ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I.CUZA" IAŞI
Tomul L, s.I a, Matematică, 2004, f.2.

φ-PSEUDO HARMONIC MORPHISMS

BY

DOREL FETCU

Abstract. First, we give some characterizations for φ-pseudo horizontally weakly conformal (φ-[PHWC]) maps. For such a map, $f : (M, g) \to (N, \varphi, \xi, \eta, h)$, between a Riemannian manifold and an almost contact metric manifold with some properties, we define a metric framed ψ-structure on M. Using this structure we find some properties of φ-[PHWC] maps and of φ-pseudo harmonic morphisms. We extend this results for the more general case, where the target manifold is a framed φ-manifold. Finally, we obtain a ψ-structure on the tangent bundle of an almost contact metric manifold, such that the projection map is a harmonic morphism.

Mathematics Subject Classification 2000: 53C15, 53C25, 53C43.
Key words: Framed φ-manifold, harmonic maps, φ-pseudo harmonic morphisms.

1. Introduction. Let M be an m-dimensional smooth manifold endowed with a tensor field φ of type $(1, 1)$, satisfying the algebraic condition

$$\varphi^3 + \varphi = 0. \tag{1.1}$$

The geometric structure on M defined by φ is called a φ-structure of rank r if the rank r of φ is constant on M and, in this case, M is called a φ-manifold. It follows easily that r is an even number.

If M is a φ-manifold and if there are $m - r$ vector fields ξ_i and $m - r$ differential 1-forms η_i satisfying

$$\varphi^2 = -I + \sum_{i=1}^{m-r} \eta_i \otimes \xi_i, \tag{1.2}$$

$$\eta_i(\xi_j) = \delta^i_j, \tag{1.3}$$

where $i, j = 1, 2, ..., m - r$, M is said to be globally framed or to have a framed φ-structure. In this case M is called a globally framed φ-manifold or, simply, a framed φ-manifold. From (1.2) and (1.3), one obtains by some algebraic computations

$$\varphi \xi_i = 0, \quad \eta_i \circ \varphi = 0, \quad \varphi^3 + \varphi = 0. \tag{1.4}$$

If $m = 2n+1$ and $rank\ \varphi = 2n$ one obtains an almost contact structure on M.

Let M be an m-dimensional globally framed φ-manifold with structure tensors (φ, ξ_i, η_i) with $rank\ \varphi = r$, and consider the manifold $M \times \mathbb{R}^{m-r}$. A vector field on $M \times \mathbb{R}^{m-r}$ will be denoted by $(X, \sum_{i=1}^{m-r} f_i \frac{\partial}{\partial t^i})$, where X is tangent to M, $\{t^1, ..., t^{m-r}\}$ are the coordinates on $\mathbb{R}^{m-r}$ and $\{f_1, ..., f_{m-r}\}$ are functions on $M \times \mathbb{R}^{m-r}$. Define an almost complex structure on $M \times \mathbb{R}^{m-r}$ by

$$J\Big(X, \sum_{i=1}^{m-r} f_i \frac{\partial}{\partial t^i}\Big) = \Big(\varphi X - \sum_{i=1}^{m-r} f_i \xi_i, \sum_{i=1}^{m-r} \eta_i(X) \frac{\partial}{\partial t^i}\Big).$$

It is easy to check that $J^2 = -I$. If J is integrable we say that the framed φ-structure is normal. A framed φ-structure is normal if the tensor field S of type (1,2) defined by

$$S = N_\varphi + \sum_{i=1}^{m-r} d\eta_i \otimes \xi_i, \tag{1.5}$$

vanishes, (see [7]), where

(1.6)
$$N_\varphi(X, Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X, Y], X, Y \in \chi(M),$$

is the Nijenhuis tensor field of φ.

If g is a (semi-)Riemannian metric on M such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \sum_{i=1}^{m-r} \eta_i(X)\eta_i(Y), \tag{1.7}$$

then we say that $(\varphi, \xi_i, \eta_i, g)$ is a metric framed φ-structure and M is called a metric framed φ-manifold.

The metric g is called an associated (semi-)Riemannian metric.

The fundamental 2-form Ω of the considered metric framed φ-manifold M, is defined just like in the case of the almost Hermitian and almost contact metric manifold, by $\Omega = g(X, \varphi Y)$, for any $X, Y \in \chi(M)$.

The framed φ-manifold M with structure tensors (φ, ξ_i, η_i) is called a $\mathcal{C}$-manifold if it is normal, $d\Omega = 0$ and $d\eta_i = 0$; (1,2)-symplectic ((1,2)-D-symplectic) manifold if

$$d\Omega(X, \varphi X, Y) = 0, \quad X \in \Gamma(D), Y \in \chi(M)(Y \in \Gamma(D)),$$

and (1,2)-symplectic like ((1,2)-D-symplectic like) manifold if

$$\sum_{k=1}^{n} d\Omega(e_k, \varphi e_k, Y) = 0, \quad Y \in \chi(M)(Y \in \Gamma(D)),$$

where $\{e_1, ..., e_n, \varphi e_1, ..., \varphi e_n\}$ is a local orthonormal φ-basis in $\Gamma(D)$.

If on an almost contact manifold (M, φ, ξ, η) it is defined an associated Riemannian metric g then $(M, \varphi, \xi, \eta, g)$ is called an almost contact metric manifold. If on an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ we have $\Omega = d\eta$, where Ω is the fundamental 2-form on M, then we say that $(M, \varphi, \xi, \eta, g)$ is a contact metric manifold.If for an almost contact metric structure (φ, ξ, η, g) which is normal we have $d\eta = 0$ and $d\Omega = 0$ then $(N, \varphi, \xi, \eta, g)$ is called a cosymplectic manifold.

In [2] the following two results are proved

Lemma 1.1. *For an almost contact metric structure (φ, ξ, η, g), the covariant derivative of φ is given by*

$$2g((\nabla_X \varphi)Y, Z)) = 3d\Omega(X, \varphi Y, \varphi Z) - 3d\Omega(X, Y, Z) + g(N_\varphi(Y, Z), \varphi X) +$$
$$+((L_{\varphi Y}\eta)(Z) - (L_{\varphi Z}\eta)(Y))\eta(X) + 2d\eta(\varphi Y, X)\eta(Z) - 2d\eta(\varphi Z, X)\eta(Y).$$

where L denotes the Lie derivative.

Obviously, in Lemma 1.1 the author uses the "Alt" convention for calculus of $d\Omega$ and $d\eta$.

