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PSEUDO-BL ALGEBRA OF FRACTIONS RELATIVE TO AN $\wedge$-CLOSED SYSTEM

BY

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Abstract. The aim of this paper is to introduce the notion of pseudo-BL algebra of fractions relative to an $\wedge$-closed system. For the case of Hilbert algebras, MV-algebras, pseudo - MV algebras and BL algebras see [2], [3], [12] and [4].

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1. Definitions and first properties

Definition 1. A BL-algebra ([8]-[13]) is an algebra

$$(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$$

of type (2,2,2,2,0,0) satisfying the following:

(a_1) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice,

(a_2) $(A, \odot, 1)$ is a commutative monoid,

(a_3) $\odot$ and $\rightarrow$ form an adjoint pair, i.e. $c \leq a \rightarrow b$ iff $a \odot c \leq b$ for all $a, b, c \in A$,

(a_4) $a \wedge b = a \odot (a \rightarrow b)$,

(a_5) $(a \rightarrow b) \vee (b \rightarrow a) = 1$, for all $a, b \in A$.

The origin of BL-algebras is in Mathematical Logic; they where invented by HÁJEK in [8] in order to study the ,,Basic Logic" (BL, for short) arising from the continuous triangular norms, familiar in the framework of fuzzy set theory. They play the role of Lindenbaum algebras from classical Propositional calculus. Apart from their logical interest, BL-algebras have important algebraic properties (see [9]-[13]). In [10]-[11], DI NOLA, GEORGESCU and IORGULESCU defined the pseudo-BL algebras as a non-commutative extension of BL algebras; the class of pseudo-BL algebras contains the pseudo-MV algebras.

Definition 2. A pseudo-BL algebra ([10]) is an algebra

$$(A, \wedge, \vee, \odot, \rightarrow, \rightsquigarrow 0, 1)$$

of type (2,2,2,2,2,0,0) satisfying the following:

(a_6) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice,

(a_7) $(A, \odot, 1)$ is a monoid,

(a_8) $a \odot b \leq c$ iff $a \leq b \rightarrow c$ iff $b \leq a \rightsquigarrow c$ for all $a, b, c \in A$,

(a_9) $a \wedge b = (a \rightarrow b) \odot a = a \odot (a \rightsquigarrow b)$,

(a_{10}) $(a \rightarrow b) \vee (b \rightarrow a) = (a \rightsquigarrow b) \vee (b \rightsquigarrow a) = 1$, for all $a, b \in A$.

We shall agree that the operations $\wedge, \vee, \odot$ have priority towards the operations $\rightarrow, \rightsquigarrow$.

Examples.

(E_1) Let $(A, \odot, \oplus, ^-, ^\sim, 0, 1)$ a pseudo-MV algebra and let $\rightarrow, \rightsquigarrow$ be two implications defined by

$$x \rightarrow y = y \oplus x^-$$

$$x \rightsquigarrow y = x^\sim \oplus y.$$

Then $(A, \wedge, \vee, \odot, \rightarrow, \rightsquigarrow 0, 1)$ is a pseudo-BL algebra.

(E_2) Let us consider an arbitrary l-group $(G, \wedge, \vee, +, -, 0, 1)$ and let $u' \in G, u' \leq 0$. We put by definition:

$$x' \odot y' = (x' + y') \vee u'$$

$$(x')^- = u' - x'$$

$$(x')^\sim = -x' + u'$$

$$x' \oplus y' = (x' - u' + y') \wedge 0.$$

Then $(A = [u', 0], \odot, \oplus, ^-, ^\sim, \mathbf{0} = u', \mathbf{1} = 0)$ is a pseudo-MV algebra and we define two implications:

$$x' \to y' = (y' - x') \wedge 0$$

$$x' \rightsquigarrow y' = (-x' + y') \wedge 0.$$

Then $(A = [u', 0], \wedge, \vee, \odot, \to, \rightsquigarrow, \mathbf{0} = u', \mathbf{1} = 0)$ is a pseudo-BL algebra.

A pseudo-BL algebra is *nontrivial* iff $0 \neq 1$. For any pseudo-BL algebra A, the reduct $L(A) = (A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice. For any $a \in A$, we define

$$a^- = a \to 0$$

and

$$a^\sim = a \rightsquigarrow 0.$$

We shall write $a^=$ instead of $(a^-)^-$ and we shall write $a^\approx$ instead of $(a^\sim)^\sim$.

Definition 3. A pseudo-BL algebra A is commutative iff $x \odot y = x \odot y$, for any $x, y \in A$.

Remark 1. A pseudo-BL algebra A is commutative iff $x \rightsquigarrow y = x \to y$, for all $x, y \in A$. Any commutative pseudo-BL algebra A is a BL algebra.

