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REGULARITY AND O-CONTINUITY FOR MULTISUBMEASURES

BY

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Abstract. In the present paper we study the relationship between regularity and o-continuity for multisubmeasures. Namely, the main result states that a multisubmeasure is R'-regular on the Baire δ-ring (respectively, σ-ring) $\mathcal{B}_0$(respectively, $\mathcal{B}'_0$) generated by the compact sets which are G_δ (that is, countable intersection of open sets) of a Hausdorff locally compact space, if and only if it is o-continuous on $\mathcal{B}_0$(respectively, $\mathcal{B}'_0$).

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1. Basic notions, terminology and notations. Let T be an abstract set, $\mathcal{C}$ a ring of subsets of T, X a real normed space, $\mathcal{P}_0(X)$ the family of all nonempty subsets of X, $\mathcal{P}_f(X)$ the family of closed, nonvoid sets of X, h the Hausdorff pseudometric on $\mathcal{P}_f(X)$. We define $|M| = h(M, \{0\})$, for every $M \in \mathcal{P}_f(X)$, where 0 is the origin of X. If, in addition, X is complete, then so is $\mathcal{P}_f(X)$. It is known that $h(M,N) = \max\{e(M,N\}, e(N,M)\}$, for every $M,N \subset X$, where $e(M,N) = \sup\{d(x,N); x \in M\}$, d being the distance from x to N with respect to the metric induced by the norm of X.

On $\mathcal{P}_0(X)$ we introduce the Minkowski addition, denoted by the symbol $\overset{\bullet}{+}$, defined by:

$$M \overset{\bullet}{+} N = \overline{M + N}, \tag{1}$$

for every $M, N \in \mathcal{P}_0(X)$.

We recall the following classical notions:

I) A set function $m : \mathcal{C} \longrightarrow \overline{R}_+$ is said to be a **submeasure** (respectively, a **strictly submeasure**) in Drewnowski's sense [6], if $m(\emptyset) = 0$, $m(A \cup B) \leq m(A)+m(B)$, for every $A, B \in \mathcal{C}$, $A \cap B = \emptyset$ (respectively, $m(A \cup B) \lneqq m(A)+m(B)$, for every $A, B \in \mathcal{C}$, $A \cap B = \emptyset$, $A, B \neq \emptyset$), and $m(A) \leq m(B)$, for every $A, B \in \mathcal{C}$, with $A \subseteq B$.

In the sequel let μ: $\mathcal{C} \rightarrow \mathcal{P}_f(X)$ be a multivalued set function.

II) μ is said to be **(finite) additive** if $\mu(\emptyset) = \{0\}$ and $\mu(A \cup B) = \mu(A) \overset{\bullet}{+} \mu(B)$, for every $A, B \in \mathcal{C}$, $A \cap B = \emptyset$.

III) μ : $\mathbf{C} \longrightarrow \mathcal{P}_f(X)$ is said to be **order-continuous** (briefly, **o-continuous**) with respect to h, if $\lim\limits_{n\to\infty} \mu(A_n)| = 0$, for every sequence $(A_n)_{n\in N^*} \subset \mathcal{C}$ such that $A_n \searrow \emptyset$ (that is, $A_n \supset A_{n+1}$, for every $n \in N^*$ and $\bigcap\limits_{n=1}^{\infty} A_n = \emptyset$).

IV) $\mu : \mathcal{C} \longrightarrow \mathcal{P}_f(X)$ is said to be **exhaustive** with respect to h, if $\lim\limits_{n\to\infty} |\mu(A_n)| = 0$, for every disjoint sequence $(A_n)_{n\in N^*} \subset \mathcal{C}$.

V) $\mu : \mathcal{C} \longrightarrow \mathcal{P}_f(X)$ is said to be **semiexhaustive** with respect to h, if $\lim\limits_{n\to\infty} |\ \mu(A_n)| = 0$, for every disjoint sequence $(A_n)_{n\in N^*} \subset \mathcal{C}$, with $\bigcup\limits_{n=1}^{\infty} A_n \in \mathcal{C}$.