Theorem 1.2. *An almost contact metric structure (φ, ξ, η, g) is cosymplectic if and only if φ is parallel.*

Using the "Det" convention for calculus of $d\Omega$ and $d\eta_i$, $i = 1, ..., n$, just like in [2], one obtains

Lemma 1.3. *If* $(M,\varphi,\xi_i,\eta_i,g)$ *is a metric framed* φ*-manifold, where* $i=1,...n$*, then*

$$\begin{aligned}2g((\nabla_X\varphi)Y,Z)) &= d\Omega(X,\varphi Y,\varphi Z)-d\Omega(X,Y,Z)+g(N_\varphi(Y,Z),\varphi X)+\\&+\sum_{i=1}^{n}[d\eta_i(\varphi Y,X)\eta_i(Z)-d\eta_i(\varphi Z,X)\eta_i(Y)]+\\&+\sum_{i=1}^{n}[d\eta_i(\varphi Y,Z)+d\eta_i(Y,\varphi Z)]\eta_i(X).\end{aligned}$$

Remark 1.4.. If $(M,\varphi,\xi_i,\eta_i,g)$ is a normal metric framed φ-manifold, where $i=1,...n$, then

$$\begin{aligned}2g((\nabla_X\varphi)Y,Z)) = d\Omega(X,\varphi Y,\varphi Z)-d\Omega(X,Y,Z)+\\+\sum_{i=1}^{n}[d\eta_i(\varphi Y,X)\eta_i(Z)-d\eta_i(\varphi Z,X)\eta_i(Y)].\end{aligned}$$

Remark 1.5. It is easy to see that if $(M,\varphi,\xi_i,\eta_i,g)$ is a $\mathcal{C}$-manifold then φ is parallel.

2. Harmonic morphisms. Concerning the harmonic maps and the harmonic morphisms between Riemannian manifolds, we should recall some notions and results as they are presented in [13] and in [1].

Let (M,g) and (M',g') two Riemannian manifolds with dim $M=m$, and let $f:M\to M'$ be a smooth map. Define the energy density function of f, $e(f)\in C^\infty(M)$, by

$$e(f)=\frac{1}{2}tr_g(f^*g')(x)=\frac{1}{2}\sum_{i=1}^{m}(f^*g')(e_i,e_i)=\frac{1}{2}\sum_{i=1}^{m}g'(df(e_i),df(e_i)),x\in M,$$

where $\{e_1,...,e_m\}$ is an orthonormal basis for the tangent space T_xM at $x\in M$, and $df:T_xM\to T_{f(x)}M'$, is the tangent map of f. If M is compact we define the energy of f by $E(f)=\int_M e(f)\nu_g$, where ν_g is the volume form of (M,g).

Then f is called a harmonic map if f is a critical point of E in $C^\infty(M,M')$.

Let $f^{-1}(TM')$ be the induced bundle from TM' over M, defined as follows. Denote by $\pi : TM' \to M'$ the projection. Then

$$f^{-1}(TM') = \{(x,u) \in M \times TM', \pi(u) = f(x), x \in M\} = \bigcup_{x \in M} T_{f(x)}M'.$$

The set of all C^∞-sections of $f^{-1}TM'$, denoted by $\Gamma(f^{-1}TM')$ is

$$\Gamma(f^{-1}TM') = \{V : M \to TM', V \; is \; a \; C^\infty - map, \; V(x) \in T_{f(x)}M', x \in M\}.$$

Denote by ∇, ∇' the Levi-Civita connections on M and M' respectively and by $\widetilde{\nabla}$ the connection induced by the map f on the bundle $f^{-1}(TM')$. Then the second fundamental form α of f is defined by $\alpha(X,Y)=\widetilde{\nabla}_X df(Y) - df(\nabla_X Y)$, for any $X, Y \in \chi(M)$.

The tension field $\tau(f)$ of f is defined by $\tau(f)_x = \sum_{i=1}^m \alpha(e_i, e_i)(x)$, where $\{e_1, ..., e_m\}$ is an orthonormal basis for the tangent space T_xM at $x \in M$.

The map $f : M \to M'$ is a harmonic map if and only if $\tau(f) = 0$, (see [13]).

Let M be a Kähler manifold with the complex structure J and let (N, h) be a Riemannian manifold. A map $f : M \to N$ is called pluriharmonic if the second fundamental form α of f satisfies (see [10])

$$\alpha(X,Y) + \alpha(JX, JY) = 0, X, Y \in \chi(M).$$

For the framed φ-manifolds we consider a similar concept.

Definition 2.1. Let $(M, \varphi, \xi_i, \eta_i, g)$ be a metric framed φ-manifold with $rank \; \varphi = r$, and let (N, h) be a Riemannian manifold. If $f : M \to N$ is a smooth map we say that f is φ-pluriharmonic if

$$\alpha(X,Y) + \alpha(\varphi X, \varphi Y) = 0, \tag{2.1}$$

for any $X, Y \in \chi(M)$, where α is the second fundamental form of f. If equation (2.1) holds for $X, Y \in \Gamma(D)$, where D denote the distribution in M orthogonal to $span\{\xi_1, ..., \xi_{m-r}\}$, where $dim \; M = m$, then we say that the map f is D-pluriharmonic.

Definition 2.2. Let $f : M \to N$ be a smooth map between Riemannian manifolds. Then f is called a harmonic morphism if, for every harmonic function $\pi : V \to \mathbb{R}$ defined on an open subset V of N with $\pi^{-1}(V)$ non-empty, the composition $\pi \circ f$ is harmonic on $\pi^{-1}(V)$.

Definition 2.3. Let $f : (M, g) \to (N, h)$ be a smooth map between Riemannian manifolds, and let $x \in M$. Then f is said to be (weakly) conformal at x if there is a number $\Lambda(x)$ such that

$$h(df_x(X), df_x(Y)) = \Lambda(x) g(X, Y), \quad X, Y \in T_x M.$$

For any smooth map $f : M \to N$ between Riemannian manifolds, and any point $x \in M$, set $\mathcal{V}_{fx} = \ker df_x$ and $\mathcal{H}_{fx} = \{\ker df_x\}^{\perp}$; then $\mathcal{V}_{fx}$ is called the vertical space and $\mathcal{H}_{fx}$ the horizontal space of f at x.

Definition 2.4. Let $f : M \to N$ be a smooth map between Riemannian manifolds, and let $x \in M$. Then f is called a horizontally weakly conformal or semiconformal at x if either

(i) $df_x = 0$, or

(ii) df_x maps the horizontal space $\mathcal{H}_{fx} = \{\ker df_x\}^{\perp}$ conformally onto $T_{f(x)}N$.

The map f is called horizontally weakly conformal or semiconformal on M if it is horizontally weakly conformal at every point of M.

In [1] the next result is proved

Theorem 2.1. *A smooth map $f : M \to N$ between Riemannian manifolds is a harmonic morphism if and only if is both harmonic and horizontally weakly conformal.*

3. φ-pseudo horizontally weakly conformal (φ-[PHWC]) maps. Let $f : (M, g) \to (N, h)$ be a smooth map between Riemannian manifolds and let $df : TM \to TN$ be the induced tangent map. The adjoint of df_x, where $x \in M$, is the linear map $df_x^* : T_{f(x)}N \to T_x M$ characterized by

$$g(X, df_x^*(Y)) = h(df_x(X), Y), \quad X \in T_x M, \ Y \in T_{f(x)}N.$$

For any point $x \in M$, set the vertical space of f, $\mathcal{V}_{fx} = \ker df_x$ and the horizontal space of f, $\mathcal{H}_{fx} = \{\ker df_x\}^{\perp}$. Also set $\mathcal{V}_{f^*x} = \ker df_x^*$ and $\mathcal{H}_{f^*x} = \{\ker df_x^*\}^{\perp}$. Then the maps $df_x : \mathcal{H}_{fx} \to \mathcal{H}_{f^*x}$ and $df_x^* : \mathcal{H}_{f^*x} \to \mathcal{H}_{fx}$ are one-to-one.