In [10]-[11] it is proved that if A is a pseudo-BL algebra and $a, b, c, b_i \in A$, $(i \in I)$ then we have the following rules of calculus:

(c_1) $a \odot (a \rightsquigarrow b) \leq b \leq a \rightsquigarrow (a \odot b)$ and $a \odot (a \rightsquigarrow b) \leq a \leq b \rightsquigarrow (b \odot a)$,

(c_2) $(a \to b) \odot a \leq a \leq b \to (a \odot b)$ and $(a \to b) \odot a \leq b \leq a \to (b \odot a)$,

(c_3) if $a \leq b$ then $a \odot c \leq b \odot c$ and $c \odot a \leq c \odot b$,

(c_4) if $a \le b$ then $c \rightsquigarrow a \le c \rightsquigarrow b$ and $c \to a \le c \to b$,

(c_5) if $a \le b$ then $b \rightsquigarrow c \le a \rightsquigarrow c$ and $b \to c \le a \to c$,

(c_6) $a \le b$ iff $a \to b = 1$ iff $a \rightsquigarrow b = 1$,

(c_7) $a \rightsquigarrow a = a \to a = 1$,

(c_8) $1 \rightsquigarrow a = 1 \to a = a$,

(c_9) $b \le a \rightsquigarrow b$ and $b \le a \to b$,

(c_{10}) $a \odot b \le a \wedge b$ and $a \odot b \le a, b$,

(c_{11}) $a \rightsquigarrow 1 = a \to 1 = 1$,

(c_{12}) $a \rightsquigarrow b \le (c \odot a) \rightsquigarrow (c \odot b)$,

(c_{13}) $a \to b \le (a \odot c) \to (b \odot c)$,

(c_{14}) if $a \le b$ then $a \le c \rightsquigarrow b$ and $a \le c \to b$,

(c_{15}) $(b \rightsquigarrow c) \odot a \le b \rightsquigarrow (c \odot a)$ and $a \odot (b \to c) \le b \to (a \odot c)$,

(c_{16}) if $a \le b$ then $b \rightsquigarrow 0 \le a \rightsquigarrow 0$ and $b \to 0 \le a \to 0$,

(c_{17}) $0 \odot a = a \odot 0 = 0$,

(c_{18}) $(a \rightsquigarrow b) \odot (b \rightsquigarrow c) \le a \rightsquigarrow c$ and $(b \to c) \odot (a \to b) \le a \to c$,

(c_{19}) $(a_1 \rightsquigarrow a_2) \odot (a_2 \rightsquigarrow a_3) \odot ... \odot (a_{n-1} \rightsquigarrow a_n) \le a_1 \rightsquigarrow a_n$,
$(a_{n-1} \to a_n) \odot ... \odot (a_2 \to a_3) \odot (a_1 \to a_2) \le a_1 \to a_n$,

(c_{20}) $a \vee (b \odot c) \ge (a \vee b) \odot (a \vee c)$,

(c_{21}) $a \vee b = ((a \rightsquigarrow b) \to b) \wedge ((b \rightsquigarrow a) \to a)$,

(c_{22}) $a \vee b = ((a \to b) \rightsquigarrow b) \wedge ((b \to a) \rightsquigarrow a)$,

(c_{23}) $a \rightsquigarrow (b \rightsquigarrow c) = (b \odot a) \rightsquigarrow c$ and $a \to (b \to c) = (a \odot b) \to c$,

(c_{24}) $a \rightsquigarrow b = a \rightsquigarrow (a \wedge b)$ and $a \to b = a \to (a \wedge b)$,

(c_{25}) $(a \rightsquigarrow b) \rightsquigarrow (a \rightsquigarrow c) = (b \rightsquigarrow a) \rightsquigarrow (b \rightsquigarrow c)$ and $(a \to b) \to (a \to c) = (b \to a) \to (b \to c)$,

(c_{26}) $c \odot (a \wedge b) = (c \odot a) \wedge (c \odot b)$ and $(a \wedge b) \odot c = (a \odot c) \wedge (b \odot c)$,

(c_{27}) $(a \rightsquigarrow b) \odot (a' \rightsquigarrow b') \leq (a \vee a') \rightsquigarrow (b \vee b')$,

(c_{28}) $(a \rightsquigarrow b) \odot (a' \rightsquigarrow b') \leq (a \wedge a') \rightsquigarrow (b \wedge b')$,

(c_{29}) $(a \to b) \odot (a' \to b') \leq (a \vee a') \to (b \vee b')$,

(c_{30}) $(a \to b) \odot (a' \to b') \leq (a \wedge a') \to (b \wedge b')$,

(c_{31}) $(a \rightsquigarrow b) \rightsquigarrow c \leq ((b \rightsquigarrow a) \rightsquigarrow c) \rightsquigarrow c$ and $(a \to b) \to c \leq ((b \to a) \to c) \to c$,

