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# EXISTENCE AND UNIQUENESS THEOREMS FOR NONLINEAR BOUNDARY VALUE PROBLEMS WITH DEVIATING ARGUMENTS

BY

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**Abstract.** In this paper an existence theorem is proved for a certain nonlinear second order two point boundary value problem of ordinary differential equations with deviating arguments. The uniqueness theorem is also proved by the method of comparison principle.

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**Key words:** Boundary value problems, existence theorems.

**1. Introduction.** Let $\mathbb{R}$ denote the real line and let $I = [t_0, t_1], I_0 = [a, t_0]$, and $I_1 = [t_1, b]$ denote the closed and bounded intervals in $\mathbb{R}$ for $-\infty < a \leq t_0 < t_1 \leq b < +\infty$. Consider the real nonlinear boundary value problem (in short BVP) of second order differential equations with deviating arguments

$$-x''(t) = f(t, x(t), x(\phi(t)), x'(t), x'(\psi(t))), \quad \text{a.}e. \ t \in I \tag{1}$$

satisfying the boundary conditions

$$\begin{cases} x(t) = q_0(t), & t \in I_0, \\ x'(t) = q_1(t), & t \in I_1, \end{cases} \tag{2}$$

where

(i) $f : I \times \mathbb{R}^4 \to \mathbb{R}$ is a function which is not necessarily continuous,

(ii) $\phi, \psi : I \to \mathbb{R}$ are two continuous functions which may delay or advance and satisfy $\phi(I) \subseteq I_0 \cup I \cup I_1$ and $\psi(I) \subseteq I \cup I_1$, and

(iii) the boundary functions $q_0, q_1 : \mathbb{R} \to \mathbb{R}$ are continuous.

Denote $J = I_0 \cup I \cup I_1$, and we use the following notations in the sequel. For any closed and bounded subset $\Omega$ of $\mathbb{R}$ we denote

$$\begin{aligned} C(\Omega,\mathbb{R}) &= \{x : \Omega \to \mathbb{R} \mid \ x \text{ is continuous}\}, \\ C^1(\Omega,\mathbb{R}) &= \{x : \Omega \to \mathbb{R} \mid \ x \text{ is continuously differentiable}\}, \\ AC(\Omega,\mathbb{R}) &= \{x : \Omega \to \mathbb{R} \mid \ x \text{ is absolutely continuous}\}, \\ AC^1(\Omega,\mathbb{R}) &= \{x : \Omega \to \mathbb{R} \mid \ x' \text{ is absolutely continuous}\}. \end{aligned}$$

**Definition 1.1.** A function $x \in C(J,\mathbb{R}) \cap C^1(I \cup I_1,\mathbb{R}) \cap AC^1(I,\mathbb{R})$ which satisfies (1)-(2) on $J$ is called a solution of the BVP (1)-(2).

The BVP (1)-(2) has got impotance since is it more general and includes some of the earlier well-known BVPs as special cases. See Agarwal [1], Dhage and Heikkila [4] and the references therein. The BVP (1)-(2) has been discussed by Dhage [3] for the existence of extremal solutions via generalized iteration method developed in Heikkila and Lakshmikantham [7], and by Dhage and Heikkila [4] for the existence of extremal solutions in Banach spaces using the properties of the cones. In this paper we shall prove the existence and uniqueness of the solution of the BVP (1)-(2) via different fixed point principles.

The existence and uniqueness theorems for nonlinear BVPs are generally obtained by employing the fixed point principles such as Banach and Schauder (see for details Lusternik and Sobolev [8]) under suitable conditions. To prove the existence theorems via Schauder fixed point principle, one needs the compactness type condition on the nonlinearity $f$ involved in equation (1), and to prove the existence theorems via Banach fixed point principle, one needs the Lipschitz type condition on the nonlinearity $f$ involved in the differential equation in question. In this paper we shall prove the existence and uniqueness theorems for the BVP (1)-(2) under certain generalized measurability and Lipschitzicity conditions. We do not assume the nonlinearity $f$ to be continuous on its domain of definition, but we only need the mild continuity conditions on the nonlinearity such as piecewise continuity or Carathéodory conditions etc. Before going to the main existence theorems, we give some preliminaries that will be used in the subsequent sections of this paper.

Our first task is to find an operator whose fixed points are the solutions of the BVP (1)-(2). Therefore, we shall consider first the case when the right hand side of the differential equation (1) depends only on independent variable $t \in I$.

**Lemma 1.2.** *The BVP*

$$-x''(t) = 0, \quad a.e. \ t \in I, \tag{3}$$

$$\begin{cases} x(t) = q_0(t), & t \in I_0, \\ x'(t) = q_1(t), & t \in I_1, \end{cases} \tag{4}$$

*has a unique solution* $z \in C^1(J, \mathbb{R})$ *given by*

$$z(t) = \begin{cases} q_0(t), & t \in I_0, \\ q_0(t_0) + q_1(t_1)(t - t_0), & t \in I, \\ q_0(t_0) + q_1(t_1)(t - t_0) + \int_{t_1}^{t} q_1(s)ds, & t \in I_1. \end{cases} \tag{5}$$

**Proof.** The proof is simple and may be obtained by using the theory of ordinary differential equations [2]. Hence we omit the details. □

**Lemma 1.3.** *If* $h : I \to \mathbb{R}$, *then the BVP*

$$-x''(t) = h(t), \quad a.e. \ t \in I, \tag{6}$$

$$\begin{cases} x(t) = q_0(t), & t \in I_0, \\ x'(t) = q_1(t), & t \in I_1, \end{cases} \tag{7}$$

*has a unique solution* $x = H(h)$, *where*

$$H(h)(t) = z(t) + \begin{cases} 0, & t \in I_0, \\ \int_{t_0}^{t} (s - t_0)h(s)ds + (t - t_0) \int_{t}^{t_1} h(s)ds, & t \in I, \\ \int_{t_0}^{t_1} (s - t_0)h(s)ds, & t \in I_1, \end{cases} \tag{8}$$

*if and only if* $h$ *is Lebesgue integrable on* $I$.

