

FURTHER RESULTS ON THE SPATIAL BEHAVIOUR IN LINEAR ELASTODYNAMICS

BY

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Abstract. In this paper we study the spatial behaviour of dynamic processes in an isotropic and homogeneous material whose elasticity tensor is strongly elliptic. In terms of the Lamé moduli this class of materials is characterized by the inequalities $\mu > 0$, $\lambda + 2\mu > 0$. To cover this class of materials, we introduce two appropriate measures associated with the dynamic process in question and then we establish some first-order differential inequalities leading to a complete description of the spatial behaviour. The first measure allows us to describe the spatial behaviour in the class of isotropic materials for which the Lamé moduli obey the inequalities $\mu > 0$, $3\lambda + 4\mu > 0$. The second measure is intended for covering the class of isotropic materials for which the Lamé moduli obey the inequalities $\mu > 0$, $-2\mu < \lambda < 0$. To have a good description of the spatial behaviour for the isotropic elastic materials for which the elasticity tensor is strongly elliptic, we have to combine the previous known results in the field with those obtained in the present paper.

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1. Introduction. Saint-Venant's principle is one of the most successful tool in the theory and applications of elasticity. A comprehensive review of the contemporary research on the spatial behaviour of solutions for statical and dynamical problems of continua was given by Horgan and Knowles [1] and updated successively by Horgan [2,3].

Some useful information on the spatial behaviour of solution in the linear theory of elastodynamics is furnished by the well-known domain of influence theorem, provided to assume that the elasticity tensor is positive definite (see, for example, Gurtin [4]). The domain of influence of the external given

data at time $T > 0$ represents the set D_T of all points of the body B that can be reached by signals propagating from the support \hat{D}_T , of the external given data on the time interval $[0, T]$, which speeds equal to or less than the maximum speed of propagation c_M . The domain of influence theorem shows that on $[0, T]$ the external given data have no effect on points outside of D_T .

Under the assumption that the elasticity tensor is positive definite, it was further shown by CHIRIȚĂ and QUINTANILLA [5] and CHIRIȚĂ and CIARLETTA [6] that, for each $t \in [0, T]$, the displacement becomes null in the part of the body in which we have $r \geq ct$, where r represents the distance from the support \hat{D}_T and $c = \sqrt{\frac{\mu_M}{\rho}}$, μ_M is the maximum elastic modulus and ρ is the mass density. Furthermore, within the domain of influence, that is for $0 \leq r \leq ct$, it is established a spatial decay estimate of exponential type whose decay rate is described by a factor independent of time. It is given in this way a complete information about the spatial behaviour of the solution in the outside of the support of the external given data.

We recall here that for an isotropic body the elasticity tensor is positive definite if the Lamé moduli λ and μ obey the inequalities $\mu > 0$, $3\lambda + 2\mu > 0$.

As it was pointed out in many works (see, for example, KNOPS and PAYNE [7,8], KNOPS and WILKES [9], IEŞAN [10], IEŞAN and SCALIA [11]), when the elasticity tensor C_{ijkl} arises from the theory of small deformations superposed upon one large static deformation, there is no a priori reason to expect that the strain energy form should be definite.

In this connection a natural question is to try to study the spatial behaviour of the elastic processes under the assumption that the elasticity tensor is strongly elliptic. To this end we introduce two appropriate measures associated with the dynamic process in question and then we establish some first-order differential inequalities in terms of such measures. The integration of these differential inequalities gives information on the spatial behaviour of the considered dynamic process. In fact, the first measure allows us to obtain the spatial behaviour of the dynamic processes in the class of elastic materials characterized by the inequalities $\mu > 0$, $3\lambda + 4\mu > 0$. While the second measure makes possible the description of the spatial behaviour for the elastic materials characterized by $\mu > 0$, $-2\mu < \lambda < 0$.

To obtain a good description of the spatial behaviour for the isotropic elastic materials with a strongly elliptic elasticity tensor, we have to com-

bine the present results with those previously established by CHIRIȚĂ and QUINTANILLA [5] and CHIRIȚĂ and CIARLETTA [6].