(c_{32}) $(a \rightsquigarrow b) \rightsquigarrow c \leq ((b \rightsquigarrow a) \rightsquigarrow c) \to c$ and $(a \to b) \to c \leq ((b \to a) \to c) \rightsquigarrow c$,

(c_{33}) $a \rightsquigarrow b \leq (b \rightsquigarrow c) \to (a \rightsquigarrow c)$ and $a \to b \leq (b \to c) \rightsquigarrow (a \to c)$,

(c_{34}) $a \rightsquigarrow b \leq (c \rightsquigarrow a) \rightsquigarrow (c \rightsquigarrow b)$ and $a \to b \leq (c \to a) \to (c \to b)$,

(c_{35}) if $x \vee y = 1$ then $x \odot y = x \wedge y$,

(c_{36}) if $x \vee y = 1$ then, for each $n \in \omega, n \geq 1, x^n \vee y^n = 1$,

(c_{37}) for each $n \in \omega, n \geq 1$,

$$(x \to y)^n \vee (y \to x)^n = 1,$$

$$(x \rightsquigarrow y)^n \vee (y \rightsquigarrow x)^n = 1,$$

(c_{38}) $a \wedge (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \wedge b_i)$,

$a \odot (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \odot b_i)$,

$(\bigvee_{i \in I} b_i) \odot a = \bigvee_{i \in I} (b_i \odot a)$,

$a \rightsquigarrow (\bigwedge_{i \in I} b_i) = \bigwedge_{i \in I} (a \rightsquigarrow b_i)$,

$a \to (\bigwedge_{i \in I} b_i) = \bigwedge_{i \in I} (a \to b_i)$,

$(\bigvee_{i \in I} b_i) \rightsquigarrow a = \bigwedge_{i \in I} (b_i \rightsquigarrow a)$,

$(\bigvee_{i \in I} b_i) \to a = \bigwedge_{i \in I} (b_i \to a)$,

(whenever the arbitrary meets and unions exist)

(c_{39}) $1^{\sim} = 1^{-} = 0, 0^{\sim} = 0^{-} = 1,$

(c_{40}) $a \odot a^{\sim} = a^{-} \odot a = 0,$

(c_{41}) $b \leq a^{\sim}$ iff $a \odot b = 0,$

(c_{42}) $b \leq a^{-}$ iff $b \odot a = 0,$

(c_{43}) $a \leq a^{-} \rightsquigarrow b, a \leq a^{\sim} \rightarrow b,$

(c_{44}) $a \leq (a^{\sim})^{-}, a \leq (a^{-})^{\sim},$

(c_{45}) $a \rightsquigarrow b \leq b^{\sim} \rightarrow a^{\sim}, a \rightarrow b \leq b^{-} \rightsquigarrow a^{-},$

(c_{46}) $a \rightarrow b^{\sim} = b \rightsquigarrow a^{-}, a \rightsquigarrow b^{-} = b \rightarrow a^{\sim},$

(c_{47}) $a \leq b$ implies $b^{\sim} \leq a^{\sim}$ and $b^{-} \leq a^{-},$

(c_{48}) $((a^{\sim})^{-})^{\sim} = a^{\sim}, ((a^{-})^{\sim})^{-} = a^{-},$

(c_{49}) $a \rightarrow a^{\sim} = a \rightsquigarrow a^{-},$

(c_{50}) $(b \odot a)^{\sim} = a \rightsquigarrow b^{\sim}, (a \odot b)^{-} = a \rightarrow b^{-},$

(c_{51}) $(a \wedge b)^{\sim} = a^{\sim} \vee b^{\sim}, (a \vee b)^{\sim} = a^{\sim} \wedge b^{\sim},$

(c_{52}) $(a \wedge b)^{-} = a^{-} \vee b^{-}, (a \vee b)^{-} = a^{-} \wedge b^{-},$

(c_{53}) $(a \wedge b)^{\approx} = a^{\approx} \wedge b^{\approx}, (a \vee b)^{\approx} = a^{\approx} \vee b^{\approx},$

(c_{54}) $(a \wedge b)^{=} = a^{=} \wedge b^{=}, (a \vee b)^{=} = a^{=} \vee b^{=},$

(c_{55}) $(a \vee c) \odot (b \vee c \vee a^{\sim}) \leq (a \odot b) \vee c,$

(c_{56}) $(b \vee c \vee a^{-}) \odot (a \vee c) \leq (b \odot a) \vee c.$

For any pseudo-BL algebra A, let us denote

$$G(A) = \{x \in A : x \odot x = x\},$$

$$M(A) = \{x \in A : x = (x^{\sim})^{-} = (x^{-})^{\sim}\}$$

and let $B(A)$ be the Boolean algebra of all complemented elements in the distributive lattice $L(A)$ (hence $B(A) = B(L(A))$).