A a direct consequence of Lemma 1.3 we obtain the following conversion of the BVP (1)-(2) to the fixed point problem.

**Lemma 1.4.** *If $f : I \times \mathbb{R}^4 \to \mathbb{R}$, then $x : J \to \mathbb{R}$ is a solution of the BVP (1)-(2) if and only if the function $Fx : I \to \mathbb{R}$ defined by*

$$Fx(t) = f(t, x(t), x(\phi(t)), x'(t), x'(\psi(t))), \ \ t \in I \tag{9}$$

*is Lebesgue integrable on $I$, and $x$ is a fixed point of the operator equation*

$$Qx(t) = H(Fx)(t), \ \ t \in J. \tag{10}$$

**2. Existence theorems via Lipschitz and compactness conditions.** We equip the space $C(J, \mathbb{R})$ of all continuous real-valued functions on $J$ with $\|\cdot\|_C$, the space $AC(J, \mathbb{R})$ of all absolutely continuous real-valued functions on $J$ with $\|\cdot\|_{AC}$. Let $M(J, \mathbb{R})$ and $BM(J, \mathbb{R})$ denote the spaces of measurable and boundendly measurable (i.e. bounded and measurable) real-valued functions on $J$ respectively. Define a norm $\|\cdot\|_{BM}$ on $BM(J, \mathbb{R})$ by

$$\|x\|_{BM} = \sup_{t \in J} |x(t)|.$$

Denote $BM^1(J, \mathbb{R}) = \{x \in AC(J, \mathbb{R}) | x' \in BM(J, \mathbb{R})\}$.

We define a metric $d$ on $BM^1(J, \mathbb{R})$ by

$$d(x, y) = \|x - y\|_{BM^1} \tag{11}$$

where the norm $\|\cdot\|_{BM^1}$ in $BM^1(J, \mathbb{R})$ is given by

$$\|x\|_{BM^1} = \|x\|_C + \|x'\|_{BM}. \tag{$\star$}$$

Clearly $BM^1(J, \mathbb{R})$ is a complete metric space with respect to this metric $d$. We shall obtain the solution of the operator equation $x = Qx$ in the space $BM^1(J, \mathbb{R})$ and hence the solution of the BVP (1)-(2) on $J$.

We need the following definition in the sequel.

**Definition 2.1.** A function $\beta : J \times \mathbb{R}^4 \to \mathbb{R}$ is said to satisfy the condition of Carathéodory or simply is called Carathéodory if

(i) $t \to \beta(t, x_1, x_2, x_3, , x_4)$ is measurable for all $x_i \in \mathbb{R}, 1 = 1, 2, 3, 4$.

(ii) $(x_1, x_2, x_3, , x_4) \to \beta(t, x_1, x_2, x_3, , x_4)$ is continuous almost everywhere for $t \in J$.

Further a Carathéodory function $\beta(t, x_1, x_2, x_3, , x_4)$ is called an $L^1$-Carathéodory if for every real number $r > 0$ there exists a function $h_r \in L^1(J, R)$ such that

$$|\beta(t, x_1, x_2, x_3, , x_4)| \leq h_r(t) \quad \text{a.e.}\, t \in J$$

for all $x_i \in \mathbb{R}$ with $|x_i| \leq r$ , $i = 1, 2, 3, 4$.

We consider the following set of assumptions:

$(f_1)$ The function $f(t, x_1, x_2, x_3, x_4)$ is continuous on $J \times \mathbb{R}^4$.

$(f_2)$ There exist functions $k_1 \in L^1(J, \mathbb{R}_+), i = 1, 2, 3, 4$, such that

$$\begin{aligned} |f(t, x_1, x_2, x_3, x_4) - f(t, y_1, y_2, y_3, y_4)| \leq \\ \leq k_1(t)|x_1 - y_1| + k_2(t)|x_2 - y_2| + \\ + k_3(t)|x_3 - y_3| + k_4(t)|x_4 - y_4|, \quad \text{a.e. } t \in I, \end{aligned}$$

for all $x_i, y_i \in \mathbb{R}, i = 1, 2, 3, 4$.

$(f_3)$ The function $f(t, x_1, x_2, x_3, x_4)$ is Carathéodory.

$(f_4)$ The function $f(t, x_1, x_2, x_3, x_4)$ is $L^1$-Caratheodory.

**Theorem 2.2.** *Assume $(f_1)$ and $(f_2)$ hold. Then the BVP (1)-(2) has a unique solution on J.*

**Proof.** The hypotheses $(f_1)$ guarantees that the function $Fx : I \to \mathbb{R}$ defined by (9) is Lebesgue integrable and so by Lemma 1.4, the solution of the operator equation (10) implies the solution of the BVP (1)-(2).

To do so, we simply show that the operator $Q$ is a contraction mapping on $BM^1(J, \mathbb{R})$.

Let $x, y \in BM^1(J, \mathbb{R})$. Then if $t \in I_0$,

$$|Qx(t) - Qy(t)| = |q_0(t) - q_0(t)| = 0,$$

and if $t \in I$, one has

$$\begin{aligned} |Qx(t) - Qy(t)| &= |H(F)x(t) - H(F)y(t)| = \\ &= \Big| \int_{t_0}^{t} (s - t_0) \big[ f(t, x(t), x(\phi(t)), x'(t), x'(\psi(t))) - \\ &- f(t, y(t), y(\phi(t)), y'(t), y'(\psi(t))) \big] \, ds + \\ &+ (t - t_0) \int_{t}^{t_1} \big[ f(t, x(t), x(\phi(t)), x'(t), x'(\psi(t))) - \end{aligned}$$

$$-f(t,y(t),y(\phi(t)),y'(t),y'(\psi(t)))\big]\,ds\Big| \le$$

$$\le \int_{t_0}^{t}(s-t_0)\big(k_1(s)|x(s)-y(s)|+k_2(s)|x(\phi(s))-y(\phi(s))|+$$