It is worth to outline that various results have been established concerning the uniqueness and continuous dependence of solutions in linear elastodynamics when the elasticity tensor is not defined. The first investigation on the spatial behaviour of solutions in elastostatics when the internal energy is not positive there seems to be that by FLAVIN, KNOPS and PAYNE [12]. In fact, there are established some decay estimates in a linear isotropic homogeneous elastic nonprismatic cylinder loaded by prescribed end displacements and with fixed curved lateral surface, for the static problem when the Lamé constants range so that $\mu > 0$ and $\lambda + \mu > 0$.

An idea to extend the class of materials for which the growth and decay properties hold was developed recently by PAYNE and SONG [13], in treating of a linear generalized thermoelasticity theory of hyperbolic type.

2. State of art of the spatial behaviour. Throughout this paper we assume that B is a bounded or unbounded regular region of the physical space \mathbb{R}^3 , whose boundary surface is ∂B . We select a rectangular system of coordinates so that vectors and tensors will have components denoted by latin subscripts ranging over 1, 2, 3. Summation over repeated subscripts and other typical conventions for differential operations are implied such as a superposed dot or a comma followed by a subscript to denote partial derivative with respect to time or the corresponding cartesian coordinate.

We suppose that B is filled with a homogeneous elastic material. According to the linear theory of elastodynamics, the fundamental system of field equations consists [4] of the strain–displacement relations

$$(2.1) \quad e_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \quad \text{in } \bar{B} \times [0, \infty),$$

the stress–strain relation

$$(2.2) \quad S_{ij} = C_{ijkl} e_{kl} \quad \text{in } \bar{B} \times [0, \infty),$$

and the equations of motion

$$(2.3) \quad S_{ji,j} + b_i = \rho \ddot{u}_i \quad \text{in } B \times (0, \infty).$$

Here u_i are the components of the displacement vector, e_{ij} are the components of the strain tensor, S_{ij} are the components of the stress tensor and

b_i are the components of the body force vector. Further, ρ is the constant mass density and C_{ijkl} are the components of the constant elasticity tensor and satisfy the symmetry relations: $C_{ijkl} = C_{klij} = C_{jikl}$.

We associate to the above equations the following initial conditions

$$(2.4) \quad u_i(\mathbf{x}, 0) = u_i^0(\mathbf{x}), \quad \dot{u}_i(\mathbf{x}, 0) = v_i^0(\mathbf{x}), \quad \mathbf{x} \in \bar{B},$$

and the mixed boundary conditions

$$u_i(\mathbf{x}, t) = \tilde{u}_i(\mathbf{x}, t) \quad \text{on} \quad \sum_1 \times [0, \infty),$$

$$(2.5) \quad S_{ji}(\mathbf{x}, t)n_j(\mathbf{x}) = \tilde{s}_i(\mathbf{x}, t) \quad \text{on} \quad \sum_2 \times [0, \infty),$$

where u_i^0 , v_i^0 , \tilde{u}_i , \tilde{s}_i are prescribed functions, \sum_1 and \sum_2 are parts of the boundary such that $\sum_1 \cup \bar{\sum}_2 = \partial B$, $\sum_1 \cap \sum_2 = \emptyset$ and n_j are the components of the outward unit normal vector to the boundary.

Let us now consider a fixed time $T > 0$. We denote by \hat{D}_T the set of all $\mathbf{x} \in \bar{B}$ such that:

(i) if $\mathbf{x} \in B$, then

$$(2.6) \quad u_i^0(\mathbf{x}) \neq 0 \quad \text{or} \quad v_i^0(\mathbf{x}) \neq 0$$

or

$$(2.7) \quad b_i(\mathbf{x}, \tau) \neq 0 \quad \text{for some} \quad \tau \in [0, T];$$

or

(ii) if $\mathbf{x} \in \sum_1$, then

$$(2.8) \quad \tilde{u}_i(\mathbf{x}, \tau) \neq 0 \quad \text{for some} \quad \tau \in [0, T],$$

and if $\mathbf{x} \in \sum_2$, then

$$(2.9) \quad \tilde{s}_i(\mathbf{x}, \tau) \neq 0 \quad \text{for some} \quad \tau \in [0, T].$$

Roughly speaking, \hat{D}_T represents the support of the initial and boundary data and the body force on the time interval $[0, T]$. In what follows we assume that \hat{D}_T is a bounded set.