Lemma 1. ([11]) *If A is a pseudo-BL algebra, then*

$$B(A) = M(A) \cap G(A).$$

Proposition 2. ([11]) *If $a \in G(A)$ and $b \in A$, then*

(i) $a \odot b = a \wedge b = b \odot a$,

(ii) $a \wedge a^{\sim} = 0 = a \wedge a^{-}$,

(iii) $a \rightsquigarrow b = a \to b$,

(iv) $a^{\sim} = a^{-}$.

Proposition 3. ([10]-[11]) *For $e \in A$, the following are equivalent:*

(i) $e \in B(A)$,

(ii) $e \odot e = e$ *and* $e = (e^{\sim})^{-} = (e^{-})^{\sim}$,

(iii) $e \odot e = e$ *and* $e^{-} \to e = e$,

(iii') $e \odot e = e$ *and* $e^{\sim} \rightsquigarrow e = e$,

(iv) $e \vee e^{-} = 1$,

(iv') $e \vee e^{\sim} = 1$.

Remark 2. If $a \in A$, and $e \in B(A)$, then $e \odot a = e \wedge a = e \odot a$ and $e^{\sim} = e^{-}$.

Proposition 4. *If $a \in A$, and $e \in B(A)$, then*

(c_{57}) $a \to e = (a \odot e^{\sim})^{-} = a^{-} \vee e$,

(c_{58}) $a \rightsquigarrow e = (e^{-} \odot a)^{\sim} = e \vee a^{\sim}$.

Proof. We have

$$a \to e = a \to (e^{\sim})^{-} \overset{c_{50}}{=} (a \odot e^{\sim})^{-} = (a \wedge e^{\sim})^{-} \overset{c_{52}}{=} a^{-} \vee (e^{\sim})^{-} = a^{-} \vee e$$

and

$$a \rightsquigarrow e = a \rightsquigarrow (e^{-})^{\sim} \overset{c_{50}}{=} (e^{-} \odot a)^{\sim} = (e^{-} \wedge a)^{\sim} \overset{c_{51}}{=} (e^{-})^{\sim} \vee a^{\sim} = e \vee a^{\sim}. \square$$

Proposition 5. *For $e \in A$, the following are equivalent:*

(*i*) $e \in B(A)$,

(*ii*) $(e \to x) \to e = e$, *for every* $x \in A$,

(*iii*) $(e \rightsquigarrow x) \rightsquigarrow e = e$, *for every* $x \in A$.

Proof. $(i) \Rightarrow (ii), (iii)$. If $x \in A$, then from $0 \le x$ we deduce $e \to 0 \le e \to x$ and $e \rightsquigarrow 0 \le e \rightsquigarrow x$, so $e^- \le e \to x$ and $e^\sim \le e \to x$ hence $(e \to x) \to e \le e^- \to e = e$ and $(e \rightsquigarrow x) \rightsquigarrow e \le e^\sim \rightsquigarrow e = e$. Since $e \le (e \to x) \to e$, $e \le (e \rightsquigarrow x) \rightsquigarrow e$ (by c_9) we obtain $(e \to x) \to e = (e \rightsquigarrow x) \rightsquigarrow e = e$.

$(ii) \Rightarrow (i)$. If $x \in A$, then from $(e \to x) \to e = e$ we deduce $[(e \to x) \to e] \odot (e \to x) = e \odot (e \to x)$, hence $(e \to x) \wedge e = (e \to x) \odot e$ so $(e \to x) \wedge e = e \wedge x$. For $x = 0$ we obtain that $e^- \wedge e = 0$. Also, from hypothesis (for $x = 0$) we obtain by Proposition 2, $e^- \to e = e^- \rightsquigarrow e = e$. So, from c_{22} we obtain

$$\begin{aligned} e^- \vee e &= [(e^- \to e) \rightsquigarrow e] \wedge [(e \to e^-) \rightsquigarrow e^-] \\ &= (e \rightsquigarrow e) \wedge [(e \to e^-) \rightsquigarrow e^-] \\ &= 1 \wedge [(e \to e^-) \rightsquigarrow e^-] \\ &= (e \to e^-) \rightsquigarrow e^- = (e \to e^\sim) \rightsquigarrow e^\sim \\ &= [e \odot (e \to e^\sim)]^\sim \text{ (by } (c_{58})) \\ &= [(e \to e^\sim) \odot e]^\sim = (e \wedge e^-)^\sim = 0^\sim = 1, \end{aligned}$$

hence $e \in B(A)$.