Let $(N, \varphi, \xi_i, \eta_i, h)$ be a metric framed φ-manifold and let $T^C N$ be the complexification of its tangent bundle, TN. Then φ can be uniquely extended to a complex linear endomorphism of $T^C N$, denoted also by φ, which satisfies 1.2. The eigenvalues of φ are $i, 0, -i$. Consider the usual decomposition

$$T^C N = T'N \oplus T^0 N \oplus T''N$$

of $T^C N$ in the eigenbundles corresponding to the eigenvalues $i, 0, -i$ of φ.

Let $f : (M, g) \to (N, \varphi, \xi_i, \eta_i, h)$ be a smooth map between the Riemannian manifold M, and the metric framed φ-manifold N. We can define the map $(df \circ df^*)_x : T^C_{f(x)} N \to T^C_{f(x)} N$, for any $x \in M$.

Definition 3.1. Let $f : (M, g) \to (N, \varphi, \xi_i, \eta_i, h)$ be a smooth map between the Riemannian manifold M, and the metric framed φ-manifold N. Then f is called a φ-pseudo horizontally weakly conformal (φ-[PHWC]) map if

$$[df \circ df^*, \varphi] = 0. \tag{3.1}$$

Remark 3.1. Any horizontally weakly conformal map, $f : (M, g) \to (N, \varphi, \xi_i, \eta_i, h)$, between the Riemannian manifold M, and the metric framed φ-manifold N, is a φ-[PHWC] map. In particular if f is a harmonic morphism then it is a φ-[PHWC] map.

We have

Lemma 3.2. *Let $f : (M, g) \to (N, \varphi, \xi_i, \eta_i, h)$ be a smooth map between the Riemannian manifold M, and the metric framed φ-manifold N. Then f is a φ-[PHWC] map if and only if $df \circ df^*$ map the holomorphic tangent bundle $T'N$ onto itself and $T^0 N$ onto itself.*

Proof. Consider $X \in T^{'}N \oplus T^{''}N$. Then $\varphi(df \circ df^*)X = (df \circ df^*)\varphi X$ if and only if $\varphi(df \circ df^*)X = i(df \circ df^*)X$, for any $X \in T^{'}N$, since $df \circ df^*$ is a complex linear map.

Next, for any ξ_i, we have $\varphi(df \circ df^*)\xi_i = 0$ if and only if $(df \circ df^*)\xi_i \in T^0 N$. □

Lemma 3.3. *Let $f : (M, g) \to (N, \varphi, \xi_i, \eta_i, h)$ be a smooth map between the Riemannian manifold M, and the metric framed φ-manifold N. Set, for any $x \in M$, $V^{'}_x = df^*_x(T^{'}_{f(x)}N)$ and $V^0_x = df^*_x(T^0_{f(x)}N)$. Then f is a*

φ-[PHWC] map if and only if $V_x^{'}$ is isotropic and $V_x^{'}$ and V_x^0 are orthogonal with respect g.

Proof. For $X, Y \in T_{f(x)}^{'}N$ one obtains

$$g_x(df_x^*(X), df_x^*(Y)) = h_{f(x)}(X, (df \circ df^*)_x(Y)) = 0$$

if and only if $df \circ df^*$ map the holomorphic tangent bundle $T'N$ onto itself or $(df \circ df^*)_x(Y) \in T_{f(x)}^0 N$. But for $Y \in T_{f(x)}^{'}N$ we have

$$g_x(df_x^*(Y), df_x^*(\xi_{if(x)})) = h_{f(x)}(Y, (df \circ df^*)_x(\xi_{if(x)})) =$$

$$= h_{f(x)}((df \circ df^*)_x(Y), \xi_{if(x)}) = 0$$

if and only if $(df \circ df^*)_x(\xi_{if(x)}) \in T_{f(x)}^0 N \cup T_{f(x)}^{'}N$ and $(df \circ df^*)_x(Y) \notin T_{f(x)}^0 N$. If $(df \circ df^*)_x(\xi_{if(x)}) \in T_{f(x)}^{'}N$ one obtains

$$g_x((df^*)_x(\xi_{if(x)}), (df^*)_x(\xi_{if(x)})) = h_{f(x)}(\xi_{if(x)}, (df \circ df^*)_x(\xi_{if(x)})) = 0.$$

Hence $(df^*)_x(\xi_{if(x)}) = 0$ and the conclusion follows from the previous Lemma. □

Remark 3.4. It is easy to see that if $\pi : (N, \varphi, \xi_i, \eta_i, h) \to (P, \varphi', \xi_i', \eta_i', h')$ is a (φ, φ')-holomorphic map between two metric framed φ (and φ' respectively) -manifolds, that is $d\pi \circ \varphi = \varphi' \circ d\pi$, then $d\pi^* \circ \varphi' = \varphi \circ d\pi^*$.

One obtains easily

Proposition 3.5. *Let $f : (M, g) \to (N, \varphi, \xi_i, \eta_i, h)$ be a smooth map between the Riemannian manifold M, and the metric framed φ-manifold N. Then f is a φ-[PHWC] map if and only if for any (φ, φ')-holomorphic map between two metric framed φ (and φ' respectively) -manifolds, $\pi : (N, \varphi, \xi_i, \eta_i, h) \to (P, \varphi', \xi_i', \eta_i', h')$, $\pi \circ f : (M, g) \to (P, \varphi', \xi_i', \eta_i', h')$ is a φ'-[PHWC] map.*

Definition 3.2. A φ-[PHWC] map which is harmonic is called a φ-pseudo harmonic morphism.

In [4] the author studies the φ-[PHWC] maps between a Riemannian manifold and a metric φ-manifold, $f : (M, g) \to (N, \varphi, h)$, which satisfies the condition $df(TM) \subset \Gamma(D^N)$, where $\Gamma(D^N) = (\ker \varphi)^{\perp}$. In the following we study the φ-[PHWC] maps between a Riemannian manifold and a metric framed φ-manifold, $f : (M, g) \to (N, \varphi, \xi_i, \eta_i, h)$, for which $df(TM) \nsubseteq \Gamma(D^N)$.

First, consider the φ-[PHWC] map $f : (M, g) \to (N, \varphi, \xi, \eta, h)$ between a Riemannian manifold and an almost contact metric manifold. It is easy to see that $df(TM) \subset \Gamma(D^N)$ if and only if $\xi_{f(x)} \in \mathcal{V}_{f^*x}$, for any $x \in M$. In the following suppose that $\xi_{f(x)} \notin \mathcal{V}_{f^*x}$, for any $x \in M$.