We consider next a nonempty set \hat{D}_T^* so that $\hat{D}_T \subset \hat{D}_T^* \subset \bar{B}$ and such that:

(i) if $\hat{D}_T \cap B \neq \emptyset$ then we choose \hat{D}_T^* to be the smallest bounded regular region in \bar{B} that includes \hat{D}_T ; in particular, we set $\hat{D}_T^* = \hat{D}_T$ it also happens \hat{D}_T to be a regular region;

(ii) if $\emptyset \neq \hat{D}_T \subset \partial B$, then we choose \hat{D}_T^* to be the smallest regular subsurface of ∂B that includes \hat{D}_T ; in particular, we set $\hat{D}_T^* = \hat{D}_T$ if \hat{D}_T is a regular subsurface of ∂B ;

(iii) if $\hat{D}_T = \emptyset$, then we choose \hat{D}_T^* to be an arbitrary nonempty regular subsurface of ∂B .

On this basis we then introduce the set D_r , $r \geq 0$, by

$$(2.10) \quad D_r = \left\{ \mathbf{x} \in \bar{B} : \hat{D}_T^* \cap \overline{\sum(\mathbf{x}, r)} \neq \emptyset \right\},$$

where $\sum(\mathbf{x}, r)$ is the open ball with radius r and center at \mathbf{x} . Further, we shall use the notation B_r for the part of B contained in $B \setminus D_r$ and we set $B(r_1, r_2) = B_{r_2} \setminus B_{r_1}$, $r_1 > r_2$. Moreover, we shall denote by S_r the subsurface of ∂B_r contained into inside of B and whose outward unit normal vector is forwarded to the exterior of D_r .

We associate with the solution of the initial-boundary value problem \mathcal{P} defined by the relations (2.1)–(2.5), the following function

$$(2.11) \quad P(r, t) = - \int_0^t \int_{S_r} e^{-\sigma s} \dot{u}_i(s) S_{ji}(s) n_j da ds, \quad r \geq 0, \quad 0 \leq t \leq T,$$

where σ is a prescribed positive parameter.

Under the assumption that the elasticity tensor is positive definite, for all fixed $t \in [0, T]$ it was shown that [6] $P(r, t) \geq 0$ for $r \geq 0$ and, moreover,

(i) for $r \geq ct$

$$(2.12) \quad u_i(\mathbf{x}, t) = 0;$$

(ii) for $0 \leq r \leq ct$

$$(2.13) \quad P(r, t) \leq P(0, t) e^{-\frac{\sigma}{c} r},$$

where

$$(2.14) \quad c = \sqrt{\frac{\mu_M}{\rho}},$$

and μ_M is the maximum elastic modulus for C_{ijkl} .

It is worth to recall that, when the body is isotropic, that is

$$(2.15) \quad C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}),$$

where λ and μ are the Lamé constants and δ_{ij} is the Kronecker symbol, then the condition that C_{ijkl} to be positive definite is

$$(2.16) \quad \mu > 0, \quad 3\lambda + 2\mu > 0,$$

and the maximum and minimum elastic moduli are $\mu_M, \mu_m \in \{2\mu, 3\lambda + 2\mu\}$. It follows then for $\lambda \geq 0$ that $\mu_M = 3\lambda + 2\mu$ and therefore we have

$$(2.17) \quad c = \sqrt{\frac{3\lambda + 2\mu}{\varrho}},$$

while for $-\frac{2}{3}\mu < \lambda \leq 0$ it follows that $\mu_M = 2\mu$ and hence,

$$(2.18) \quad c = \sqrt{\frac{2\mu}{\varrho}}.$$

Our main aim in this paper is to prove that the condition of positive definiteness on C_{ijkl} can be relaxed and therefore, the class of materials for which we have the conclusion described by the relations (2.12) and (2.13) can be extended.

In what follows we consider the basic equations of the linear elastodynamics for an isotropic and homogeneous elastic solid

$$(2.19) \quad \mu u_{i,jj} + (\lambda + \mu) u_{r,ri} + b_i = \varrho \ddot{u}_i \quad \text{in } B \times (0, \infty),$$

with the initial conditions

$$(2.20) \quad u_i(\mathbf{x}, 0) = u_i^0(\mathbf{x}), \quad \dot{u}_i(\mathbf{x}, 0) = v_i^0(\mathbf{x}), \quad \mathbf{x} \in \bar{B}.$$