$(iii) \Rightarrow (i)$. If $x \in A$, then from $(e \rightsquigarrow x) \rightsquigarrow e = e$ we deduce $(e \rightsquigarrow x) \odot [(e \rightsquigarrow x) \rightsquigarrow e] = (e \rightsquigarrow x) \odot e$, hence $(e \rightsquigarrow x) \wedge e = (e \rightsquigarrow x) \odot e$ so $(e \rightsquigarrow x) \wedge e = e \wedge x$. For $x = 0$ we obtain that $e \wedge e^\sim = 0$. Also, from hypothesis (for $x = 0$) we obtain by Proposition 2, $e^\sim \rightsquigarrow e = e^\sim \to e = e$. So, from c_{22} we obtain

$$\begin{aligned} e^\sim \vee e &= [(e^\sim \to e) \rightsquigarrow e] \wedge [(e \to e^\sim) \rightsquigarrow e^\sim] \\ &= (e \rightsquigarrow e) \wedge [(e \to e^\sim) \rightsquigarrow e^\sim] \\ &= 1 \wedge [(e \to e^\sim) \rightsquigarrow e^\sim] \\ &= (e \to e^\sim) \rightsquigarrow e^\sim = (e \to e^-) \rightsquigarrow e^\sim \\ &= [e \odot (e \to e^-)]^\sim \text{ (by } (c_{58})) \\ &= (e \wedge e^-)^\sim = 0^\sim = 1, \end{aligned}$$

hence $e \in B(A)$. $\square$

Lemma 6. *If* $e, f \in B(A)$ *and* $x, y \in A$, *then:*

(c_{59}) $e \vee (x \odot y) = (e \vee x) \odot (e \vee y)$,

(c_{60}) $e \wedge (x \odot y) = (e \wedge x) \odot (e \wedge y)$,

(c_{61}) $e \odot (x \rightsquigarrow y) = e \odot [(e \odot x) \rightsquigarrow (e \odot y)]$ *and* $(x \to y) \odot e = [(x \odot e) \to (y \odot e)] \odot e$,

(c_{62}) $x \odot (e \rightsquigarrow f) = x \odot [(x \odot e) \rightsquigarrow (x \odot f)]$ *and* $(e \to f) \odot x = [(e \odot x) \to (f \odot x)] \odot x$,

(c_{63}) $e \to (x \to y) = (e \to x) \to (e \to y)$ *and* $e \rightsquigarrow (x \rightsquigarrow y) = (e \rightsquigarrow x) \rightsquigarrow (e \rightsquigarrow y)$.

Proof. (c_{59}). We have

$$(e \vee x) \odot (e \vee y) \overset{c_{38}}{=} [(e \vee x) \odot e] \vee [(e \vee x) \odot y] = [(e \vee x) \odot e] \vee [(e \odot y) \vee (x \odot y)]$$

$$= [(e \vee x) \wedge e] \vee [(e \odot y) \vee (x \odot y)] = e \vee (e \odot y) \vee (x \odot y) = e \vee (x \odot y).$$

(c_{60}). We have

$$(e \wedge x) \odot (e \wedge y) = (e \odot x) \odot (e \odot y) = (e \odot e) \odot (x \odot y) = e \odot (x \odot y) = e \wedge (x \odot y).$$

(c_{61}). By c_{13} we have $x \to y \leq (x \odot e) \to (y \odot e)$, hence by c_3, $(x \to y) \odot e \leq [(x \odot e) \to (y \odot e)] \odot e$. Conversely, $[(x \odot e) \to (y \odot e)] \odot e \leq e$ and $[(x \odot e) \to (y \odot e)] \odot (x \odot e) \leq y \odot e \leq y$ so $[(x \odot e) \to (y \odot e)] \odot e \leq x \to y$. Hence $[(x \odot e) \to (y \odot e)] \odot e \leq (x \to y) \wedge e = (x \to y) \odot e$.

By c_{12} we have $x \rightsquigarrow y \leq (e \odot x) \rightsquigarrow (e \odot y)$, hence by c_3, $e \odot (x \rightsquigarrow y) \leq e \odot [(e \odot x) \rightsquigarrow (e \odot y)]$. Conversely, $e \odot [(e \odot x) \rightsquigarrow (e \odot y)] \leq e$ and $(e \odot x) \odot [(e \odot x) \rightsquigarrow (e \odot y)] \leq e \odot y \leq y$ so $e \odot [(e \odot x) \rightsquigarrow (e \odot y)] \leq x \rightsquigarrow y$.

Hence $e \odot [(e \odot x) \rightsquigarrow (e \odot y)] \leq e \wedge (x \rightsquigarrow y) = e \odot (x \rightsquigarrow y)$.