$$+k_3(s)|x'(s)-y'(s)|+k_4(s)|x'(\psi(s))-y'(\psi(s))|\big)\,ds+$$

$$+(t-t_0)\int_{t}^{t_1}\big(k_1(s)|x(s)-y(s)|+k_2(s)|x(\phi(s))-y(\phi(s))|+$$

$$+k_3(s)|x'(s)-y'(s)|+k_4(s)|x'(\psi(s))-y'(\psi(s))|\big)\,ds \le$$

$$\le (t_1-t_0)\int_{t_0}^{t_1}\big(k_1(s)|x(s)-y(s)|+k_2(s)|x(\phi(s))-y(\phi(s))|+$$

$$+k_3(s)|x'(s)-y'(s)|+k_4(s)|x'(\psi(s))-y'(\psi(s))|\big)\,ds \le$$

$$\le (t_1-t_0)\left(\sum_{i=1}^{4}\|k_i\|\right)\|x-y\|_{BM^1}.$$

Similarly if $t \in I_1$,

$$|Qx(t)-Qy(t)| = \textstyle\int_{t_0}^{t_1}(s-t_0)|f(t,x(t),x(\phi(t)),x'(t),x'(\psi(t)))-$$

$$-f(t,y(t),y(\phi(t)),y'(t),y'(\psi(t)))|ds \le$$

$$\le \int_{t_0}^{t_1}(s-t_0)(k_1(s)|x(s)-y(s)|+k_2(s)|x(\phi(s))-y(\phi(s))|+$$

$$+k_3(s)|x'(s)-y'(s)|+k_4(s)|x'(\psi(s))-y'(\psi(s))|)ds \le$$

$$\le (t_1-t_0)\left(\sum_{i=1}^{4}\|k_i\|\right)\|x-y\|_{BM^1}.$$

Again for $t \in I$, one has

$$|(Qx)'(t)-(Qy)'(t)| \le \left(\sum_{i=1}^{4}\|k_i\|\right)\|x-y\|_{BM^1}.$$

Therefore

$$\|Qx-Qy\|_{BM^1} = \|Qx-Qy\|_C + \|(Qx)'-(Qy)'\|_{BM} =$$

$$= \sup_{t\in J}|Qx(t)-Qy(t)| + \sup_{t\in J}|(Qx)'(t)-(Qy)'(t)| \le \alpha\|x-y\|_{BM^1}$$

where

$$\alpha = (1+t_1-t_0)\left(\sum_{i=1}^{4}\|k_i\|\right) < 1.$$

Now an application of Banach fixed point theorem yields the desired result. This completes the proof. □

Now a small generalization of Peano's theorem gives the following existence theorem for local solution.

**Theorem 2.3.** *Assume that* $(f_4)$ *holds. Then the BVP (1)-(2) has a solution on* $J$.

**Proof.** Let $r > \|z\|_{BM^1} + |q_1(t_1)|$ and choose the interval $[t_0, t_1]$ so small that

$$\|z\|_{BM^1} + (1 + t_1 - t_0)\|h_r\|_{L^1} + |q_1(t_1)| < r.$$

Define for $n \in \mathbb{N}$, a sequence

$$x_n(t) = z(t) + \begin{cases} 0, & t \in I_0, \\ \displaystyle\int_{t_0}^{t} (s - t_0) F_n x_n(s) ds + (t - t_0) \int_{t}^{t_1} F_n x_n(s) ds, & t \in I, \\ \displaystyle\int_{t_0}^{t_1} (s - t_0) F_n x_n(s) ds, & t \in I_1, \end{cases}$$

where

$$F_k x_n(s) = f\left(s, x_n\left(s - \frac{1}{k}\right), x_n\left(\phi\left(s - \frac{1}{k}\right)\right),\right.$$
$$\left. x_n'\left(s - \frac{1}{k}\right), x_n'\left(\psi\left(s - \frac{1}{k}\right)\right)\right), \quad s \in I.$$

We note that

$$|F_n x_n(s)| \le h_r(s), \text{ a.e. } s \in I,$$

for all $n \in \mathbb{N}$.

Then for $t \in (t_0, t_1)$,

$$|x_n(t)| \le |z(t)| + \begin{cases} 0, & t \in I_0, \\ \int_{t_0}^{t} (s - t_0) h_r(s) ds + (t - t_0) \int_{t}^{t_1} h_r(s) ds, & t \in I, \\ \int_{t_0}^{t_1} (s - t_0) h_r(s) ds, & t \in I_1, \end{cases}$$

$$\le |z(t)| + (t_1 - t_0)\|h_r\|_{L^1}.$$

Again

$$|x_n'(t)| \leq |q_1(t_1)| + \int_t^{t_1} |F_n x_n(s)| ds \leq |q_1(t_1)| + \|h_r\|_{L^1}.$$

Therefore

$$\begin{aligned} \|x_n\|_{BM^1} &= \sup_{t\in J} |x_n(t)| + \sup_{t\in J} |x_n'(t)| \leq \\ &\leq \|z\|_{BM^1} + |q_1(t_1)| + (1 + t_1 - t_0)\|h_r\|_{L^1} \leq r. \end{aligned}$$

This shows that the sequence $\{x_n\}$ is uniformly bounded on $J$. Next we show that the sequence $\{x_n\}$ is equi-continuous on $J$.