We have,

Lemma 3.6. *For f being as above, one obtains*

$$(df \circ df^*)_x \xi_{f(x)} = \Lambda(x)\xi_{f(x)},$$

for any $x \in M$, where $\Lambda(x) = \lambda^2(x) \gneq 0$.

Proof. Since f is a φ-[PHWC] map, from Lemma 3.2, we have at any $x \in M$, that $(df \circ df^*)_x \xi_{f(x)} = \Lambda(x)\xi_{f(x)}$. Then one obtains

$$0 \lneq g_x(df^*\xi_{f(x)}, df^*\xi_{f(x)}) = h_x(\xi_{f(x)}, (df \circ df^*)_x \xi_{f(x)}) = \Lambda(x).$$

□

Hence we can define the vector field ζ_1 on M by

$$\zeta_{1x} = \frac{1}{\lambda(x)} df^*_x \xi_{f(x)}, \quad x \in M, \tag{3.2}$$

where $\lambda(x)$ is defined in the previous Lemma.

Define a tensor field, ψ, of type (1,1), on M by

$$\psi(X) = \begin{cases} df^* \circ \varphi \circ (df^*)^{-1} X & , X \in \mathcal{H}_f \\ 0 & , X \in \mathcal{V}_f, \end{cases} \tag{3.3}$$

where $\mathcal{H}_f = \cup_{x \in M} \mathcal{H}_{fx}$ and $\mathcal{V}_f = \cup_{x \in M} \mathcal{V}_{fx}$.

Note that if $X \in \mathcal{H}_f \setminus span\{\zeta_1\}$ then $\psi X \in \mathcal{H}_f \setminus span\{\zeta_1\}$.

Next consider the 1-form, θ_1, on M, by

$$\theta_{1x}(X_x) = \frac{1}{\lambda(x)} \eta_{f(x)}(df_x(X_x)), \tag{3.4}$$

for any $x \in M$, and $X \in \chi(M)$.

Finally let $\{\zeta_2, ..., \zeta_r\}$ be an orthonormal frame field on $\mathcal{V}_f$ and let $\{\theta_2, ..., \theta_r\}$ be the 1-forms defined by $\theta_k(X) = g(X, \zeta_k)$, $X \in \chi(M)$, for any $k = 2, ..., r$.

Obviously one obtains

$$\psi^2 X = -X + \sum_{k=1}^{r} \theta_k(X)\zeta_k,\ \theta_k \circ \psi = 0,\ \theta_k(\zeta_l) = \delta_{kl},$$

for any $k, l = 1, ..., r$ and $X \in \chi(M)$. Hence

Proposition 3.7. *$(M, \psi, \zeta_k, \theta_k, g)$ is a metric framed ψ-manifold.*

In the following we call the framed ψ-structure the related (to f) framed ψ-structure on M.

Proposition 3.8. *Let $f : (M, \psi, \zeta_k, \theta_k, g) \to (N, \varphi, \xi, \eta, h)$ be as above. Then f is a (ψ, φ)-holomorphic map.*

Definition 3.3. Let (M, g) be a Riemannian manifold. If $\mathcal{V}$ is a distribution on M then $\mathcal{V}$ is said to be minimal if $\sum_{i=1}^{s} \nabla_{v_i} v_i \in \mathcal{V}$, where $\{v_i\}_{i=1}^{s}$ is a local orthonormal basis in $\mathcal{V}$, *dim* $\mathcal{V} = s$, and ∇ is the Levi-Civita connection on M.

Using the related framed ψ-structure one obtains

Theorem 3.9. *Let $f : (M, g) \to (N, \varphi, \xi, \eta, h)$ be a φ-[PHWC] map between a Riemannian manifold and a (1,2)-symplectic manifold, for which $\mathcal{V}_f = \ker f$ is minimal, $\xi_{f(x)} \notin \mathcal{V}_{f^*x}$, for any $x \in M$, and $d\eta(\widetilde{X}, \xi) = 0$, for any $\widetilde{X} \in \chi(M)$. If ψ is parallel and $\theta_1 \wedge d\theta_1 \neq 0$ then f is a harmonic map.*

Proof. Denote by ∇^M and ∇^N the Levi-Civita connections on M and N respectively. Since ψ is parallel one obtains easily that

$$\nabla^M_X X + \nabla^M_{\psi X} \psi X = \psi[\psi X, X],$$

for any $X \in \mathcal{H}_f \setminus span\{\zeta_1\}$.

Since N is (1,2)-symplectic we have, from Lemma 1.3

$$\nabla^N_{\widetilde{X}} \widetilde{X} + \nabla^N_{\varphi \widetilde{X}} \varphi \widetilde{X} = \varphi[\varphi X, X],$$

for any $\widetilde{X} \in \Gamma(D^N)$, where D^N is the distribution on N orthogonal to $span\{\xi\}$.

If we denote by α the second fundamental form of f, since f is a (ψ,φ)-holomorphic map, one obtains

$$\sum_{i=1}^{m}[\alpha(e_i,e_i)+\alpha(\psi e_i,\psi e_i)]=0,$$

where $\{e_1,...,e_m,\psi e_1,...,\psi e_m,\zeta_1,...,\zeta_r\}$ is a local orthonormal ψ-basis on M.

Next, for $X=Y=\zeta_1$ in Lemma 1.3, we have $d\theta_1(\psi Z,\zeta_1)=0$ for any $Z\in\chi(M)$. From definition of θ_1 it follows easily that $d\theta_1(\zeta_k,\zeta_1)=0$ for any $k=2,...,r$. Using this results one obtains, in particular, that $\nabla^M_{\zeta_1}\zeta_1=0$.

Since $d\eta(\widetilde{X},\xi)=0$, for any $\widetilde{X}\in\chi(N)$, and $\lambda\theta_1(X)=\eta(df(X))$ one obtains by differentiation that

$$(d\lambda\wedge\theta_1)(X,Y)+\lambda d\theta_1(X,Y)=d\eta(df(X),df(Y)),$$

for any $X,Y\in\chi(M)$. If we set $Y=\zeta_1$ one obtains $d\lambda=L_{\zeta_1}\lambda\theta_1$, where L denote the Lie derivative. It follows that $d\lambda\wedge\theta_1=0$ and then $d\lambda\wedge d\theta_1=0$. Hence $L_{\zeta_1}\lambda\theta_1\wedge d\theta_1=0$. So, by hypothesis we have that $L_{\zeta_1}\lambda=0$. That is $d\lambda=0$ and then λ is a constant.

Thus we have $\nabla^N_{df(\zeta_1)}df(\zeta_1)=\Lambda\nabla^N_\xi\xi=0$, and $df(\sum_{k=2}^r\nabla^M_{\zeta_k}\zeta_k)=0$, since $\mathcal{V}_f=\ker f$ is minimal.