3. First extension. $\mu > 0, 3\lambda + 4\mu > 0$.

It is easy to see that the basic equations (2.19) can be written in the following form

$$(3.1) \quad T_{ji,j} + b_i = \varrho \ddot{u}_i \quad \text{in } B \times (0, \infty),$$

where

$$(3.2) \quad T_{ji} = \mu u_{i,j} + (\lambda + \mu) u_{r,r} \delta_{ij}.$$

In this section we consider the initial-boundary value problem \mathcal{P}_1 defined by the relations (3.1), (3.2), (2.20) and by the following boundary conditions

$$(3.3) \quad \begin{aligned} u_i(\mathbf{x}, t) &= \tilde{u}_i(\mathbf{x}, t) \quad \text{on} \quad \sum_1 \times [0, \infty), \\ T_{ji}(\mathbf{x}, t) n_j(\mathbf{x}) &= \tilde{t}_i(\mathbf{x}, t) \quad \text{on} \quad \sum_2 \times [0, \infty). \end{aligned}$$

We introduce the surface S_r like in the above section and define the function

$$(3.4) \quad I(r, t) = - \int_0^t \int_{S_r} e^{-\sigma s} \dot{u}_i(s) T_{ji}(s) n_j ds, \quad r \geq 0, \quad 0 \leq t \leq T.$$

Then we note that for $0 < r_2 < r_1$, by means of the boundary condition (3.3) and the definition for S_r , we have

$$(3.5) \quad I(r_1, t) - I(r_2, t) = - \int_0^t \int_{\partial B(r_1, r_2)} e^{-\sigma s} \dot{u}_i(s) T_{ji}(s) n_j ds,$$

and hence, by means of the divergence theorem, we deduce

$$(3.6) \quad \begin{aligned} I(r_1, t) - I(r_2, t) &= - \int_0^t \int_{B(r_1, r_2)} e^{-\sigma s} [\dot{u}_i(s) T_{ji,j}(s) + \\ &\quad + \dot{u}_{i,j}(s) T_{ji}(s)] dv ds. \end{aligned}$$

By taking into account the relations (3.1) and (3.2) and the definition for $B(r_1, r_2)$, we get

$$(3.7) \quad \begin{aligned} I(r_1, t) - I(r_2, t) &= - \int_0^t \int_{B(r_1, r_2)} e^{-\sigma s} [\varrho \dot{u}_i(s) \ddot{u}_i(s) + \\ &\quad + \mu u_{i,j}(s) \dot{u}_{i,j}(s) + (\lambda + \mu) u_{r,r}(s) \dot{u}_{s,s}(s)] dv ds. \end{aligned}$$

Further, by using an integration by parts and by means of the definition for $B(r_1, r_2)$, we can write

$$I(r_1, t) - I(r_2, t) = - \int_{B(r_1, r_2)} e^{-\sigma t} \frac{1}{2} [\varrho \dot{u}_i \dot{u}_i + \mu u_{i,j} u_{i,j} + (\lambda + \mu) u_{r,r} u_{s,s}] dv -$$

$$(3.8) \quad - \int_0^t \int_{B(r_1, r_2)} e^{-\sigma s} \frac{\sigma}{2} [\rho \dot{u}_i(s) \dot{u}_i(s) + \mu u_{i,j}(s) u_{i,j}(s) + (\lambda + \mu) u_{r,r}(s) u_{s,s}(s)] dv ds.$$

Thus, we deduce that

$$(3.9) \quad \begin{aligned} \frac{\partial I}{\partial r}(r, t) &= - \int_{S_r} e^{-\sigma t} \frac{1}{2} [\rho \dot{u}_i \dot{u}_i + \mu u_{i,j} u_{i,j} + (\lambda + \mu) u_{r,r} u_{s,s}] da - \\ &- \int_0^t \int_{S_r} e^{-\sigma s} \frac{\sigma}{2} [\rho \dot{u}_i(s) \dot{u}_i(s) + \mu u_{i,j}(s) u_{i,j}(s) + (\lambda + \mu) u_{r,r}(s) u_{s,s}(s)] da ds. \end{aligned}$$

On the other hand, we note that the relation (3.4) implies

$$(3.10) \quad \frac{\partial I}{\partial t}(r, t) = - \int_{S_r} e^{-\sigma t} \dot{u}_i T_{ji} n_j da.$$

We proceed now to obtain an estimate for the terms $I(r, t)$ and $\frac{\partial I}{\partial t}(r, t)$. In this aim we introduce the following quadratic form

$$(3.11) \quad \mathcal{F}(\xi) = \mu \xi_{ij} \xi_{ij} + (\lambda + \mu) \xi_{rr} \xi_{ss}, \quad \forall \xi_{ij},$$

and note that it is positive definite if and only if we have

$$(3.12) \quad \mu > 0, \quad 3\lambda + 4\mu > 0.$$

This corresponds to the Poisson's ratio ν ranging on the interval $(-\infty, \frac{1}{2}) \cup (2, \infty)$.