For this, let $t, \bar{t} \in J, t < \bar{t}$ then we have

$$\begin{aligned} |x_n(t) - x_n(\bar{t})| &\leq |z(t) - z(\bar{t})| + \\ &+ \Big| \int_{t_0}^{t} (s - t_0) F_n x_n(s) ds + (t - t_0) \int_t^{t_1} F_n x_n(s) ds - \\ &- \int_{t_0}^{\bar{t}} (s - t_0) F_n x_n(s) ds + (\bar{t} - t_0) \int_{\bar{t}}^{t_1} F_n x_n(s) ds \Big| \leq \\ &\leq |z(t) - z(\bar{t})| + \\ &+ \Big| \int_{t_0}^{t} (s - t_0) F_n x_n(s) ds - \int_{t_0}^{\bar{t}} (s - t_0) F_n x_n(s) ds \Big| + \\ &+ \Big| (t - t_0) \int_t^{t_1} F_n x_n(s) ds - (\bar{t} - t_0) \int_{\bar{t}}^{t_1} F_n x_n(s) ds \Big| \leq \\ &\leq |z(t) - z(\bar{t})| + \left( \int_{t_0}^{t} - \int_{t_0}^{\bar{t}} \right) (s - t_0) |F_n x_n(s)| ds + \\ &+ \left| (t - t_0) \int_t^{t_1} F_n x_n(s) ds - (\bar{t} - t_0) \int_t^{t_1} F_n x_n(s) ds \right| + \\ &+ \left| (\bar{t} - t_0) \int_t^{t_1} F_n x_n(s) ds - (\bar{t} - t_0) \int_{\bar{t}}^{t_1} F_n x_n(s) ds \right| \leq \\ &\leq |z(t) - z(\bar{t})| + \int_t^{\bar{t}} h_r(s) ds + \\ &+ (t - \bar{t}) \int_t^{t_1} h_r(s) ds + (t - t_0) \int_t^{\bar{t}} h_r(s) ds = \\ &= |z(t) - z(\bar{t})| + |w(t) - w(\bar{t})|, \end{aligned}$$

where

$$w(t) = (1 + t_1 - t_0) \int_{t_0}^{t} h_r(s)ds + t\|h_r\|_{L^1}, \ \ t \in J. \tag{12}$$

Similarly

$$\begin{aligned} |x_n'(t) - x_n'(\bar{t})| &\le \left| \int_t^{t_1} F_n x_n(s)ds - \int_{\bar{t}}^{t_1} F_n x_n(s)ds \right| \le \\ &\le \int_t^{\bar{t}} |F_n x_n(s)|ds \le \int_t^{\bar{t}} h_r(s)ds \end{aligned}$$

and thus equi-continuity follows. By Arzelá-Ascoli Theorem, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{h_k} \to x \in BM^1(J, \mathbb{R})$.

Next passing to the limit as $k \to \infty$ in the expression

$$x_{n_k}(t) = z(t) + \begin{cases} 0, \ \ t \in I_0, \\ \displaystyle\int_{t_0}^{t} (s - t_0) F_{n_k} x_{n_k}(s)ds + (t - t_0) \int_t^{t_1} F_{n_k} x_{n_k}(s)ds, \ \ t \in I, \\ \int_{t_0}^{t_1} (s - t_0) F_{n_k} x_{n_k}(s)ds, \ \ t \in I_1, \end{cases}$$

we obtain

$$x(t) = z(t) + \begin{cases} 0, & t \in I_0, \\ \displaystyle\int_{t_0}^{t} (s - t_0) Fx(s)ds + (t - t_0) \int_t^{t_1} Fx(s)ds, & t \in I, \\ \displaystyle\int_{t_0}^{t_1} (s - t_0) Fx(s)ds, & t \in I_1, \end{cases}$$

since

$$\begin{aligned} \lim_{k\to\infty} F_{n_k} x_{n_k}(s) &= \lim_{k\to\infty} f\left(s, x_{n_k}\left(s - \frac{1}{n_k}\right), x_{n_k}\left(\phi\left(s - \frac{1}{n_k}\right)\right),\right. \\ &\qquad \left. x_{n_k}'\left(s - \frac{1}{n_k}\right), x_{n_k}'\left(\psi\left(s - \frac{1}{n_k}\right)\right)\right) = \\ &= f(t, x(t), x(\phi(t)), x'(t), x'(\psi(t))) = \\ &= Fx(s), \ \ s \in I. \end{aligned}$$

This shows that the BVP (1)-(2) has a solution on $J$. This completes the proof. □

**3. Existence via topological transversality method.** It is quite known that *a priori bound method* is very much effective in proving the existence theorems for differential equations. In this section we prove some results in this direction. We use the following nonlinear alternative of Leray-Schauder type which follows immediately from the topological transversality theorem of Granas et. al. [6].

By a map being compact we mean it is continuous with compact range. A map is said to be completely continuous, if it is continuous and the image of every bounded set in the domain is contained in the compact set of the range.

**Theorem 3.1.** *Let $U$ be a relatively open and bounded subset of a convex set in $V$ in a Banach space $E$ and let $G : \bar{U} \to V$ be a compact map. If $p^* \in U$ and $N_\lambda(x) := N(\lambda, x) : [0, 1] \times \bar{U} \to V$ is a family of compact maps with $N_0 = p^*$, the constant map, and $N_1 = G$, then*

*(i) $G$ has a fixed point in $\bar{U}$, or*

*(ii) there is a point $u \in \partial U$ and $\lambda \in (0, 1)$ such that $N_\lambda u = u$.*

Before going to the main existence theorem we prove a useful lemma.

**Lemma 3.2.** *Assume that hypothesis $(f_4)$ holds. Then for any bounded subset $S$ of $BM^1(J, \mathbb{R})$, the set $Q(S)$ is equi-continuous on $J$.*

**Proof.** Now for any $x \in S$ and for $t, \bar{t} \in J$, one has

$$\begin{aligned}
|Qx(t) - Qx(\bar{t})| &\leq |z(t) - z(\bar{t})| + \\
&+ \Big| \int_{t_0}^{t} (s - t_0) Fx(s) ds + (t - t_0) \int_{t}^{t_1} Fx(s) ds - \\
&- \int_{t_0}^{\bar{t}} (s - t_0) Fx(s) ds - (\bar{t} - t_0) \int_{\bar{t}}^{t_1} Fx(s) ds \Big| \leq \\
&\leq |z(t) - z(\bar{t})| + \\
&+ \Big| \int_{t_0}^{t} (s - t_0) Fx(s) ds - \int_{t_0}^{\bar{t}} (s - t_0) Fx(s) ds \Big| + \\
&+ \Big| (t - t_0) \int_{t}^{t_1} Fx(s) ds - (\bar{t} - t_0) \int_{\bar{t}}^{t_1} Fx(s) ds \Big| \leq
\end{aligned}$$

$$\le |z(t)-z(\bar t)| + \left(\int_{t_0}^{t} - \int_{t_0}^{\bar t}\right)(s-t_0)|Fx(s)|ds +$$
$$+ \left|(t-t_0)\int_t^{t_1} Fx(s)ds - (\bar t - t_0)\int_t^{t_1} Fx(s)ds\right| +$$
$$+ \left|(\bar t-t_0)\int_t^{t_1} Fx(s)ds - (\bar t - t_0)\int_{\bar t}^{t_1} Fx(s)ds\right| =$$
$$= |z(t)-z(\bar t)| + |w(t)-w(\bar t)|,$$

where $w$ is defined in (12).