Hence

$$\sum_{k=1}^{r}\alpha(\zeta_k,\zeta_k)=0,$$

and then $\tau(f)=0$, where $\tau(f)$ is the tension field of f. That is f is a harmonic map. □

Remark 3.10. If *rank* $f=$ *dim* M then $\mathcal{V}_f=\ker f=\{0\}$. Thus the condition for $\mathcal{V}_f$ to be minimal is not necessarily in this case.

Proposition 3.11. *Let $f:(M,g)\to(N,\varphi,\xi,\eta,h)$ be a φ-[PHWC] map between a Riemannian manifold and a contact metric manifold, for which $\mathcal{V}_f=\ker f$ is minimal, and $\xi_{f(x)}\notin\mathcal{V}_{f^*x}$, for any $x\in M$. If ψ is parallel then f is a harmonic map.*

Indeed the condition for the (2n+1)-dimensional manifold, N, to be a contact manifold is $\eta\wedge(d\eta)^n\neq0$, which, in particular gives $\theta_1\wedge d\theta_1\neq0$. Hence the result follows in the same way as in the previous Theorem.

Proposition 3.12. *The condition in the Theorem 3.9 and in Proposition 3.11 for ψ to be parallel can be substituted with the following two conditions, for M, endowed with the related framed ψ-structure, to be a (1,2)-D^M-symplectic like manifold, where D^M is the distribution on M orthogonal to span$\{\zeta_1, ..., \zeta_r\}$, and $d\theta_1(X, \zeta_1) = 0$, for any $X \in \Gamma(D^M)$.*

Proof. Since $d\theta_1(X, \zeta_1) = 0$, for any $X \in \Gamma(D^M)$, and by definition of θ_1, it follows that $d\theta_1(X, \zeta_1) = 0$, for any $X \in \chi(M)$, and more $\nabla^M_{\zeta_1}\zeta_1 = 0$. In the same way as in the proof of the Theorem 3.9 one obtains that $\sum_{k=1}^{r} \alpha(\zeta_k, \zeta_k) = 0$.

Since M is (1,2)-D^M-symplectic like it follows, from Lemma 1.3, that

$$\sum_{i=1}^{m}[(\nabla_{e_i}\psi)\psi e_i + (\nabla_{\psi e_i}\psi)e_i] = 0,$$

where $\{e_1, ..., e_m, \psi e_1, ..., \psi e_m\}$ is a local orthonormal φ-basis in $\Gamma(D^M)$.

From here one obtains

$$\sum_{i=1}^{m}[\alpha(e_i, e_i) + \alpha(\psi e_i, \psi e_i)] = 0,$$

just like in the proof of the Theorem 3.9.

It follows that $\tau(f) = 0$, and then f is a harmonic map. □

In the following we extend the results above in the case where the target manifold is a metric framed φ-manifold.

Consider the φ-[PHWC] map $f : (M, g) \to (N, \varphi, \xi_i, \eta_i, h)$ between a Riemannian manifold and a metric framed φ-manifold. Suppose that $\xi_{i_k f(x)} \notin \mathcal{V}_{f^*x}$, $k = 1, ..., s$, for any $x \in M$. We denote by $E_x = \mathcal{H}_{f^*x} \cap span\{\xi_{i_1 f(x)}, ..., \xi_{i_s f(x)}\}$, for any $x \in \chi(M)$. Then $(df \circ df^*)_x : E_x \to E_x$ is a self-adjoint endomorphism. So, there exists an orthonormal basis, $\{\xi'_{1f(x)}, ..., \xi'_{sf(x)}\}$, in E_x such that the matrix of $(df \circ df^*)_x$ is a diagonal one. That is

$$(df \circ df^*)_x \xi'_{kf(x)} = \Lambda_k(x)\xi'_{k(x)}, \quad k = 1, ..., s.$$

In the same way as for the case of almost contact metric manifolds we see that $\Lambda_k(x) = \lambda_k^2(x) \gneq 0$. It is easy to see that we have

$$\xi'_{kf(x)} = \sum_{j=1}^{s} c_{kj}(x)\xi_{jf(x)}, \quad k = 1, ..., s,$$

where $c_{kj} : M \to \mathbb{R}$ are functions on M. Define the 1-forms, η'_k, on N, by

$$\eta'_k = \sum_{j=1}^{s} c_{jk}\eta_k, \quad k = 1, ..., s.$$

One obtains

$$\eta'_k \circ \varphi = 0, \quad \eta'_k(\xi'_l) = \delta_{kl}, \quad k, l = 1, ..., s.$$

Set the matrix

$$C = \begin{pmatrix} c_{11} & c_{12} & ... & c_{1s} \\ c_{21} & c_{22} & ... & c_{2s} \\ ... & ... & ... & ... \\ c_{s1} & c_{s2} & ... & c_{ss} \end{pmatrix}.$$

Since $\{\xi_{1f(x)}, ..., \xi_{sf(x)}\}$,$\{\xi'_{1f(x)}, ..., \xi'_{sf(x)}\}$ are orthonormal basis it follows that C is an orthogonal matrix.

We can define the vector fields ζ_k on M by

$$\zeta_{kx} = \frac{1}{\lambda_k(x)} df^*_x \xi'_{f(x)}, \quad k = 1, ..., s, \quad x \in M. \tag{3.5}$$

Then the tensor field, ψ, of type (1,1), on M, can be taken as

$$\psi(X) = \begin{cases} df^* \circ \varphi \circ (df^*)^{-1}X & , X \in \mathcal{H}_f \\ 0 & , X \in \mathcal{V}_f, \end{cases} \tag{3.6}$$

where $\mathcal{H}_f = \cup_{x\in M}\mathcal{H}_{fx}$ and $\mathcal{V}_f = \cup_{x\in M}\mathcal{V}_{fx}$. Note that if $X \in \mathcal{H}_f \setminus span\{\zeta_1, ..., \zeta_s\}$ then $\psi X \in \mathcal{H}_f \setminus span\{\zeta_1, ..., \zeta_s\}$.

Next consider the 1-forms, θ_k, $k = 1, ..., s$, on M, by

$$\theta_{kx}(X_x) = \frac{1}{\lambda_k(x)} \eta'_{kf(x)}(df_x(X_x)), \tag{3.7}$$

for any $x \in M$, and $X \in \chi(M)$.

Finally let $\{\zeta_{s+1}, ..., \zeta_r\}$ be an orthonormal frame field on $\mathcal{V}_f$ and let $\{\theta_{s+1}, ..., \theta_r\}$ be the 1-forms defined by

$$\theta_k(X) = g(X, \zeta_k), \quad X \in \chi(M),$$

for any $k = s + 1, ..., r$.