The eigenvalues of the matrix of the above quadratic form are μ and $3\lambda + 4\mu$ and therefore we get

$$(3.13) \quad \mathcal{F}(\xi) \leq \max\{\mu, 3\lambda + 4\mu\} \xi_{ij} \xi_{ij}, \quad \forall \xi_{ij}.$$

Then, by the relations (3.2), (3.11) and (3.13), we deduce

$$(3.14) \quad \begin{aligned} T_{ij} T_{ij} &= \mu u_{j,i} T_{ij} + (\lambda + \mu) u_{r,r} T_{ss} \leq \\ &\leq [\mu u_{i,j} u_{i,j} + (\lambda + \mu) u_{r,r} u_{s,s}]^{\frac{1}{2}} [\mu T_{ij} T_{ij} + (\lambda + \mu) T_{rr} T_{ss}]^{\frac{1}{2}} \leq \\ &\leq [\max\{\mu, 3\lambda + 4\mu\} T_{ij} T_{ij}]^{\frac{1}{2}} [\mu u_{i,j} u_{i,j} + (\lambda + \mu) u_{r,r} u_{s,s}]^{\frac{1}{2}}, \end{aligned}$$

so that we obtain

$$(3.15) \quad T_{ij}T_{ij} \leq \max\{\mu, 3\lambda + 4\mu\} [\mu u_{i,j}u_{i,j} + (\lambda + \mu)u_{r,r}u_{s,s}].$$

Thus, we deduce that

$$(3.16) \quad \begin{aligned} \left| \frac{\partial I}{\partial t}(r, t) \right| &= \left| \int_{S_r} e^{-\sigma t} (\sqrt{\varrho} \dot{u}_i) \left(\frac{1}{\sqrt{\varrho}} T_{ji} n_j \right) da \right| \leq \\ &\leq \frac{1}{2} \int_{S_r} e^{-\sigma t} [\varepsilon \varrho \dot{u}_i \dot{u}_i + \frac{1}{\varepsilon \varrho} T_{ij} T_{ij}] da \leq \\ &\leq \frac{1}{2} \int_{S_r} e^{-\sigma t} \{ \varepsilon \varrho \dot{u}_i \dot{u}_i + \frac{1}{\varepsilon \varrho} \max\{\mu, 3\lambda + 4\mu\} [\mu u_{i,j} u_{i,j} + \\ &\quad + (\lambda + \mu) u_{r,r} u_{s,s}] \} da, \quad \forall \varepsilon > 0. \end{aligned}$$

Choosing ε in such a way that

$$(3.17) \quad \varepsilon = \frac{1}{\varepsilon \varrho} \max\{\mu, 3\lambda + 4\mu\},$$

that is by setting

$$(3.18) \quad \varepsilon = c_1 = \sqrt{\frac{\max\{\mu, 3\lambda + 4\mu\}}{\varrho}},$$

we get

$$(3.19) \quad \left| \frac{\partial I}{\partial t}(r, t) \right| \leq c_1 \frac{1}{2} \int_{S_r} e^{-\sigma t} [\varrho \dot{u}_i \dot{u}_i + \mu u_{i,j} u_{i,j} + (\lambda + \mu) u_{r,r} u_{s,s}] da.$$

In a similar way we get

$$(3.20) \quad \begin{aligned} |I(r, t)| &\leq c_1 \frac{1}{2} \int_0^t \int_{S_r} e^{-\sigma s} [\varrho \dot{u}_i(s) \dot{u}_i(s) + \\ &\quad + \mu u_{i,j}(s) u_{i,j}(s) + (\lambda + \mu) u_{r,r}(s) u_{s,s}(s)] da ds. \end{aligned}$$