Similarly

$$\begin{aligned} |(Qx)'(t)-(Qx)'(\bar t)| &\le |z'(t)-z'(\bar t)| + \left|\int_t^{t_1} Fx(s)ds - \int_{\bar t}^{t_1} Fx(s)ds\right| \le \\ &\le |z'(t)-z'(\bar t)| + \int_t^{\bar t} |Fx(s)|ds \le \\ &\le |z'(t)-z'(\bar t)| + \int_t^{\bar t} h_r(s)ds = \\ &= |z'(t)-z'(\bar t)| + |w'(t)-w'(\bar t)|. \end{aligned}$$

Therefore the set $Q(S) \subset Q(BM^1(J,\mathbb{R}))$ is equi-continuous on $J$ and the proof of the lemma is complete. □

**Theorem 3.3.** *Assume that* $(f_4)$ *holds. In addition assume there is a constant* $K^* > 0$ *independent of* $\lambda$ *such that*

$$\|x\|_{BM^1} \le K^* \tag{13}$$

*for any solution* $x$ *of the BVP*

$$-x''(t) = \lambda f(t, x(t), x(\phi(t)), x'(t), x'(\psi(t))), \quad a.e.\ t \in I \tag{14}$$

$$\begin{cases} x(t) = q_0(t), & t \in I_0 \\ x'(t) = q_1(t), & t \in I_1 \end{cases} \tag{15}$$

*for each* $\lambda \in (0,1)$. *Then the BVP (1)-(2) has a solution.*

**Proof.** By Lemma 1.4, the BVP (14)-(15) has a unique solution $x$ given by

(16)

$$x(t) = z(t) + \lambda \begin{cases} 0, & t \in I_0, \\ \displaystyle\int_{t_0}^{t} (s - t_0) Fx(s) ds + (t - t_0) \int_{t}^{t_1} Fx(s) ds, & t \in I, \\ \displaystyle\int_{t_0}^{t_1} (s - t_0) Fx(s) ds, & t \in I_1 \end{cases}$$

for each $\lambda \in (0, 1)$.

We define the operator $N_\lambda : BM_C^1(J, \mathbb{R}) \to BM_C^1(J, \mathbb{R})$ by the right hand side of (14), where $BM_C^1(J, \mathbb{R}) \subset BM^1(J, \mathbb{R})$ and is given by

$$BM_C^1(J, \mathbb{R}) = \{x \in BM^1(J, \mathbb{R}) \mid x \text{ satisfies boundary conditions } (2)\}.$$

Then the equation (16) reduces to fixed point problem $x = N_\lambda x$. Let $S$ be a bounded subset of $BM_C^1(J, \mathbb{R})$. Then there exists a real number $r > 0$ such that $\|x\| \leq r$ for all $x \in S$. First we claim that the operator $N_\lambda$ is continuous on $BM_C^1(J, \mathbb{R})$. Let there be a sequence $\{u_n\}$ in $S$ converging to a point $u \in S$. Hence we have $|u_n(s)| \leq r$, $|u(s)| \leq r$ for $s \in J$. By uniform convergence we have $Fu_n(s) \to Fu(s)$ pointwice almost everywhere on $J$. Also there exists a Lebesgue integrable function $h_r$ with

$$|Fu_n(s)| \leq h_r(s)$$

for almost all $s \in I$.

Now Lebesgue dominated convergence theorem implies that $N_\lambda u_n(t) \to N_\lambda u(t)$ and $(N_\lambda u_n)'(t) \to (N_\lambda u)'(t)$ pointwise for each $t \in J$. To prove that $N_\lambda$ is continuous we only need to show that the above convergence is uniform.

Let $t, \bar{t} \in J$, then by Lemma 3.2 we have

$$|N_\lambda u_n(t) - N_\lambda u_n(\bar{t})| \leq |z(t) - z(\bar{t})| + |w(t) - w(\bar{t})|$$

and

$$|(N_\lambda u_n)'(t) - (N_\lambda u_n)'(\bar{t})| \leq |z'(t) - z'(\bar{t})| + |w'(t) - w'(\bar{t})|$$

where $w : J \to \mathbb{R}$ is a continuous function given by (12).

We have the similar bounds for

$$|N_\lambda u(t) - N_\lambda u(\bar{t})| \text{ and } |(N_\lambda u)'(t) - (N_\lambda u)'(\bar{t})|.$$

Let $\epsilon > 0$ be given. Then there exists $\delta > 0$ such that

$$\begin{aligned} &|N_\lambda u_n(t) - N_\lambda u_n(\bar{t})| < \tfrac{\epsilon}{3}, \\ &|(N_\lambda u_n)'(t) - (N_\lambda u_n)'(\bar{t})| < \tfrac{\epsilon}{3} \end{aligned} \tag{17}$$

for all $n \in \mathbb{N}$ and

$$\begin{aligned} &|N_\lambda u(t) - N_\lambda u(\bar{t})| < \tfrac{\epsilon}{3}, \\ &|(N_\lambda u)'(t) - (N_\lambda u)'(\bar{t})| < \tfrac{\epsilon}{3} \end{aligned} \tag{18}$$

whenever $|t - \bar{t}| < \delta$.