(c_{62}). We have

$$[(e \odot x) \to (f \odot x)] \odot x = [(e \odot x) \to (f \wedge x)] \odot x$$

$$\overset{c_{38}}{=} [((e \odot x) \to f) \wedge ((e \odot x) \to x)] \odot x$$

$$= [((e \odot x) \to f) \wedge 1] \odot x = [(e \odot x) \to f] \odot x = [(x \odot e) \to f] \odot x$$

$$\overset{c_{23}}{=} [x \to (e \to f)] \odot x = x \wedge (e \to f) = x \odot (e \to f).$$

We have

$$x \odot [(x \odot e) \rightsquigarrow (x \odot f)] = x \odot [(x \odot e) \rightsquigarrow (x \wedge f)]$$

$$\overset{c_{38}}{=} x \odot [((x \odot e) \rightsquigarrow x) \wedge ((x \odot e) \rightsquigarrow f)] = x \odot [1 \wedge ((x \odot e) \rightsquigarrow f)]$$

$$= x \odot [(x \odot e) \rightsquigarrow f] = x \odot [(e \odot x) \rightsquigarrow f] \overset{c_{23}}{=} x \odot [x \rightsquigarrow (e \rightsquigarrow f)]$$

$$= x \wedge (e \rightsquigarrow f) = x \odot (e \rightsquigarrow f).$$

(c_{63}).We have

$$(e \to x) \to (e \to y) \overset{c_{23}}{=} [(e \to x) \odot e] \to y =$$

$$= (e \wedge x) \to y = (e \odot x) \to y \overset{c_{23}}{=} e \to (x \to y)$$

and

$$(e \rightsquigarrow x) \rightsquigarrow (e \rightsquigarrow y) \overset{c_{23}}{=} [e \odot (e \rightsquigarrow x)] \rightsquigarrow y =$$

$$= (e \wedge x) \rightsquigarrow y = (x \odot e) \rightsquigarrow y \overset{c_{23}}{=} e \rightsquigarrow (x \rightsquigarrow y). \square$$

Definition 4. ([10]-[11]) Let A and B be a pseudo-BL algebras. A function $f : A \to B$ is a morphism of pseudo -BL algebras iff it satisfies the following conditions, for every $x, y \in A$:

(a_{11}) $f(0) = 0$,

(a_{12}) $f(x \odot y) = f(x) \odot f(y)$,

(a_{13}) $f(x \to y) = f(x) \to f(y)$,

(a_{14}) $f(x \rightsquigarrow y) = f(x) \rightsquigarrow f(y)$.

Remark 3. ([10]-[11]) It follows that:

$$f(1) = 1, \;\; f(x^-) = [f(x)]^-, \;\; f(x^\sim) = [f(x)]^\sim,$$
$$f(x \vee y) = f(x) \vee f(y), \;\; f(x \wedge y) = f(x) \wedge f(y),$$

for every $x, y \in A$.

2. Pseudo-BL algebra of fractions relative to an $\wedge-$ closed system

Definition 5. A nonempty subset $S \subseteq A$ is called $\wedge-$closed system in A if $1 \in S$ and $x, y \in S$ implies $x \wedge y \in S$.

We denote by $S(A)$ the set of all $\wedge-$closed system of A (clearly $\{1\}, A \in S(A)$).

For $S \in S(A)$, on the pseudo - BL algebra A we consider the relation θ_S defined by

$$(x, y) \in \theta_S \text{ iff there exists } e \in S \cap B(A) \text{ such that } x \wedge e = y \wedge e.$$

Lemma 7. *θ_S is a congruence on A.*

Proof. The reflexivity (since $1 \in S \cap B(A)$) and the symmetry of θ_S are immediately. To prove the transitivity of θ_S, let $(x, y), (y, z) \in \theta_S$. Thus there exists $e, f \in S \cap B(A)$ such that $x \wedge e = y \wedge e$ and $y \wedge f = z \wedge f$. If denote $g = e \wedge f \in S \cap B(A)$, then $g \wedge x = (e \wedge f) \wedge x = (e \wedge x) \wedge f = (y \wedge e) \wedge f = (y \wedge f) \wedge e = (z \wedge f) \wedge e = z \wedge (f \wedge e) = z \wedge g$, hence $(x, z) \in \theta_S$.

To prove the compatibility of θ_S with the operations $\wedge, \vee, \odot, \rightarrow$ and $\rightsquigarrow$, let $x, y, z, t \in A$ such that $(x, y) \in \theta_S$ and $(z, t) \in \theta_S$. Thus there exists $e, f \in S \cap B(A)$ such that $x \wedge e = y \wedge e$ and $z \wedge f = t \wedge f$; we denote $g = e \wedge f \in S \cap B(A)$.

We obtain:

$$(x \wedge z) \wedge g = (x \wedge z) \wedge (e \wedge f) = (x \wedge e) \wedge (z \wedge f) = (y \wedge e) \wedge (t \wedge f) = (y \wedge t) \wedge g,$$

hence $(x \wedge z, y \wedge t) \in \theta_S$ and

$$(x \vee z) \wedge g = (x \vee z) \wedge (e \wedge f) = [(e \wedge f) \wedge x] \vee [(e \wedge f) \wedge z] = [(e \wedge x) \wedge f] \vee [e \wedge (f \wedge z)]$$

$$= [(e \wedge y) \wedge f] \vee [e \wedge (f \wedge t)] = [(e \wedge f) \wedge y] \vee [(e \wedge f) \wedge t] = (y \vee t) \wedge (e \wedge f) = (y \vee t) \wedge g,$$

hence $(x \vee z, y \vee t) \in \theta_S$.