VI) $\mu : \mathcal{C} \longrightarrow \mathcal{P}_f(X)$ is said to be **increasing convergent** if $\lim h(\mu(A_n), \mu(A)) = 0$, for every increasing sequence $(A_n)_n \subset \mathcal{C}$ such that $A_n \nearrow A = \bigcup\limits_{n=1}^{\infty} A_n \in \mathcal{C}$.

VII) $\mu : \mathcal{C} \longrightarrow \mathcal{P}_f(X)$ is said to be h-σ-**subadditive** if $|\mu(A)| \leq \sum\limits_{n=1}^{\infty} |\mu(A_n)|$, for every sequence $(A_n)_n \subset \mathcal{C}$ such that $A_n \cap A_m = \emptyset$, $m \neq n$ and $A = \bigcup\limits_{n=1}^{\infty} A_n \in \mathcal{C}$.

VIII) If $\mu : \mathcal{C} \longrightarrow \mathcal{P}_f(X)$, we call the **semivariation** of μ, the set function $\hat{\mu}$ defined by:

$$\hat{\mu}(A) = \sup\{|\mu(B)|;\ B \subset A,\ B \in \mathcal{C}\},\ \text{for every } A \subset T. \tag{2}$$

In the sequel we shall use the special type of multivalued set function, which we have introduced in [8] and was called there a multisubmeasure.

Definition 1.1. I) A multivalued set function μ from $\mathcal{C}$ to $\mathcal{P}_f(X)$ is said to be a **multisubmeasure** if:

$$\begin{cases} a)\ \mu(\emptyset) = \{0\}, \\ b)\ \mu(A \cup B) \subseteq \mu(A) \overset{\bullet}{+} \mu(B), \text{for every } A, B \in \mathcal{C}, \text{ with } A \cap B = \emptyset, \\ c)\ \mu(A) \subseteq \mu(B), \text{for every } A, B \in \mathcal{C}, \text{ with } A \subseteq B. \end{cases}$$

II) A multivalued set function μ from $\mathcal{C}$ to $\mathcal{P}_f(X)$ is said to be a **strictly multisubmeasure** if μ satisfies the conditions $a)$ and $c)$ from definition 1.1.I) and instead of $b)$ we have $b')$ $\mu(A \cup B) \subsetneq \mu(A) \overset{\bullet}{+} \mu(B)$, for every $A, B \in \mathcal{C}$, $A \cap B = \emptyset$, $A, B \neq \emptyset$.

Remark 1.2. The condition $\mu(A \cup B) \subseteq \mu(A) \overset{\bullet}{+} \mu(B)$, for every $A, B \in \mathcal{C}$ with $A \cap B = \emptyset$ is equivalent to the condition $\mu(A \cup B) \subseteq \mu(A) \overset{\bullet}{+} \mu(B)$, for every $A, B \in \mathcal{C}$.

We have presented in [8] the relationships among o-continuity, exhaustivity, h-σ-subadditivity and increasing convergence: o-continuity is equivalent to h-σ-subadditivity, it implies increasing convergence and if, supplementary, $\mathcal{C}$ is a σ-ring, it implies exhaustivity. Also, if $\mathcal{C}$ is simply a ring, then a multisubmeasure may be o-continuous and not exhaustive, and conversely, it can be exhaustive and not o-continuous.

In the following we recall different definitions for the regularity of multisubmeasures, which have been introduced in [8]. The same as there, in this section we shall suppose, additionally, that T is a Hausdorff locally compact space. Let $\mathcal{B}_0$ (respectively, $\mathcal{B}'_0$) be the Baire δ-ring (respectively, σ-ring) generated by the compact sets which are G_δ and $\mathcal{B}$ (respectively, $\mathcal{B}'$) the borelian δ-ring (respectively, σ-ring) generated by the compact sets of T.

Let $\mathcal{C}$ be a ring of subsets of T.

Definition 1.3. I) A set $A \in \mathcal{C}$ is said to be R **- regular** with respect to the multisubmeasure μ if for every $\varepsilon >0$, there exist a compact set $K \subset A$, $K \in \mathcal{C}$ and an open set $D \supset A$, $D \in \mathcal{C}$ such that $h(\mu(A), \mu(B)) < \varepsilon$, for every $B \in \mathcal{C}$, $K \subset B \subset D$.