Then the inequalities (17)-(18) together with the fact that $N_\lambda u_n(t) \to N_\lambda u(t)$ and $(N_\lambda u_n)'(t) \to (N_\lambda u)'(t)$ pontwise imply that the convergence is uniform. Therefore $N_\lambda : S \to BM_C^1(J, \mathbb{R})$ is continuous.

Again it can be shown, as in the proof of Theorem 2.3, that $N_\lambda(S)$ is uniformly bounded. The equi-continuity of $N_\lambda(S)$ follows from Lemma 3.2. Hence an application of Arzelá-Ascoli theorem yields that the operator, $N_\lambda : BM_C^1(J, \mathbb{R}) \to BM_C^1(J, \mathbb{R})$ is completely continuous. Set

$$U = \{x \in BM_C^1(J, \mathbb{R}) \mid \|x\|_{BM^1} < K + 1\},$$

$$V = BM_C^1(J, \mathbb{R}) \text{ and } E = BM^1(J, \mathbb{R}).$$

Now $N([0, 1] \times \overline{U})$ is bounded in a compact subset of $BM_C^1(J, \mathbb{R})$. So by an application of Theorem 3.1 with $p^* = z(t), t \in J$ yields that the possibility (ii) of Theorem 3.1 is ruled out and we deduce that the BVP (1)-(2) has a solution on $J$. This completes the proof. □

**Theorem 3.4.** *Suppose that* $(f_1)$ *holds. In addition assume*

$(f_5)$ *there exist functions* $p_j \in L^1(I, \mathbb{R}_+)$ *such that*

$$|f(t, x_1, x_2, x_3, x_4)| \leq \sum_{j=1}^{4} p_j(t)|x_j| + p_5(t) \ \ a.e. \ t \in I$$

*for* $x_j \in \mathbb{R}, j = 1, 2, 3, 4$ *holds. Then the BVP (1)-(2) has a solution on* $J$ *whenever*

$$(1 + t_1 - t_0)\left(\sum_{j=1}^{4} \|p_j\|_{L^1}\right) < 1.$$

**Proof.** By Theorem 3.3 we need only establish *a priori bound* for the solution $x \in BM_C^1(J,\mathbb{R})$ to the BVP (14)-(15). Therefore by $(f_5)$ we get

$$|x(t)| \leq |z(t)| + \begin{cases} 0, t \in I_0, \\ \int_{t_0}^{t} (s-t_0)|f(s,x(s),x(\phi(s)),x'(s),x'(\psi(s)))|ds \\ +(t-t_0)\int_{t}^{t_1} |f(s,x(s),x(\phi(s)),x'(s),x'(\psi(s)))|ds, t \in I, \\ \int_{t_0}^{t_1} (s-t_0)|f(s,x(s),x(\phi(s)),x'(s),x'(\psi(s)))|\, ds, t \in I_1, \end{cases}$$

from which, by applying $(f_5)$

$$\begin{aligned} \|x\|_C \leq \|z\|_C + (t_1-t_0)[\|p_1\|_C + \|p_2\|_{L^1})\|x\|_C + \\ + (\|p_3\|_{L^1} + \|p_4\|_{L^1})\|x'\|_{BM} + \|p_5\|_{L^1}]. \end{aligned}$$

Similarly we obtain

$$\begin{aligned} \|x'\|_{BM} \leq \|z'\|_{BM} + (\|p_1\|_{L^1} + \|p_2\|_{L^1})\|x\|_C + \\ + (\|p_3\|_{L^1} + \|p_4\|_{L^1})\|x'\|_{BM} + \|p_5\|_{L^1}. \end{aligned}$$

Therefore

$$\begin{aligned} \|x\|_{BM^1} &= \|x\|_C + \|x'\|_{BM} \leq \\ &\leq (\|z\|_C + \|z'\|_{BM}) + (1+t_1-t_0)(\|p_1\|_C + \|p_2\|_{L^1})\|x\|_C + \\ &+(1+t_1-t_0)(\|p_3\|_{L^1} + \|p_4\|_{L^1})\|x'\|_{BM} + \|p_5\|_{L^1} = \\ &= \|z\|_{BM^1} + (1+t_1-t_0)\left(\sum_{j=1}^{4}\|p_j\|_{L^1}\right)(\|x\|_C + \|x'\|_{BM}) + \|p_5\|_{L^1} = \\ &= \|z\|_{BM^1} + (1+t_1-t_0)\left(\sum_{j=1}^{4}\|p_j\|_{L^1}\right)\|x\|_{BM^1} + \|p_5\|_{L^1}, \end{aligned}$$

or

$$\|x\|_{BM^1} \leq \frac{\|z\|_{BM^1} + \|p_5\|_{L^1}}{1-(1+t_1-t_0)\left(\sum_{j=1}^{4}\|p_j\|_{L^1}\right)} := K.$$

Now the desired result follows by an application of Theorem 3.1. This completes the proof. □

**4. Existence and uniqueness via monotone iterative technique.** In this section we prove the existence and uniqueness of the following nonlinear discontinuous and deviating BVP via comparison principle.

Consider the BVP

$$-x''(t) = f(t, x(t), x(\phi(t))), \quad \text{a.}e. \ t \in I \tag{19}$$

satisfying the boundary conditions

$$\begin{cases} x(t) = q_0(t), & t \in I_0, \\ x'(t) = q_1(t), & t \in I_1, \end{cases} \tag{20}$$

where

(i) $f : I \times \mathbb{R}^2 \to \mathbb{R}$ is a function which is not necessarily continuous,

(ii) $\phi : I \to \mathbb{R}$ are two continuous functions which may delay or advance and satisfy $\phi(I) \subseteq I_0 \cup I \cup I_1$ and

(iii) the boundary functions $q_0, q_1 : \mathbb{R} \to \mathbb{R}$ are continuous.