In view of the relations (3.9), (3.19) and (3.20), we arrive at the following differential inequalities

$$(3.21) \quad \left| \frac{\partial I}{\partial t}(r, t) \right| + c_1 \frac{\partial I}{\partial r}(r, t) \leq 0, \quad r \geq 0, \quad t \in [0, T],$$

and

$$(3.22) \quad \frac{\sigma}{c_1} |I(r, t)| + \frac{\partial I}{\partial r}(r, t) \leq 0, \quad r \geq 0, \quad t \in [0, T].$$

Following the procedure developed in [5,6], from (3.21) and (3.22), we deduce that, for all $t \in [0, T]$, we have $I(r, t) \geq 0$ and, moreover, we deduce that

(i) for $r \geq c_1 t$

$$(3.23) \quad u_i(x, t) = 0,$$

while

(ii) for $0 \leq r \leq c_1 t$

$$(3.24) \quad I(r, t) \leq I(0, t) e^{-\frac{\sigma}{c_1} r}.$$

We have to note that for $\lambda > -\mu$ we have

$$(3.25) \quad c_1 = \sqrt{\frac{3\lambda + 4\mu}{\varrho}},$$

while for $-\frac{4}{3}\mu < \lambda \leq -\mu$ we have

$$(3.26) \quad c_1 = \sqrt{\frac{\mu}{\varrho}}.$$

4. Second Extension. $\mu > 0, -2\mu < \lambda < 0$.

In this section we describe a second method for discussing the spatial behaviour of the solution and it allows us to extend the class of materials by relaxing the range of elastic moduli. In this aim we write the basic equations (2.19) in the following form

$$(4.1) \quad R_{ji,j} + b_i = \varrho \ddot{u}_i \quad \text{in } B \times (0, \infty),$$

where

$$(4.2) \quad R_{ji} = \mu u_{i,j} + (\lambda + \mu) u_{j,i},$$

and consider the initial-boundary value problem \mathcal{P}_2 defined by the relations (4.1), (4.2), (2.20) and by the following boundary conditions

$$u_i(\mathbf{x}, t) = \tilde{u}_i(\mathbf{x}, t) \quad \text{on } \sum_1 \times [0, \infty),$$

$$(4.3) \quad R_{ji}(\mathbf{x}, t)n_j(\mathbf{x}) = \tilde{r}_i(\mathbf{x}, t) \quad \text{on} \quad \sum_2 \times [0, \infty).$$

Introducing now the notations D_r , B_r , S_r in the same manner like those of the second section, we define now the function

$$(4.4) \quad J(r, t) = - \int_0^t \int_{S_r} e^{-\sigma s} \dot{u}_i(s) R_{ji}(s) n_j da ds, \quad r \geq 0, \quad 0 \leq t \leq T.$$

Further, we use the boundary conditions (4.3) and the definition for $B(r_1, r_2)$ in order to write, for $0 < r_2 < r_1$, $t \in [0, T]$,

$$(4.5) \quad J(r_1, t) - J(r_2, t) = - \int_0^t \int_{\partial B(r_1, r_2)} e^{-\sigma s} \dot{u}_i(s) R_{ji}(s) n_j da ds.$$

Moreover, by using the divergence theorem, we obtain

$$(4.6) \quad J(r_1, t) - J(r_2, t) = - \int_0^t \int_{B(r_1, r_2)} e^{-\sigma s} [\dot{u}_i(s) R_{ji,j}(s) + \dot{u}_{i,j}(s) R_{ji}(s)] dv ds,$$

so that, on the basis of the equations (4.1) and the definition for $B(r_1, r_2)$, we get

$$(4.7) \quad J(r_1, t) - J(r_2, t) = - \int_0^t \int_{B(r_1, r_2)} e^{-\sigma s} [\rho \dot{u}_i(s) \ddot{u}_i(s) + \mu u_{i,j}(s) \dot{u}_{i,j}(s) + (\lambda + \mu) u_{j,i}(s) \dot{u}_{i,j}(s)] dv ds,$$

and hence, we obtain

$$(4.8) \quad \begin{aligned} \frac{\partial J}{\partial r}(r, t) &= - \int_{S_r} e^{-\sigma t} \frac{1}{2} [\rho \dot{u}_i \dot{u}_i + \mu u_{i,j} u_{i,j} + (\lambda + \mu) u_{i,j} u_{j,i}] da - \\ &- \int_0^t \int_{S_r} e^{-\sigma s} \frac{\sigma}{2} [\rho \dot{u}_i(s) \dot{u}_i(s) + \mu u_{i,j}(s) u_{i,j}(s) + \\ &+ (\lambda + \mu) u_{i,j}(s) u_{j,i}(s)] da ds. \end{aligned}$$

