\right.$$
$$-\frac{v(v^2a-vc+e)}{\mu_1\mu_2}+\frac{v(v^2a-vc+e)}{\mu_2\mu_3}-$$
$$\left.-\frac{v({\tau_2}^2+v{\tau_3}^2)}{4\mu_1\tau_2\tau_3}-\frac{({\tau_1}^2-v^2{\tau_3}^2)}{4\mu_2\tau_1\tau_3}-\frac{({\tau_1}^2+v{\tau_2}^2)}{4\mu_3\tau_1\tau_2}\right)+$$
$$+\frac{1}{|\Delta|^4}\left(\frac{v(v^2a-vc+e)({\tau_1}^2+v{\tau_2}^2)}{4\tau_1\tau_2}-\frac{v(v^2-vb+d)({\tau_1}^2+v^2{\tau_3}^2)}{4\tau_1\tau_3}-\right.$$
$$\left.-\frac{(v^2a-vc+e)({\tau_2}^2+v{\tau_3}^2)}{4\tau_2\tau_3}-\frac{v^2(v^2-vb+d)^2+v(v^2a-vc+e)^2}{2\mu_2}\right)-$$
$$\left.-\frac{v^2(v^2-vb+d)}{|\Delta|^6}\left(2(v^2-vb+d)-v(v^2-vb+d)^2-(v^2a-vc+e)^2\right)\right]>0$$

For $v\neq 0$, $det\pi(\omega)>0$ is equivalent to showing that

(3.24)

$$\frac{1}{P}\left[|\Delta|^6+|\Delta|^4\left(v\left((v^2-vb+d)\,\mu_2-(v^2a-vc+e)(\mu_3-\mu_1)\right)-\right.\right.$$
$$-\frac{({\tau_1}^2+v{\tau_2}^2)\mu_1\mu_2}{4\tau_1\tau_2}-\frac{({\tau_1}^2-v^2{\tau_3}^2)\mu_1\mu_3}{4\tau_1\tau_3}-\frac{v({\tau_2}^2+v{\tau_3}^2)\mu_2\mu_3}{4\tau_2\tau_3}+$$
$$+\frac{v(\tau_1-v\tau_3)^2(v^2a-vc+e)^2(v^2-vb+d)^2}{2\tau_1\tau_3}\times$$
$$\times v(v^2a-vc+e)\left(\frac{{\tau_2}^2-\tau_1\tau_3}{\tau_2\tau_3}+\frac{v(\tau_1\tau_3-{\tau_2}^2)}{\tau_1\tau_2}\right)\mu_1\mu_2\mu_3-$$
$$\left.-\frac{v^2(\tau_1+v\tau_2)^2(v^2-vb+d)\left((v^2a-vc+e)^2-v(v^2-vb+d)^2\right)^2}{4\tau_1\tau_3}\mu_1\mu_2\mu_3\right)+$$
$$+|\Delta|^2\left(\frac{v(v^2a-vc+e)({\tau_1}^2+v{\tau_2}^2)}{4\tau_1\tau_2}-\frac{v(v^2-vb+d)({\tau_1}^2+v^2{\tau_3}^2)}{4\tau_1\tau_3}-\right.$$
$$\left.-\frac{v|\Delta|^2\mu_1\mu_3}{2}--\frac{(v^2a-vc+e)({\tau_2}^2+v{\tau_3}^2)}{4\tau_1\tau_3}\right)\mu_1\ \mu_2\mu_3]>v^2(v^2-vb+d)\mu_1\mu_2\mu_3$$

where $P=v(v^2-vb+d)^2-(v^2a-vc+e)^2$

The maximum of the right hand side of inequality (3.34) is

$$\left(\frac{b(32d-9b^2)}{512}(3b+(9b^2-32d)^{\frac{1}{2}}-\frac{d^2}{4}\right)\mu_1\mu_2\mu_3$$

and the minimum of the left hand side is

$$\frac{\tau_1\mu_1 e^2}{4}\left(\frac{\mu_3}{\tau_3}+\frac{\mu_2}{\tau_2}\right)+\frac{{\tau_2}^2\mu_1\mu_2\mu_3 e}{4\tau_1\tau_3}-e^4$$

Hence, if

$$\frac{e}{\mu_2\mu_3}\left(e(\lambda_1\mu_2+\lambda_2\mu_3+\lambda_3\mu_2\mu_2-4\frac{e^4}{\mu_1}\right)>$$

$$>\left(\frac{b(32d-9b^2)}{512}(3b+(9b^2-32d)^{\frac{1}{2}}-\frac{d^2}{4}\right)$$