II) A set $A \in \mathcal{C}$ is said to be R_l-**regular** with respect to the multisubmeasure μ if for every $\varepsilon >0$, there exists a compact set $K \subset A, K \in \mathcal{C}$ such that $h(\mu(A), \mu(B)) < \varepsilon$, for every $B \in \mathcal{C}, K \subset B \subset A$.

III) A set $A \in \mathcal{C}$ is said to be R_r- **regular** with respect to the multisubmeasure μ if for every $\varepsilon > 0$, there exists an open set $D \supset A$, $D \in \mathcal{C}$ such that $h(\mu(A), \mu(B)) < \varepsilon$, for every $B \in \mathcal{C}, A \subset B \subset D$.

Definition 1.4. A multisubmeasure $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is said to be R **- regular** (R_l-**regular,** R_r **- regular)** if every $A \in \mathcal{C}$ is a R-regular (R_l - regular, R_r - regular, respectively) set with respect to μ.

It is easy to see that every compact set $K \in \mathcal{C}$ is R_l - regular (with respect to μ) and every open set $D \in \mathcal{C}$ is R_r - regular (with respect to μ).

Definition 1.5. I) A set $A \in \mathcal{C}$ is said to be R' **- regular** with respect to the multisubmeasure μ if for every ε >0, there are a compact set $K \subset A$, $K \in \mathcal{C}$ and an open set $D \supset A, D \in \mathcal{C}$ such that $|\mu(B)| < \varepsilon$, for every $B \in \mathcal{C}$, $B \subset D \backslash K$.

II) A set $A \in \mathcal{C}$ is said to be R'_l **- regular** with respect to the multisubmeasure μ if for every ε >0, there exists a compact set $K \subset A, K \in \mathcal{C}$ such that $|\mu(B)| < \varepsilon$, for every $B \in \mathcal{C}$, $B \subset A \backslash K$.

III) A set $A \in \mathcal{C}$ is said to be R'_r **- regular** with respect to the multisubmeasure μ if for every $\varepsilon > 0$, there exists an open set $D \supset A, D \in \mathcal{C}$ such that $|\mu(B)| < \varepsilon$, for every $B \in \mathcal{C}$, $B \subset D \backslash A$.

It is easy to see that every compact set $K \in \mathcal{C}$ is R'_l - regular and every open set $D \in \mathcal{C}$ is R'_r - regular.

Definition 1.6. A multisubmeasure $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is said to be R' **- regular** (R'_l **- regular,** R'_r **- regular)** if every $A \in \mathcal{C}$ is a R'- regular (R'_l - regular, R'_r - regular, respectively) set.

We have proved in [8] that if $\mathcal{C}$ is the δ-ring generated by the compact sets which are G_δ, or simply by the compact sets, then a multisubmeasure $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is R'-regular if and only if it is R'_l - regular (or R'_r - regular).

2. Regularity and o-continuity. It is natural to ask ourselves if there exists any relationship between a certain type of regularity and o-continuity, without any special assumption on the definition ring or space. We show that any R'_l-regular multisubmeasure on a ring $\mathcal{C}$ is o-continuous.

On the other hand, in the following example we present an o-continuous multisubmeasure which is not regular (in a certain sense).

Example 2.1. Let the ring $\mathcal{C} = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ in $T = R$, $\nu : \mathcal{C} \longrightarrow R_+$, defined by:

$$\nu(A) = \begin{cases} 0, A = \emptyset, \\ 1, A = \{1\} \text{ or } A = \{2\}, \\ \frac{3}{2}, A = \{1,2\}, \end{cases}$$

for every $A \in \mathcal{C}$ and $\mu : \mathcal{C} \longrightarrow \mathcal{P}_f([0,\infty))$, defined by:

$\mu(A) = [0, \nu(A)]$, for every $A \in \mathcal{C}$.

It easily gets (see [8]) that μ is a strictly multisubmeasure which is o-continuous and exhaustive. On the other hand, μ is R'_l and R_l - regular and it is not R'_r or R_r - regular. Consequently, it is not R'- regular.