On the other hand, a differentiation of the relation (4.4) implies

$$(4.9) \quad \frac{\partial J}{\partial t}(r, t) = - \int_{S_r} e^{-\sigma t} \dot{u}_i R_{ji} n_j da.$$

In order to obtain an estimate for $J(r, t)$ and $\frac{\partial J}{\partial t}(r, t)$, we have to use the following quadratic form

$$(4.10) \quad \mathcal{G}(\xi) = \mu \xi_{ij} \xi_{ij} + (\lambda + \mu) \xi_{ij} \xi_{ji}, \quad \forall \xi_{ij},$$

which is positive definite when we have

$$(4.11) \quad \mu > 0, \quad -2\mu < \lambda < 0.$$

The eigenvalues of the matrix of the above quadratic form are $\lambda + 2\mu$ and $-\lambda$, and therefore, we have

$$(4.12) \quad \mathcal{G}(\xi) \leq \max\{-\lambda, \lambda + 2\mu\} \xi_{ij} \xi_{ij}, \quad \forall \xi_{ij}.$$

In view of the relations (4.2), (4.10) and (4.12), we obtain

$$(4.13) \quad \begin{aligned} R_{ij} R_{ij} &= \mu u_{j,i} R_{ij} + (\lambda + \mu) u_{i,j} R_{ij} \leq \\ &\leq [\mu u_{i,j} u_{i,j} + (\lambda + \mu) u_{i,j} u_{j,i}]^{\frac{1}{2}} [\mu R_{ij} R_{ij} + (\lambda + \mu) R_{ij} R_{ij}]^{\frac{1}{2}} \leq \\ &\leq [\max\{-\lambda, \lambda + 2\mu\} R_{ij} R_{ij}]^{\frac{1}{2}} [\mu u_{i,j} u_{i,j} + (\lambda + \mu) u_{i,j} u_{j,i}]^{\frac{1}{2}}, \end{aligned}$$

so that, we get

$$(4.14) \quad R_{ij} R_{ij} \leq \max\{-\lambda, \lambda + 2\mu\} [\mu u_{i,j} u_{i,j} + (\lambda + \mu) u_{i,j} u_{j,i}].$$

Therefore, the relations (4.4) and (4.14) give

$$(4.15) \quad \begin{aligned} |J(r, t)| &\leq \frac{1}{2} \int_0^t \int_{S_r} e^{-\sigma s} [\varepsilon \varrho \dot{u}_i(s) \dot{u}_i(s) + \frac{1}{\varepsilon \varrho} R_{ij}(s) R_{ij}(s)] d\alpha ds \leq \\ &\leq \frac{1}{2} \int_0^t \int_{S_r} e^{-\sigma s} \{ \varepsilon \varrho \dot{u}_i(s) \dot{u}_i(s) + \\ &+ \frac{\max\{-\lambda, \lambda + 2\mu\}}{\varepsilon \varrho} [\mu u_{i,j}(s) u_{i,j}(s) + (\lambda + \mu) u_{i,j}(s) u_{j,i}(s)] \} d\alpha ds, \end{aligned}$$

so that, if we set

$$(4.16) \quad \varepsilon = c_2 = \sqrt{\frac{\max\{-\lambda, \lambda + 2\mu\}}{\varrho}},$$

we get

$$|J(r, t)| \leq \frac{c_2}{2} \int_0^t \int_{S_r} e^{-\sigma s} \{ \varrho \dot{u}_i(s) \dot{u}_i(s) + \mu u_{i,j}(s) u_{i,j}(s) + (\lambda + \mu) u_{i,j}(s) u_{j,i}(s) \} d\alpha ds$$

$$(4.17) \quad +(\lambda + \mu)u_{i,j}(s)u_{j,i}(s)]\}dads.$$

Using a similar procedure we can obtain an appropriate estimate for $\frac{\partial J}{\partial t}$ as defined by the relation (4.9) and so we get

$$(4.18) \quad \left| \frac{\partial J}{\partial t}(r, t) \right| \leq \frac{c_2}{2} \int_{S_r} e^{-\sigma t} [\rho \dot{u}_i \dot{u}_i + \mu u_{i,j} u_{i,j} + (\lambda + \mu) u_{i,j} u_{j,i}] da.$$