By Remark 2 we obtain:

$$(x \odot z) \wedge g = (x \odot z) \odot g = (x \odot e) \odot (z \odot f) = (x \wedge e) \odot (z \wedge f) = (y \wedge e) \odot (t \wedge f)$$

$$= (y \odot e) \odot (t \odot f) = (y \odot t) \odot g = (y \odot t) \wedge g,$$

hence $(x \odot z, y \odot t) \in \theta_S$ and by (c_{61}):

$$(x \rightarrow z) \wedge g = (x \rightarrow z) \odot g = [(x \odot g) \rightarrow (z \odot g)] \odot g$$

$$= [(y \odot g) \rightarrow (t \odot g)] \odot g = (y \rightarrow t) \odot g = (y \rightarrow t) \wedge g,$$

hence $(x \to z, y \to t) \in \theta_S$ and

$$(x \rightsquigarrow z) \wedge g = g \odot (x \rightsquigarrow z) = g \odot [(g \odot x) \rightsquigarrow (g \odot z)]$$
$$= g \odot [(g \odot y) \rightsquigarrow (g \odot t)] = g \odot (y \rightsquigarrow t) = (y \rightsquigarrow t) \wedge g,$$

hence $(x \rightsquigarrow z, y \rightsquigarrow t) \in \theta_S$. □

For x we denote by x/S the equivalence class of x relative to θ_S and by

$$A[S] = A/\theta_S.$$

By $p_S : A \to A[S]$ we denote the canonical map defined by $p_S(x) = x/S$, for every $x \in A$. Clearly, in $A[S]$, $\mathbf{0} = 0/S$, $\mathbf{1} = 1/S$ and for every $x, y \in A$,

$$x/S \wedge y/S = (x \wedge y)/S$$
$$x/S \vee y/S = (x \vee y)/S$$
$$x/S \odot y/S = (x \odot y)/S$$
$$x/S \to y/S = (x \to y)/S$$
$$x/S \rightsquigarrow y/S = (x \rightsquigarrow y)/S.$$

So, p_S is an onto morphism of pseudo-BL algebras.

Remark 4. Since for every $s \in S \cap B(A)$, $s \wedge s = s \wedge 1$ we deduce that $s/S = 1/S = \mathbf{1}$, hence $p_S(S \cap B(A)) = \{\mathbf{1}\}$.

Proposition 8. *If $a \in A$, then $a/S \in B(A[S])$ iff there exists $e \in S \cap B(A)$ such that $e \wedge a \in B(A)$. So, if $e \in B(A)$, then $e/S \in B(A[S])$.*

Proof. For $a \in A$, we have $a/S \in B(A[S]) \Leftrightarrow a/S \odot a/S = a/S$ and $((a/S)^-)^\sim = ((a/S)^\sim)^- = a/S$.

From $a/S \odot a/S = a/S$ we deduce that $(a \odot a)/S = a/S \Leftrightarrow$ there exists $g \in S \cap B(A)$ such that $(a \odot a) \wedge g = a \wedge g \Leftrightarrow (a \odot a) \odot g = a \wedge g \Leftrightarrow (a \odot g) \odot (a \odot g) = a \wedge g \Leftrightarrow (a \wedge g) \odot (a \wedge g) = a \wedge g$.

From $((a/S)^-)^\sim = ((a/S)^\sim)^- = a/S$ we deduce that exists $f, h \in S \cap B(A)$ such that $(a^-)^\sim \wedge f = a \wedge f$ and $(a^\sim)^- \wedge h = a \wedge h$. If denote $e = g \wedge f \wedge h \in S \cap B(A)$, then

$$(a \wedge e) \odot (a \wedge e) = (a \wedge g \wedge f \wedge h) \odot (a \wedge g \wedge f \wedge h) =$$

$$(a \odot g) \odot f \odot h \odot (a \odot g) \odot f \odot h = a \odot g \odot f \odot h = a \wedge g \wedge f \wedge h = a \wedge e$$

and

$$((a \wedge e)^{-})^{\sim} \overset{c_{52}}{=} (a^{-} \vee e^{-})^{\sim} \overset{c_{51}}{=} (a^{-})^{\sim} \wedge (e^{-})^{\sim} = (a^{-})^{\sim} \wedge e$$
$$= (a^{-})^{\sim} \wedge g \wedge f \wedge h = [(a^{-})^{\sim} \wedge f] \wedge g \wedge h = (a \wedge f) \wedge g \wedge h = a \wedge e$$