In the sequel, let T be a Hausdorff locally compact space and $\mathcal{C}$ a ring of subsets of T.

It is well-known from [3] that any $[0,\infty]$-valued Baire measure is regular. Using this fact, Dinculeanu and Kluvanek generalized this result, proving in [4] that a Baire measure taking values in a locally convex Hausdorff topological vector space is also regular. Lately, Dinculeanu and Lewis gave in [5] a more direct proof of this result, which also holds for the non-negative and, respectively, the vector case. In turn, Sundaresan and Day (in [14]) and Khurana (in [9]) (using a different method) gave a generalization to a Hausdorff topological group valued Baire measure. Particularly, if the group is a locally convex Hausdorff topological vector space, their result improves the theorems of Dinculeanu- Kluvanek-Lewis. A natural extension to the uniform semigroup case was obtained by Morales in [11].

In [1], Belley and Morales introduced an abstract type of regularity for topological group-valued measures. They proved that in this context, a regular finite additive set function is a measure, that is, regularity implies σ-additivity. This result was generalized in [10] by Maria Gonzalez for measures taking values in a uniform semigroup.

Taking as starting point the works of Belley and Morales [1] and Maria Gonzalez [10], Precupanu generalized in [13] their result for the case of multimeasures, proving that, with some assumptions on the definition set, a regular multimeasure is σ-additive. She also proved that a σ-additive and locally exhaustive multimeasure defined on $\mathcal{B}_0$ is regular.

Following this direction we turn now the attention to multisubmeasures, in order to see if the above mentioned results have here a correspondent. It is known, for instance from [11] and [14] that:

1) If G is a Hausdorff topological group, $\mathcal{C}$ is a ring of subsets of an abstract set and $\mu : \mathcal{C} \longrightarrow G$ is a set function, then:

a) If μ is σ-additive, then μ is exhaustive.

b) If μ is finite additive, then μ is σ-additive if and only if it is o-continuous.

2) If S is a uniform semigroup, $\mathcal{C}$ a ring of subsets of an abstract set and $\mu : \mathcal{C} \longrightarrow S$ is a set function, then:

a) If μ is finite additive and o-continuous, then it is σ-additive.

b) If μ is σ-additive and semiexhaustive, then it is o-continuous.

That is why, in the following we shall consider o-continuity, proving that it is a useful tool instead of σ-additivity for multisubmeasures. We obtain as a consequence of our main result that a multisubmeasure is R'- regular on $\mathcal{B}_0$ (respectively, $\mathcal{B}'_0$) if and only if it is o-continuous on $\mathcal{B}_0$ (respectively, $\mathcal{B}'_0$).

Lemma 2.2. *Let $\mathcal{C}$ be a $\delta-$ ring. If $\mu : \mathcal{C} \longrightarrow \mathcal{P}_f(X)$ is an o - continuous multisubmeasure and if $(A_n)_{n\in N} \subset \mathcal{C}$, $A_n \searrow \emptyset$, then:*

$$\text{(3)} \qquad \begin{array}{c} \textit{for every } \varepsilon > 0, \textit{ there exists } n_0 \in N \\ \textit{such that } |\mu(B)| < \varepsilon, \textit{ for every } B \in \mathcal{C}, B \subset A_{n_0}. \end{array}$$

Proof. Since μ is o-continuous and $(A_n)_{n\in N} \subset \mathcal{C}$, $A_n \searrow \emptyset$, then for every $\varepsilon > 0$, there exists $n_0 \in N$ such that $|\mu(A_n)| < \varepsilon$, for every $n \geq n_0$. Particularly,

$$|\mu(A_{n_0})| < \varepsilon. \tag{4}$$

Let $B \in \mathcal{C}$, $B \subset A_{n_0}$. We have then:

$$|\mu(B)| \leq |\mu(A_{n_0})|. \tag{5}$$

From (4) and (5) we draw the conclusion. □

Using an idea of SUNDARESAN and DAY in [14], we prove:

Theorem 2.3. *If $\mathcal{C}$ is the $\delta-$ring generated by a class $\mathcal{D}$ of subsets of T and if $\mu : \mathcal{C} \longrightarrow \mathcal{P}_f(X)$ is an o-continuous multisubmeasure which is R'- regular on $\mathcal{D}$, then μ is R'- regular on $\mathcal{C}$.*

Proof. Let $\mathcal{A} = \{A \in \mathcal{C}\ ;\ A$ is R'- regular with respect to $\mu\}$. We show that $\mathcal{A} = \mathcal{C}$. First, let us prove that $\mathcal{A}$ is a ring. Indeed, if $A_1, A_2 \in \mathcal{A}$, then for every $\varepsilon > 0$, there exist the compact sets $K_1,\ K_2 \in \mathcal{C}$ and the open sets $D_1,\ D_2 \in \mathcal{C}$, $K_1 \subset A_1 \subset D_1,\ K_2 \subset A_2 \subset D_2$ such that $|\mu(B)| < \frac{\varepsilon}{2}$ for every $B \in \mathcal{C},\ B \subset D_1 \backslash K_1$ or $B \subset D_2 \backslash K_2$, respectively.

Let $B \in \mathcal{C}$,

$$B \subset (D_1 \cup D_2) \backslash (K_1 \cup K_2). \tag{6}$$

It is evident that $D_1 \cup D_2$ is an open set, $K_1 \cup K_2$ is a compact set and $K_1 \cup K_2 \subset A_1 \cup A_2 \subset D_1 \cup D_2$. From (6) we get that there exist $B_1, B_2 \in \mathcal{C}$, $B_1 \subset D_1 \backslash K_1,\ B_2 \subset D_2 \backslash K_2,\ B = B_1 \cup B_2,\ B_1 \cap B_2 = \emptyset$. Hence, $|\mu(B_1)| < \frac{\varepsilon}{2}$ and $|\mu(B_2)| < \frac{\varepsilon}{2}$. We obtain that $|\mu(B)| \leq |\mu(B_1)| + |\mu(B_2)| < \varepsilon$, so $A_1 \cup A_2 \in \mathcal{A}$.

Now, let $B \in \mathcal{C}$,

$$B \subset (D_1 \backslash K_2) \backslash (K_1 \backslash D_2) \subset (D_1 \backslash K_1) \cup (D_2 \backslash K_2). \tag{7}$$

Clearly, $D_1 \backslash K_2,\ K_1 \backslash D_2 \in \mathcal{C},\ D_1 \backslash K_2$ is an open set, $K_1 \backslash D_2$ is a compact set, and $K_1 \backslash D_2 \subset A_1 \backslash A_2 \subset D_1 \backslash K_2$. From (7) we get that there exist $B_1, B_2 \in \mathcal{C},\ B_1 \subset D_1 \backslash K_1,\ B_2 \subset D_2 \backslash K_2,\ B = B_1 \cup B_2,\ B_1 \cap B_2 = \emptyset$. As before, we obtain that $|\mu(B)| < \varepsilon$, so $A_1 \backslash A_2 \in \mathcal{A}$.

Now, let us prove that $\mathcal{A}$ is a $\delta-$ ring. It is sufficient to show that if $(A_n)_{n \in N} \subset \mathcal{A},\ A_n \searrow A$, then $A \in \mathcal{A}$. Because $A_n \in \mathcal{A}$, for every $n \in N^*$, then for every $\varepsilon > 0$, there exist a compact set $K_n \in \mathcal{C}$ and an open set $D_n \in \mathcal{C}, K_n \subset A_n \subset D_n$, such that

$$|\mu(B)| < \frac{\varepsilon}{2^{n+1}},\ \text{for every } B \in \mathcal{C}\ , B \subset D_n \backslash K_n. \tag{8}$$

Let us denote $C_n = \bigcap\limits_{i=1}^{n} K_i,\ E_n = \bigcap\limits_{i=1}^{n} D_i$ and $C = \bigcap\limits_{i=1}^{n} C_n$. Since $C_n \searrow C$, from lemma 2.2., we get that there exists $n_0 \in N$ such that

$$|\mu(B)| < \frac{\varepsilon}{2},\ \text{for every } B \in \mathcal{C}\ ,\ B \subset C_{n_0} \backslash C. \tag{9}$$