On the basis of the relations (4.8), (4.17) and (4.18), we arrive at the following differential inequalities

$$(4.19) \quad \left| \frac{\partial J}{\partial t}(r, t) \right| + c_2 \frac{\partial J}{\partial r}(r, t) \leq 0, \quad r \geq 0, \quad t \in [0, T],$$

and

$$(4.20) \quad \frac{\sigma}{c_2} |J(r, t)| + \frac{\partial J}{\partial r}(r, t) \leq 0, \quad r \geq 0, \quad t \in [0, T].$$

By means of the method developed in [5,6], from the relations (4.19) and (4.20), we can conclude for all $t \in [0, T]$ that $J(r, t) \geq 0$ and, moreover,

(i) for $r \geq c_2 t$ we have

$$(4.21) \quad u_i(\mathbf{x}, t) = 0,$$

while

(ii) for $0 \leq r \leq c_2 t$ we have

$$(4.22) \quad J(r, t) \leq J(0, t) e^{-\frac{\sigma}{c_2} r}.$$

We have to note that

$$(4.23) \quad c_2 = \sqrt{\frac{-\lambda}{\rho}},$$

for $-\frac{4}{3}\mu < \lambda \leq -\mu$ and

$$(4.24) \quad c_2 = \sqrt{\frac{\lambda + 2\mu}{\rho}},$$

for $-\mu < \lambda \leq 0$.

5. Concluding remarks.

1. While the above results have been established for an extended range of elastic moduli, nevertheless the various boundary conditions that have been used in the boundary-initial value problems \mathcal{P}_1 and \mathcal{P}_2 are not of a practical use when $meas \sum_2 \neq 0$ and $(u_{j,i} - u_{s,s}\delta_{ij})n_j \neq 0$ on $\sum_2 \times [0, \infty)$. We have to note that the above problems coincide with the boundary-initial value problem \mathcal{P} when $meas \sum_2 = 0$ or $meas \sum_2 \neq 0$ and $(u_{j,i} - u_{s,s}\delta_{ij})n_j = 0$ on $\sum_2 \times [0, \infty)$ and in this respect our results complete those reported in the previous studies. In what follows we refer to the case when $meas \sum_2 = 0$ or $meas \sum_2 \neq 0$ and $(u_{j,i} - u_{s,s}\delta_{ij})n_j = 0$ on $\sum_2 \times [0, \infty)$.

2. It is also worth emphasizing that the use of the measure $I(r, t)$ in the third section offers an alternative for the measure $P(r, t)$, when $meas \sum_2 = 0$ or $meas \sum_2 \neq 0$ and $(u_{j,i} - u_{s,s}\delta_{ij})n_j = 0$ on $\sum_2 \times [0, \infty)$, but for an extended class of materials ($\lambda > -\frac{4}{3}\mu$). However, the decay rate predicted into the Section 3 for the displacement-initial boundary value problem within the class of range $\lambda > -\frac{2}{3}\mu$ deteriorates with respect to that predicted in the previous studies. A combination of use of the two measures leads to a good description of the spatial behavior in the class of materials for which $\mu > 0$ and $\lambda > -\frac{4}{3}\mu$.

Unlike the measure considered into the Section 3, the measure used in the fourth section is characteristic only for the extension range $-2\mu < \lambda < 0$. Thus, we have to use the three measures $J(r, t)$, $I(r, t)$ and $P(r, t)$ in order to obtain a good description of the spatial behavior in the class of materials for which $\mu > 0$ and $\lambda > -2\mu$.

3. It should be noted that the domain influence theorem in linear elastodynamics has been established, among other assumptions, under the hypothesis that the elasticity tensor is positive definite. Our results in the above proves that such an assumption can be relaxed and so we have established a domain influence theorem for an extended class of isotropic materials.

4. It should be noted that the above results lead to establish a uniqueness theorem for linear isotropic elastodynamics that is valid for finite or infinite domains like in [5,6], but valid for the class of isotropic materials for which $\mu > 0$ and $\lambda > -2\mu$. The advantage of such a method is that it does not involve artificial a priori assumptions on the orders of magnitude

of the velocity and stresses fields at infinity.

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