Now consider the following scalar discontinuous BVP with deviating arguments

$$-u''(t) = g(t, u(t), u(\phi(t))), \ \text{a.e.} \ t \in I, \tag{21}$$

$$\begin{cases} u(t) = \alpha_0(t), \ t \in I_0 \\ u'(t) = \alpha_1(t), \ t \in I_1, \end{cases} \tag{22}$$

where $g : I \times \mathbb{R}_+^2 \to \mathbb{R}_+$, $\mathbb{R}_+$ denotes the set of all positive real numbers, and the boundary functions satisfy

(c) $\alpha_0 : I_0 \to \mathbb{R}_+$ and $\alpha_1 : I_1 \to \mathbb{R}_+$ are continuous functions, and

(d) the condition (ii) of the BVP (19)-(20) holds.

We need the following set of hypotheses in the sequel.

($f_6$) $f(\cdot, x_1, x_2)$ is strongly measurable in $t$ for all $x_i \in \mathbb{R}, i = 1, 2$ and $|f(t, 0, 0)|$ is Lebesgue integrable on $I$.

($f_7$) For $x_i, y_i \in \mathbb{R}, i = 1, 2$ one has

$$|f(t, x_1, x_2) - f(t, y_1, y_2)|$$
$$\leq g(t, |x_1 - y_1|, |x_2 - y_2|), \text{ a.e. } t \in I,$$

and $\quad |q_i(t)| \leq \alpha_i(t), i = 1, 2.$

($g_1$) $g(t, u_1, u_2)$ is Lebesgue measurable in $t$ for all $u_i \in \mathbb{R}_+, i = 1, 2$ and is continuous and nondecreasing in each $u_i \in \mathbb{R}_+$ for almost all $t \in I$.

($g_2$) The BVP (21)-(22) has an upper solution and that (21) has a zero solution with $\alpha_0 = 0 = \alpha_1$.

**Theorem 4.1.** *Assume* ($f_6$), ($f_7$), ($g_1$) *and* ($g_2$) *hold. Then the BVP (19)-(20) has a unique solution* $x^*$ *on* $J$. *Moreover* $x^*$ *is the uniform limit of the sequence* $\{x_n\}$ *defined by*

$$x_{n+1}(t) = Qx_n(t), \;\; t \in J. \tag{23}$$

**Proof.** By ($g_2$), $u(t) \equiv 0$ satisfies the equations (21). So $g(t, 0, 0) \equiv 0$ for almost all $t \in I$. This, ($f_7$) and ($g_1$) imply that $f(t, \cdot, \cdot)$ is almost everywhere continuous for $t \in I$. Also ($f_6$) implies that the function $f$ satisfies Carathéodory condition. In particular $f(\cdot, x_1(\cdot), x_2(\cdot))$ is strongly measurable on $I$ for all $x_i \in C(J, \mathbb{R}), i = 1, 2$.

From hypotheses ($f_6$) and ($f_7$) it follows that

$$\begin{aligned} |f(t, x(t), x(\phi(t)))| &\leq |f(t, 0, 0)| + g(t, |x(t)|, |x(\phi(t))|) \leq \\ &\leq |f(t, 0, 0)| + |\bar{u}''(t)|, \end{aligned}$$

where $\bar{u}$ is an upper solution of the BVP (21)-(22) on $I$, whence

$$F_1 x(t) = f(\cdot, x(\cdot), x(\phi(\cdot)))$$

is Lebesgue integrable for all $x \in BM^1(J, \mathbb{R})$. Therefore the mapping $Q$ on $BM^1(J, \mathbb{R})$ defined by (10) ie well defined.

Now define a mapping $T : C^1(J, \mathbb{R}) \to C^1(J, \mathbb{R})$ by

$$Tu(t) = \begin{cases} 0, & t \in I_0, \\ \displaystyle\int_{t_0}^{t} (s - t_0) Gu(s) ds + (t - t_0) \int_{t}^{t_1} Gu(s) ds, & t \in I, \\ \displaystyle\int_{t_0}^{t_1} (s - t_0) Gu(s) ds, & t \in I_1, \end{cases}$$

where

$$Gu(t) = g(t, u(t), u(\phi(t))), \;\; t \in I.$$

Clearly the mapping $T$ is nondecreasing on $C^1(J, \mathbb{R})$ with respect to the order relation $\leq$ defined by the order cone $K$ in $C^1(J, \mathbb{R})$, viz.

$$K = \{x \in C^1(J, \mathbb{R}) | x(t) \geq 0 \;\text{ and }\; x'(t) \geq 0, \;\text{ for all }\; t \in J\}.$$

Then by Theorem 3.2 of Dhage [3] the operator equation

$$u(t) = u_1(t) + Tu(t), \;\; t \in J \tag{24}$$

has at least one solution for each choice of $u_1 \in \mathbb{R}_+$.

Let $y_0 \in X$ be arbitrary and let $v$ be the solution of (24) with

$$u_1(t) = |z(t) - y_0(t)| + \begin{cases} 0, \;\; t \in I_0, \\ \displaystyle\int_{t_0}^{t} (s - t_0)|F_1 y_0(s)| ds + (t - t_0) \int_{t}^{t_1} |F_1 y_0(s)| ds, \;\; t \in I, \\ \displaystyle\int_{t_0}^{t_1} (s - t_0)|F_1 y_0(s)| ds, \;\; t \in I_1. \end{cases}$$

Consider the subset $S$ of $BM^1(J, \mathbb{R})$ defined by

$$S = \{x \in BM^1(J, \mathbb{R}) \mid |x(t) - y_0(t)| \leq v(t) \;\text{ and }\; |x'(t) - y_0'(t)| \leq v'(t) \text{ for all } t \in J\}.$$