So, for every $\varepsilon > 0$, there exist $C_{n_0} = \bigcap\limits_{i=1}^{n_0} K_i$ a compact set in $\mathcal{C}$ and $E_{n_0} = \bigcap\limits_{i=1}^{n_0} D_i$ an open set in $\mathcal{C}$, $C_{n_0} \subset K_{n_0} \subset A_{n_0} = \bigcap\limits_{i=1}^{n_0} A_i \subset \bigcap\limits_{i=1}^{n_0} D_i = E_{n_0}$. This implies $C_{n_0} \subset A_{n_0} \subset E_{n_0}$.

We prove that

(10)
$$|\mu(B)| < \frac{\varepsilon}{2}, \text{ for every } B \in \mathcal{C}, \text{with}$$
$$B \subset E_{n_0} \backslash C_{n_0} = (\bigcap_{i=1}^{n_0} D_i) \backslash (\bigcap_{i=1}^{n_0} K_i) \subset \bigcap_{i=1}^{n_0} (D_i \backslash K_i).$$

Because $B \subset \bigcup_{i=1}^{n_0} (D_i \backslash K_i)$, we have that $B = B_1 \cup ... \cup B_{n_0},\ B_i \subset D_i \backslash K_i$, for every $i = \overline{1, n_0}$. From (8) it follows that

(11)
$$|\mu(B_i)| < \frac{\varepsilon}{2^{i+1}}, \text{ for every } i = \overline{1, n_0}.$$

Then $|\mu(B)| = |\mu(B_1 \cup ... \cup B_{n_0})| \leq \sum_{i=1}^{n_0} |\mu(B_i)| < \sum_{i=1}^{n_0} \frac{\varepsilon}{2^{i+1}} < \frac{\varepsilon}{2}$, as claimed.

Let us prove now that $A \in \mathcal{A}$. Taking into account that $C \subset A \subset E_{n_0}$ (clearly, C is a compact set in $\mathcal{C}$ and E_{n_0} is an open set in $\mathcal{C}$), we shall show that $|\mu(B)| < \varepsilon$, for every $B \in \mathcal{C},\ B \subset E_{n_0} \backslash C$. Indeed, $B \subset E_{n_0} \backslash C = (E_{n_0} \backslash C_{n_0}) \cup (C_{n_0} \backslash C)$. This implies that there exist $B_1, B_2 \in \mathcal{C}$, $B_1 \subset E_{n_0} \backslash C_{n_0}, B_2 \subset C_{n_0} \backslash C,\ B = B_1 \cup B_2$. From (9) and (10) we get that $|\mu(B_1)| < \frac{\varepsilon}{2}$ and $|\mu(B_2)| < \frac{\varepsilon}{2}$. Thus $|\mu(B)| < \varepsilon$ and, therefore, $A \in \mathcal{A}$. Since $\mathcal{D} \subset \mathcal{A} \subset \mathcal{C}$, $\mathcal{A}$ being a $\delta-$ring and $\mathcal{C}$ the smallest $\delta-$ ring containing $\mathcal{D}$, we obtain that $\mathcal{A} = \mathcal{C}$, which completes the proof. □

The following theorem shows that for a special ring $\mathcal{C}$, an o-continuous multisubmeasure becomes $R'-$ regular.

Theorem 2.4. *If $\mu : \mathcal{B}_0 \longrightarrow \mathcal{P}_f(X)$ is an o-continuous multisubmeasure, then μ is R'- regular on $\mathcal{B}_0$.*

Proof. Let us denote $\mathcal{D}$, the class of all compact sets of T which are G_δ; so, $\mathcal{B}_0$ is the $\delta-$ ring generated by $\mathcal{D}$. Using the theorem 2.3., we observe that it is sufficient to show that μ is R'- regular on $\mathcal{D}$. Let $K \in \mathcal{D}$. It is known that there exists a decreasing sequence of open sets $(D_n)_{n \in N} \subset \mathcal{B}_0$ such that $D_n \searrow K$ [see [3]). Let $\varepsilon > 0$. From lemma 2.2., we get that there exists $n_0 \in N$ such that $|\mu(B)| < \varepsilon$, for every $B \in \mathcal{B}_0,\ B \subset D_{n_0} \backslash K$, hence μ is R' -regular on $\mathcal{D}$, as claimed. □