Obviously one obtains

$$\psi^2 X = -X + \sum_{k=1}^{r} \theta_k(X)\zeta_k,\ \theta_k \circ \psi = 0,\ \theta_k(\zeta_l) = \delta_{kl},$$

for any $k, l = 1, ..., r$ and $X \in \chi(M)$. As in the case above, of metric contact manifolds, we have

Proposition 3.13. *$(M, \psi, \zeta_k, \theta_k, g)$ is a metric framed ψ-manifold.*

Proposition 3.14. *Let $f : (M, \psi, \zeta_k, \theta_k, g) \to (N, \varphi, \xi, \eta, h)$ be as above. Then f is a (ψ, φ)-holomorphic map.*

Theorem 3.15. *Let $f : (M, g) \to (N, \varphi, \xi_i, \eta_i, h)$, $i = 1, ..., n$, be a φ-[PHWC] map between a Riemannian manifold and a (1,2)-symplectic normal manifold, for which $\mathcal{V}_f = \ker f$ is minimal, and $\xi_{i_k f(x)} \notin \mathcal{V}_{f^*x}$, $k = 1, ..., s$, for any $x \in M$. If ψ is parallel and $\theta_k \wedge d\theta_k \neq 0$, $k = 1, ..., s$, then f is a harmonic map.*

Proof. Since N is a normal manifold it follows that $d\eta_i(X, \xi_j) = 0$, for any $i, j = 1, ..., n$. After a straightforward computation, taking account of the fact that the matrix C, defined above, is orthogonal, one obtains that $d\eta'_k(X, \xi'_k) = 0$, for any $k = 1, ..., s$. From here the proof follows just like the proof of the Theorem 3.9. □

Remark 3.16. As for the Theorem 3.9 the condition for $\mathcal{V}_f$ to be minimal is not necessarily if $rank\ f = dim\ M$.

Remark 3.17. The conditions $\theta_k \wedge d\theta_k \neq 0$, $k = 1, ..., s$, are equivalent with $(\eta_k \circ df) \wedge (d\eta_k \circ (df, df)) \neq 0$.

Finally, we can impose, also in this case, a weaker condition instead of that for ψ to be parallel

Proposition 3.18. *The condition in the Theorem 3.16 for ψ to be parallel can be substituted with the following two conditions, for M, endowed with the related framed ψ-structure, to be a (1,2)-D^M-symplectic like manifold, where D^M is the distribution on M orthogonal to $span\{\zeta_1, ..., \zeta_r\}$, and $d\theta_k(X, \zeta_k) = 0$, $k = 1, ..., s$, for any $X \in \Gamma(D^M)$.*

4. φ-pseudo harmonic morphisms.

Definition 4.1. By a local map $\pi : N \to P$ we shall mean a map defined on an open subset of N.

A particular case of a result in [4] is

Proposition 4.1. *Let* $\pi : (N, \varphi, \xi_i, \eta_i, h) \to (P, \varphi', \xi'_j, \eta'_j, k)$, $i = 1, ..., n$, $j = 1, ..., p$, *be a* (φ, φ')*-holomorphic map between two (1,2)-symplectic manifolds. Then* π *is a* D^N*-pluriharmonic map, where* D^N *is the distribution on* N *orthogonal to span*$\{\xi_i\}_{i=1}^n$.

One obtains

Proposition 4.2. *Let* $f : (M, g) \to (N, \varphi, \xi, \eta, h)$ *be a* φ*-pseudo harmonic morphism between a Riemannian manifold and a (1,2)-symplectic manifold, such that* $\xi \notin \mathcal{V}_{f*}$, *and let* $\pi : (N, \varphi, \xi, \eta, h) \to (P, \varphi', \xi', \eta', k)$ *be a local harmonic map between (1,2)-symplectic manifolds, which is* (φ, φ')*-holomorphic. Then* $\pi \circ f$ *is a local* φ'*-pseudo harmonic morphism on* M.

Proof. Suppose that f is a φ-pseudo harmonic morphism. From Proposition 3.5 we have that $\pi \circ f$ is a φ'-[PHWC] map.

Consider on M the related (to f) framed ψ-structure obtained in the previous paragraph. From the composition law

$$\tau(\pi \circ f) = d\pi(\tau(f)) + trace \nabla d\pi(df, df),$$

since f is harmonic and (ψ, φ)-holomorphic, one obtains that

$$\begin{aligned}\tau(\pi \circ f) &= trace \nabla d\pi(df, df) = \\ &= \sum_{i=1}^{m} [\alpha_\pi(df(e_i), df(e_i) + \alpha_\pi(\varphi df(e_i), \varphi df(e_i)] + \lambda^2 \alpha_\pi(\xi, \xi),\end{aligned}$$

where the function λ is the same function used in the previous paragraph, and $\{e_1, ..., e_m, \psi e_1, ..., \psi e_m\}$ is an orthonormal basis in $\Gamma(D^M) = (\ker \psi)^\perp$. Since π is (φ, φ')-holomorphic then it is D^N-pluriharmonic. Thus $\tau(\pi \circ f) = \lambda^2 \alpha_\pi(\xi, \xi)$ which vanishes because π is harmonic. Hence $\pi \circ f$ is a harmonic map and then a φ'-pseudo harmonic morphism on M. □

In [6] it is proved that any (φ, φ')-holomorphic map, $\pi : (N, \varphi, \xi, \eta, h) \to (P, \varphi', \xi', \eta', k)$, between cosymplectic manifolds is harmonic, and in [8] it is proved the same result in the case of contact metric manifolds. Then we have

Proposition 4.3. *Let $f : (M, g) \to (N, \varphi, \xi, \eta, h)$ be a φ-pseudo harmonic morphism between a Riemannian manifold and a cosymplectic (contact metric) manifold, such that $\xi \notin \mathcal{V}_{f*}$, and let $\pi : (N, \varphi, \xi, \eta, h) \to (P, \varphi', \xi', \eta', k)$ be a local (φ, φ')-holomorphic map between cosymplectic (contact metric) manifolds. Then $\pi \circ f$ is a local φ'-pseudo harmonic morphism on M.*

Proposition 4.4. *Let $f : (M, g) \to (N, \varphi, \xi, \eta, h)$ be a φ-pseudo harmonic morphism between a Riemannian manifold and an almost contact metric manifold, such that $\xi \notin \mathcal{V}_{f*}$, and let $\pi : (N, \varphi, \xi, \eta, h) \to (P, \varphi', \xi', \eta', k)$ be a local φ-pluriharmonic map between two almost contact metric manifolds. Then $\pi \circ f$ is a local harmonic map.*

Indeed, since a φ-pluriharmonic map is harmonic and D-pluriharmonic the proof is similar with that for Proposition 4.2.

Just like in the previous section we give some generalizations for this results in the case of metric framed φ-manifolds.