Now from the definition of $Q$ it follows that

$$|Qx(t) - y_0(t)| \leq |z(t) - Qy_0(t)| + |Qy_0(t) - Qx(t)| \leq$$

$$\leq |z(t) - y_0(t)| + \begin{cases} 0, \quad t \in I_0, \\ \displaystyle\int_{t_0}^{t} (s - t_0)|F_1 y_0(s)|\, ds + (t - t_0) \int_{t}^{t_1} |F_1 y_0(s)|\, ds \\ \displaystyle + \int_{t_0}^{t} (s - t_0) G(|x(t) - y_0(t)|)\, ds \\ \displaystyle + (t - t_0) \int_{t}^{t_1} G(|x(t) - y_0(t)|)\, ds, \;\; t \in I, \\ \displaystyle\int_{t_0}^{t_1} (s - t_0)|F_1 y_0(s)|\, ds \\ \displaystyle + \int_{t_0}^{t_1} (s - t_0) G(|x(t) - y_0(t)|)\, ds, \;\; t \in I_1 \end{cases}$$

$$= u_1(t) + Tv(t) = v(t)$$

for all $t \in J$. Similarly

$$|(Qx)'(t) - y_0'(t)| \leq |(Qy_0)'(t) - y_0'(t)| + |(Qx)'(t) - (Qy_0)'(t)| =$$
$$= |z'(t) - y_0'(t)| + \int_t^{t_1} |F_1 y_0(s)|ds + \int_t^{t_1} |F_1 x(s) - F_1 y_0(s)|ds =$$
$$= u_1'(t) + \int_t^{t_1} G(|x(s) - y_0(s)|)ds = v'(t)$$

for all $t \in J$.

This shows that $G(S) \subset S$. Define a mapping $|\cdot| : BM^1(J, \mathbb{R}) \to C^1(J, \mathbb{R}_+)$ by

$$|x|(t) = |x(t)|, \ t \in J.$$

Now from definition of $T$, it follows that

$$\begin{aligned} &|Qx - Qy| \leq T|x - y|, \\ &|(Qx)' - (Qy)'| \leq (T|x - y|)' \end{aligned} \tag{25}$$

for all $x, y \in BM^1(J, \mathbb{R})$.

In view of $Q(S) \subset S$, it follows that

$$\begin{aligned} &|Q^{m+n}x_0 - Q^m x_0| \leq T^m v, \\ &|(Q^{m+n}x_0)' - (Q^m x_0)'| \leq (T^m v)' \end{aligned} \tag{26}$$

for all $n \in \mathbb{N}$ holds when $m = 0$.

Since $T$ is nondecreasing, from (25) and (26) we have

$$\begin{aligned} &|Q^{m+n+1}x_0 - Q^{m+1}x_0| \leq T|Q^{m+n}x_0 - Q^m x_0| \leq T^m v, \\ &|(Q^{m+n+1}x_0)' - (Q^{m+1}x_0)'| \leq T|(Q^{m+n}x_0)' - (Q^m x_0)'| \leq (T^m v)' \end{aligned} \tag{27}$$

holds for all $n \in \mathbb{N}$ and $m \in \mathbb{N}$.

The sequences $\{T^n v\}_{n=0}^{\infty}$ and $\{(T^n v)'\}_{n=0}^{\infty}$ converge, by Theorem 3.2 of Dhage and Patil [5] uniformly on $J$ to $x^*(t)$ and $(x^*)'(t)$ respectively where $x^*$ is the greatest fixed point of the operator $T$. But by $(g_2)$, $x^* \equiv 0$. So, $T^n v \to 0$ uniformly on $J$ with respect to the metric $d$ given by (11).

From (27), we see that $\{Q^n x_0\}$ is a Cauchy sequence in $BM^1(J, \mathbb{R})$, and so converges to a point $\bar{x} \in BM^1(J, \mathbb{R})$.

Thus we have

$$\bar{x} = \lim_{n \to \infty} Q^n x_0 \quad \text{and} \quad \bar{x}' = \lim_{n \to \infty} (Q^n x_0)'.$$

From (27), it follows that when $n \to \infty$,

$$
\begin{aligned}
&|Q\bar{x} - Q^m x_0| \le T^m v, \\
&|(Q\bar{x})' - (Q^m x_0)'| \le (T^m v)'
\end{aligned} \tag{28}
$$

for all $m \in \mathbb{N}$. Then $Q\bar{x} = \lim\limits_{n\to\infty} Q^n x_0$ and $(Q\bar{x})' = \lim\limits_{n\to\infty} (Q^n x_0)'$ so that $\bar{x} = Q\bar{x}$.

If $\bar{y}$ is also a fixed point of $Q$, then from (25), we get

$$
|\bar{x} - \bar{y}| = |Q^n \bar{x} - Q^n \bar{y}| \le T^n |x - y| \tag{29}
$$

for all $n \in \mathbb{N}$. If $v$ is any solution of (24) with $u_1(t)|\bar{x}(t) - \bar{y}(t)|$, then $|\bar{x} - \bar{y}| \le Tv \le v$.

Since $T$ is nondecreasing, we have

$$
T^n |\bar{x} - \bar{y}| \le T^n v
$$

for all $n \in \mathbb{N}$.

As $\{T^n v\}_{n=0}^{\infty}$ converges on $J$ to zero function, in view of (29) we get $\bar{x} = \bar{y}$. This shows that $Q$ has a unique fixed point and consequently the BVP (19)-(20) has a unique solution on $J$. This completes the proof. □

**Remark.** Finally we remark that the existence and uniqueness theorems for the BVPs with deviating arguments of the form

$$
-x^n(t) = f(t, x(t), x(\phi_1), \ldots, x(\phi_m), x'(t), x'(\psi_1), \ldots, x'(\psi_m)), \quad \text{a.}e.\ t \in I \tag{30}
$$

satisfying the boundary conditions (in short BC)

$$
\begin{cases}
x(t) = q_1(t) & t \le t_0, \\
x'(t) = q_2(t) & t \ge t_1
\end{cases} \tag{31}
$$

may be obtained with the similar approach with appropriate modifications. This is accomplished by choosing the interval $J$ so large that

$$
I \bigcup \big( \cup_{i=1}^{m} \phi_i(I) \big) \bigcup \big( \cup_{i=1}^{n} \psi_i(I) \big) \subseteq J.
$$

Further the results of this paper can also be extended to the systems of differential equations i.e $\mathbb{R}$ replaced with $\mathbb{R}^n$ throughout this paper. This is achieved by defining a suitable norm in $\mathbb{R}^n$.

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