(i.e. minimum of l.h.s is greater than maximum of r.h.s of inequality (3.24)) where $\lambda_1=\frac{\tau_1}{\tau_2}$, $\lambda_2=\frac{\tau_1}{\tau_3}$ and $\lambda_3=\frac{\tau_2}{\tau_3}$. Then $det\pi(\omega)$ will be positive. This is possible and it follows from the inequalities (1.10)-(1.12). □

4. Proof of theorems. We shall now give the proof of the theorems stated in Section 1.

Proof of Theorem 1. Choose positive parameters c, d and e satisfying inequalities (1.10)-(2.12) and (3.2)-(3.4).Then we can rewrite the inequalities (1.7), (2.8) and (2.9) respectively for $z\neq\hat{z}$, as

$$0\leq\frac{\hat{h}(z)-\hat{h}(\bar{z})}{z-\bar{z}}\leq\mu_1 \tag{4.1}$$

$$0\leq\frac{\hat{g}(z)-\hat{g}(\bar{z})}{z-\bar{z}}\leq\mu_2 \tag{4.2}$$

$$0\leq\frac{\hat{f}(z)-\hat{f}(\bar{z})}{z-\bar{z}}\leq\mu_3 \tag{4.3}$$

By setting $x=x_1$, the equation (1.1) reduces to the equivalent form:

$$\begin{aligned}
x_1^{'} &= x_2\\
x_2^{'} &= x_3\\
x_3^{'} &= x_4\\
x_4^{'} &= -ax_4+x_5\\
x_5^{'} &= -ex_1-dx_2-cx_3-bx_4-ax_5-\hat{f}(x_3)-\hat{g}(x_2)-\hat{h}(x_1)+p(t)
\end{aligned} \tag{4.4}$$

which in the vector form (1.4) with X, A, B, C, P and $\varphi(\sigma)$ as given in the system (3.1) reduces to the matrix inequality (3.6) which is satisfied for all ω in $\mathbb{R}$. This is true by using Lemmas 1-5. Thus the conclusions of Theorem 1. □

Proof of Theorem 2. We shall reduce the equation (1.2) into a system which will be dual (in the sense of Barbălat and Halanay [8]) to system (3.1). Let

$$\int_0^x g_1(s)ds = dx + \hat{g}_1(x) \tag{4.5}$$

$f(x) = cx + \hat{f}(x)$ and $h(x) = dx + \hat{h}(x)$. Then the inequalities (1.13) can be written as

$$0 \leq \frac{\hat{g}_1(x)}{x} \leq \mu_2,\ x \neq 0 \tag{4.6}$$

Futhermore, the equation (1.2) is reduced to the following system:

$$\begin{aligned} x_1' &= -ex_5 - \hat{h}(x_5) + p(t) \\ x_2' &= x_1 - dx_5 - \hat{g}(x_5) \\ x_3' &= x_2 - cx_5 - \hat{f}(x_5) \\ x_4' &= x_3 - bx_5 \\ x_5' &= x_4 - ax_5 \end{aligned} \tag{4.7}$$

which in vector form

$$X' = A_1X - B_1\varphi(\hat{\sigma}_1) + P_1(t),\ \hat{\sigma}_1 = C_1^\star X \tag{4.8}$$

has $A_1 = A^\star$, $B_1 = C$, $C_1 = B$ and $P_1(t) = TP(t)$, with A, B, C and P as given in the system (3.1), $A^\star$ is the transpose of A and T is a non-singular matrix transformation given by

$$T = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence equation (1.2) is a dual to equation (1.1) (see [2], [3] and [8]). Thus the frequency-domain inequality for both systems (4.4) and (4.7) are equivalent and the conclusions to Theorem 2 follow from Theorem 1. □

Proof of Theorem 3. Let $f(z) = cz + \hat{f}(z)$ and $h(z) = ez + \hat{h}(z)$ with c and e as positive parameters. The equation (1.15) can be written in the vector form (1.4) with X and A as given in the system (3.1) and

$$B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}; \; C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix};$$

(4.8)

$$P(t) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ p(t) \end{pmatrix} \quad \varphi(\sigma) = \begin{pmatrix} \hat{h}(x_1) \\ \hat{f}(x_3) \end{pmatrix}$$

The frequency-domain inequality for the equation (1.15) is given as

$$\pi(\omega) = \begin{pmatrix} \pi_{11} & \pi_{13} \\ \pi_{31} & \pi_{33} \end{pmatrix} > 0 \tag{4.9}$$

with π_{11}, π_{13}, π_{31} and π_{33} as given in inequality (3.6). Inequality (4.9) is valid for all $\omega > 0$ in $\mathbb{R}$ by using lemmas 1, 3 and 4. The conclusions follow from the generalized theorem of Yacubovich. □

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