In the same way as in Proposition 4.2, using the induced framed ψ-structure on M, obtained in Proposition 3.14, we can prove

Proposition 4.5. *Let $f : (M, g) \to (N, \varphi, \xi_i, \eta_i, h)$, $i = 1, ..., n$, be a φ-pseudo harmonic morphism between a Riemannian manifold and a (1,2)-symplectic manifold, such that $\xi_i \notin \mathcal{V}_{f*}$, for any $i = 1, ..., n$, and let $\pi : (N, \varphi, \xi_i, \eta_i, h) \to (P, \varphi', \xi'_j, \eta'_j, k)$, $j = 1, ..., p$, be a local harmonic map between (1,2)-symplectic manifolds, which is (φ, φ')-holomorphic. Then $\pi \circ f$ is a local φ'-pseudo harmonic morphism on M.*

In [5] it is prove that any (φ, φ')-holomorphic map between two $\mathcal{C}$-manifolds is a harmonic map. Hence

Proposition 4.6. *Let $f : (M, g) \to (N, \varphi, \xi_i, \eta_i, h)$, $i = 1, ..., n$, be a φ-pseudo harmonic morphism between a Riemannian manifold and a $\mathcal{C}$-manifold, such that $\xi_i \notin \mathcal{V}_{f*}$, for any $i = 1, ..., n$, and let $\pi : (N, \varphi, \xi_i, \eta_i, h) \to (P, \varphi', \xi'_j, \eta'_j, k)$, $j = 1, ..., p$, be a local (φ, φ')-holomorphic map between two $\mathcal{C}$-manifolds. Then $\pi \circ f$ is a local φ'-pseudo harmonic morphism on M.*

Finally, we have

Proposition 4.7. *Let $f : (M, g) \to (N, \varphi, \xi_i, \eta_i, h)$, $i = 1, ..., n$, be a φ-pseudo harmonic morphism between a Riemannian manifold and a metric framed φ-manifold, such that $\xi_{i_k} \notin \mathcal{V}_{f*}$, for some $k = 1, ..., r$, $r \leqslant n$, and* *let* $\pi : (N, \varphi, \xi_i, \eta_i, h) \to (P, \varphi', \xi'_j, \eta'_j, k)$ *be a local* φ*-pluriharmonic map between two metric framed* $\varphi(\varphi')$*-manifolds. Then* $\pi \circ f$ *is a local harmonic map.*

5. A related framed ψ-structure on the tangent bundle of an almost contact metric manifold. Consider a differentiable manifold, M, of dimension m, and let $\pi : TM \to M$ be its tangent bundle. Then TM can be organized as a $2m$-dimensional manifold as follows. A local coordinate neighborhood $(U; x^i), i = 1, ..., m$, in M induces a local coordinate neighborhood $(\pi^{-1}(U); x^i, y^j), i, j = 1, ..., m$, on TM, where we denote $x^i \circ \pi$ by x^i and y^j are the coordinates of the vectors on $\pi^{-1}(U)$ in natural basis $\{\frac{\partial}{\partial x^i}\}_{i=1}^{m}$, (see [14]).

If ω is a differentiable 1-form on M then it can be regarded as a function on TM denoted by $\iota\omega$.

If f is a function on M, we define the vertical lift f^V of f by $f^V = f \circ \pi$, and the complete lift f^C of f by $f^C = \iota(df)$. We have $f^C = y^i \frac{\partial f}{\partial x^i} = y^i \partial_i f = \partial f$ with respect to the induced coordinates in TM, where ∂_i denote $\frac{\partial}{\partial x^i}$ and ∂ denote $y^i \partial_i$. The vertical lift $(df)^V$ of the 1-form df is defined by $(df)^V = d(f)^V$. For two function f and g on M we have $(gdf)^V = g^V (df)^V = g^V (df^V)$. The vector field $C = y^i \frac{\partial}{\partial x^i}$ is the Liouville vector field.

Let $X = X^i \frac{\partial}{\partial x^i}$ be a vector field on M. Its vertical lift X^V its defined by $X^V(\iota\omega) = (\omega(X))^V$, ω being an arbitrary 1-form on M. The complete lift X^C of X is defined by $X^C f^C = (Xf)^C$, f being an arbitrary function M. One obtains with respect to the induced coordinates in TM

$$X^V = X^i \frac{\partial}{\partial y^i}, \qquad X^C = X^i \frac{\partial}{\partial x^i} + \partial X^i \frac{\partial}{\partial y^i}.$$

Let $\eta = \eta_i dx^i$ be a differentiable 1-form on M. We define the vertical lift η^V of η by $\eta^V = (\eta_i)^V (dx_i)^V$, and the complete lift η^C of η by $\eta^C(X^C) = (\omega(X))^C$, X being an arbitrary vector field on M. Then, we have with respect to the induced coordinates in TM

$$\eta^V = \eta_i dx^i, \quad \eta^C = \partial\eta_i dx^i + \eta_i dy^i.$$

Now, we assume that there is given an affine connection, ∇, on M. Define the horizontal lift of a function, f, in M to TM, by $f^H = f^C - \gamma(\nabla f)$. From definition of f^C it follows that $f^H = 0$. Let $X \in \chi(M)$ be a vector field on M. Define the horizontal lift, X^H, of X by $X^H = X^C - \gamma(\nabla X)$. If $X = X^i\frac{\partial}{\partial x^i}$, in local coordinates, and ∇ has the components Γ^h_{ji}, then

$$X^H = X^h\frac{\partial}{\partial x^h} - \Gamma^h_i X^i\frac{\partial}{\partial y^h},$$

with respect to the induced coordinates in TM, where $\Gamma^h_i = y^j\Gamma^h_{ji}$.

Let ω be a 1-form on M. Define the horizontal lift, ω^H, of ω, by $\omega^H = \omega^C - \gamma(\nabla\omega)$. If ω has the local components ω_i, then

$$\omega^H = \Gamma^h_i\omega_h dx^i + \omega_i dy^i,$$

with respect to the induced coordinates in TM.

The horizontal lift of a tensor field on M can be defined, using the conditions

$$(P+Q)^H = P^H + Q^H, \quad (P\otimes Q)^H = P^H\otimes Q^V + P^V\otimes Q^H,$$

where P, Q are tensor fields on M.

Let $(M, g, \varphi, \xi, \eta)$ be a (2m+1)-dimensional almost contact metric manifold and let TM be its tangent bundle. Define on TM the Riemannian metric G, introduced in [11], given, by

$$G(X^H, Y^H) = G(X^V, Y^V) = g(X, Y), \quad G(X^H, Y^V) = 0,$$

for any $X, Y \in \chi(M)$.

Let $\pi : (TM, G) \to (M, g, \varphi, \xi, \eta)$ be the projection map. If we denote by $d\pi$ the tangent map and by $d\pi^*$ its adjoint map then it is easy to obtain that $d\pi^*(X) = X_i\frac{\partial}{\partial x^i}$, for any vector field on M, $X = X_i\frac{\partial}{\partial x^i}$. It follows that $d\pi\circ d\pi^*(X) = X$, for any $X \in \chi(M)$, that is π is a horizontally weakly conformal map and then a φ-[PHWC] map. Obviously the horizontal distribution, $\mathcal{H}_\pi$, and the vertical distribution, $\mathcal{V}_\pi$, are orthogonal with respect the metric G. Since $\mathcal{H}_{\pi^*} = TM\setminus\{0\}$ and $d\pi : \mathcal{H}_\pi \to \mathcal{H}_{\pi^*}$ is one-to-one and linear it follows that $dim\mathcal{H}_\pi = dim\mathcal{V}_\pi = 2m+1$. Then, one obtains easily, that $\{d\pi^*(e_1), ..., d\pi^*(e_m), d\pi^*(\varphi e_1), ..., d\pi^*(\varphi e_m), d\pi^*(\xi)\}$ and $\{e_1^V, ..., e_m^V, (\varphi e_1)^V, ..., (\varphi e_m)^V, \xi^V\}$ are local orthogonal basis in $\mathcal{H}_\pi$ and in $\mathcal{V}_\pi$, respectively, where $\{e_1, ..., e_m, \varphi e_1, ..., \varphi e_m, \xi\}$ is a local φ-basis in M.