and

$$((a \wedge e)^{\sim})^{-} \overset{c_{51}}{=} (a^{\sim} \vee e^{\sim})^{-} \overset{c_{52}}{=} (a^{\sim})^{-} \wedge (e^{\sim})^{-} = (a^{\sim})^{-} \wedge e$$
$$= (a^{\sim})^{-} \wedge g \wedge f \wedge h = [(a^{\sim})^{-} \wedge h] \wedge g \wedge f = (a \wedge h) \wedge g \wedge f = a \wedge e,$$

so,

$$((a \wedge e)^{\sim})^{-} = ((a \wedge e)^{-})^{\sim} = a \wedge e,$$

hence $a \wedge e \in B(A)$.

If $e \in B(A)$, since $1 \in S \cap B(A)$ and $1 \wedge e = e \in B(A)$ we deduce that $e/S \in B(A[S])$. □

Theorem 9. *If A' is a pseudo-BL algebra and $f : A \to A'$ is an morphism of pseudo-BL algebras such that $f(S \cap B(A)) = \{\mathbf{1}\}$, then there exists an unique morphism of pseudo-BL algebras $f' : A[S] \to A'$ such that the diagram*

$$\begin{array}{ccc} A & \xrightarrow{p_S} & A[S] \\ \searrow_{f} & & \swarrow_{f'} \\ & A' & \end{array}$$

is commutative (i.e. $f' \circ p_S = f$).

Proof. If $x, y \in A$ and $p_S(x) = p_S(y)$, then $(x, y) \in \theta_S$, hence there exists $e \in S \cap B(A)$ such that $x \wedge e = y \wedge e$. Since f is morphism of pseudo-BL-algebras, we obtain that $f(x \wedge e) = f(y \wedge e) \Leftrightarrow f(x) \wedge f(e) = f(y) \wedge f(e) \Leftrightarrow f(x) \wedge \mathbf{1} = f(y) \wedge \mathbf{1} \Leftrightarrow f(x) = f(y)$.

From this observation we deduce that the map $f' : A[S] \to A'$ defined for $x \in A$ by $f'(x/S) = f(x)$ is correctly defined. Clearly, f' is an morphism of pseudo-BL algebras. The unicity of f' follows from the fact that p_S is a onto map. □

Remark 5. Theorem 9 allows us to call $A[S]$ the pseudo-BL algebra of fractions relative to the $\wedge-$closed system S.

REFERENCES

1. Balbes, R.; Dwinger, Ph. – *Distributive Lattices*, University of Missouri Press, 1974.
2. Buşneag, D. – *Hilbert algebra of fractions relative to an $\veebar$-closed system*, Analele Universităţii din Craiova, Seria Matematica-Fizica-Chimie, vol. XVI, 1998, pp. 34-38.
3. Buşneag, D.; Piciu, D. – *MV-algebra of fractions relative to an $\wedge$-closed system*, Annals of University of Craiova, Math. Comp. Sci. series, Volume 31, 2003, pp. 1-5.
4. Buşneag, D.; Piciu, D. – *BL-algebra of fractions relative to an $\wedge$-closed system*, Analele Ştiinţifice ale Universităţii Ovidius Constanţa, Seria Matematica, vol. XI, Fascicula 1, 2003, pp. 39-48.
5. Buşneag, D.; Piciu, D. – *On the lattice of deductive systems of a BL-algebra*, Central European Journal of Mathematics, Volume 1, Number 2, April 2003, pp. 221-238.
6. Chang, C.C. – *Algebraic analysis of many valued logics*, Trans. Amer. Math. Soc., 88, 1958, pp. 467-490.
7. Cignoli, R.; D'Ottaviano, I.M.L.; Mundici, D. – *Algebraic foundation of many -valued Reasoning*, Kluwer Academic Publishers, Dordrecht, 2000.
8. Hájek, P. – *Metamathematics of Fuzzy Logic*, Kluwer Acad. Publ., Dordrecht, 1998.
9. Iorgulescu, A. – *Iséki algebras. Connections with BL-algebras,* to appear in Soft Computing.
10. Di Nola, A.; Georgescu, G.; Iorgulescu, A. – *Pseudo-BL algebras: Part I,* Multiple Valued Logic, 8, No.5-6, (2002), pp. 673-714.
11. Di Nola, A.; Georgescu, G.; Iorgulescu, A. – *Pseudo-BL algebras: Part II,* Multiple Valued Logic, 8, No.5-6, (2002), pp. 717-750.
12. Piciu, D. – *Pseudo MV-algebra of fractions relative to an $\wedge$-closed system*, Annals of University of Craiova, Math. Comp. Sci. series, Volume 31, 2003, pp. 6-11.
13. Turunen, E. – *Mathematics Behind Fuzzy Logic*, Physica-Verlag, 1999.

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