Suppose that M is a (1,2)-symplectic manifold, $d\eta(X, \xi) = 0$, for any $X \in \chi(M)$, and $\eta_x([\xi, \frac{\partial}{\partial x_i}]_x) = 0$, for any $i = 1, ..., 2m+1$ and $x \in M$, where $(U; x^i), i = 1, ..., 2m+1$, is a local coordinate neighborhood. Then, after a straightforward computation, in local coordinates, we have that $\mathcal{V}_\pi$ is minimal.

Next, we can consider a related framed ψ-structure on TM in the same way as in the general case, in the third paragraph.

If we denote by ω and Ω the fundamental 2-forms on M and TM, respectively, one obtains that $\Omega(d\pi^*(X), d\pi^*(Y)) = \omega(X, Y)$, for any $X, Y \in \chi(M)$. Thus, we have $d\Omega(d\pi^*(X), \psi(d\pi^*(X)), d\pi^*(Z)) = 0$, for any $X, Z \in \chi(M)$, since M is (1,2)-symplectic. It follows, from Lemma 1.3, that

$$G((\nabla^G_{d\pi^*(e_i)}\psi)\psi(d\pi^*(e_i)) - (\nabla^G_{d\pi^*(\varphi e_i)}\psi)d\pi^*(e_i), d\pi^*(Z)) = 0,$$

for any $Z \in \chi(M)$, where ∇^G is the Levi-Civita connections on (TM, G). Hence

$$d\pi((\nabla^G_{d\pi^*(e_i)}\psi)\psi(d\pi^*(e_i)) - (\nabla^G_{d\pi^*(\varphi e_i)}\psi)d\pi^*(e_i)) = 0,$$

and, from this,

$$d\pi(\nabla^G_{d\pi^*(e_i)}d\pi^*(e_i) + \nabla^G_{d\pi^*(\varphi e_i)}d\pi^*(\varphi e_i)) = d\pi(\psi[\psi d\pi^*(e_i), d\pi^*(e_i)]),$$

since π is a (ψ, φ)-holomorphic map. But M is a (1,2)-symplectic manifold and then, by Lemma 1.3, one obtains that

$$\nabla_{e_i}e_i + \nabla_{\varphi e_i}\varphi e_i = \varphi[\varphi e_i, e_i].$$

Next, it is easy to see that $d\pi(\nabla^G_{d\pi^*(\xi)}d\pi^*(\xi)) = 0$.

From all this one obtains $\tau(\pi) = 0$. Hence π is harmonic. We can state the following

Proposition 5.1. *For M and TM as above, the projection π is a harmonic morphism.*

Remark 5.2. We can obtain a manifold M with the necessarily properties as follows, (see [2]). Let (M', J, h) be an almost Kähler manifold with local coordinates $\{x^i\}_{i=1}^{2n}$ and let t the coordinate on $\mathbb{R}$. On $M = M' \times \mathbb{R}$ set $\eta = dt$ and $\xi = \frac{\partial}{\partial t}$. Define φ by $\varphi\xi = 0$, $\varphi X = JX$, for X orthogonal to ξ. Finally, set the Riemannian metric g on M, $g = h_{ij}dx^i dx^j + dtdt$, where h_{ij} are the components of the Riemannian metric h. It is easy to see that $(M, \varphi, \xi, \eta, g)$ is an almost contact metric manifold with all the properties we need.

Acknowledgments. The author wants to thank to professor Vasile Oproiu for his many valuable advices.

REFERENCES

1. Baird, P.; Wood, J.C. – *Harmonic Morphisms Between Riemannian Manifolds*, Clarendon Press Oxford, London Mathematical Society Monographs, New Series, Volume 29(2003), pp. xi, 520.
2. Blair, D.E. – *Riemannian Geometry of Contact and Symplectic Manifolds*, Birkhäuser Boston, Progress in Mathematics, Volume 203(2002), pp. x, 260.
3. Eells, J.; Lemaire, L. – *Selected topics in harmonic maps*, Am. Math. Soc, Conf. Board Math. Sci. 50 (1983), pp. 83.
4. Erdem, S. – *Harmonicity of maps between (indefinite) metric f-manifolds and φ-pseudo harmonic morphisms*, preprint, 1999, pp. 25.
5. Fetcu, D. – *Harmonic and φ-pluriharmonic maps between $\mathcal{C}$-manifolds*, to appear in Italian Journal of Pure and Applied Mathematics.
6. Gherghe, C.; Boeckx, E. – *Harmonic maps and cosymplectic manifolds*, to appear in Journal of the Australian Mathemathical Society.
7. Goldberg, S.I.; Yano, K. – *On normal globally framed $f-$manifold*, Tôhoku Math.Journ. 22(1970), pp. 362-370.
8. Ianus, S.; Pastore, A.M. – *Harmonic maps on contact metric manifold*, Ann. Math. Blaise Pascal, Vol2, No2(1995), pp. 43-53.
9. Loubeau, E. – *Pseudo harmonic morphisms*, Internat. J. Math., 8(1997), pp. 943-957.
10. Loubeau, E. – *Pluriharmonic morphisms*, Math. Scand., 84(1999), pp. 165-178.
11. Oniciuc, C. – *Tangency and Harmonicity Properties*(in Roumanian), Geometry Balkan Press, E-version, (2003), 87pp.
12. Oproiu, V. – *Harmonic maps between tangent bundles*, Rend. Sem. Mat., Univers. Politecn. Torino, Vol47, 1(1989), pp. 47-55.
13. Urakawa, H. – *Calculus of Variation and Harmonic Maps*, American Mathematical Society, Providence, Rhode Island, Translations of Mathematical Monographs, 1993, pp viii, 251.
14. Yano, K.; Ishihara, S. – *Tangent and Cotangent Bundles*, M.Dekker, New York, Pure and Applied Mathematics (1973), pp iv, 421.
15. Xin, Y.L. – *Geometry of Harmonic Maps*, Birkhäuser Boston, Progress in Nonlinear Differential Equations and Their Aplications, 1996, pp x, 241.

Received: 15.IV.2004

Department of Mathematics,
Technical University, Iaşi,
ROMÂNIA,
dfetcu@math.tuiasi.ro