Corollary 2.5. *According to the preceding theorem we obtain that, particularly, μ is R - regular and R'_l (R'_r) - regular on $\mathcal{B}_0$.*

Using the above mentioned results, we can give the following example of a R'- regular, strictly multisubmeasure.

Example 2.6. Let T be a locally compact space, $\mathcal{C}$ a ring of subsets of T and $\nu : \mathcal{C} \longrightarrow R_+$ a strictly submeasure which is o-continuous on $\mathcal{C}$. We define the multivalued set function
$\mu : \mathcal{C} \longrightarrow \mathcal{P}_f([0,\infty))$, by $\mu(A) = [0, \nu(A)]$, for every $A \in \mathcal{C}$.
Then μ is an o-continuous strictly multisubmeasure. If, particularly, $\mathcal{C}$ is the Baire δ-ring $\mathcal{B}_0$, from theorem 2.4. we get that μ is R'- regular on $\mathcal{B}_0$.

We prove now that it also holds a converse of theorem 2.4. in the general setting of a ring:

Theorem 2.7. *If $\mu : \mathcal{C} \longrightarrow \mathcal{P}_f(X)$ is a R'_l - regular multisubmeasure on $\mathcal{C}$, then μ is o-continuous.*

Proof. Let $\varepsilon > 0$ and $(A_n)_{n \in N} \subset \mathcal{C}$, $A_n \searrow \emptyset$. We have to prove that there exists $n_0 \in N$ such that $|\mu(A_n)| < \varepsilon$, for every $n \geq n_0$. Since every A_n is R'_l - regular, we obtain that for every $n \in N$, there exists a compact set $K_n \in \mathcal{C}$, $K_n \subset A_n$ such that $|\mu(B)| < \frac{\varepsilon}{2^n}$, for every $B \in \mathcal{C}$, $B \subset A_n \backslash K_n$. Then we also have that

$$(12) \qquad |\mu(A_n \backslash K_n)| < \frac{\varepsilon}{2^n}, \text{ for every } n \in N.$$

In the following we construct by induction a decreasing sequence of compact sets $(K'_n)_{n \in N}$ of $\mathcal{C}$, such that $K'_n \subset A_n$, for every $n \in N$ and $|\mu(B)| < \varepsilon$, for every $B \in \mathcal{C}$, $B \subset A_n \backslash K'_n$. Let $K'_1 = K_1 \in \mathcal{C}$. Then K'_1 is a compact set, $K'_1 \subset A_1, |\mu(B)| < \frac{\varepsilon}{2} < \varepsilon$, for every $B \in \mathcal{C}$, $B \subset A_1 \backslash K'_1$. Let $K'_2 = K_1 \cap K_2 \in \mathcal{C}$. Then $K'_2 \subset A_2$, K'_2 is a compact set and if we consider $B \in \mathcal{C}$, $B \subset A_2 \backslash K'_2 = (A_2 \backslash K_1) \cup (A_2 \backslash K_2) \subset (A_1 \backslash K_1) \cup (A_2 \backslash K_2)$, then there exist T_1, $T_2 \in \mathcal{C}$, $T_1 \subset A_1 \backslash K_1$, $T_2 \subset A_2 \backslash K_2$, $T_1 \cap T_2 = \emptyset$, $B = T_1 \cup T_2$. From (12) we get that $|\mu(B)| \leq |\mu(T_1)| + |\mu(T_2)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} < \varepsilon$.

In the following we suppose that we have constructed K'_1, $K'_2, ... K'_{p-1}$. Let then $K'_p = K_1 \cap ... K_p \in \mathcal{C}$. We have that $K'_p \subset A_p$ and K'_p is a compact set in $\mathcal{C}$. Let $B \in \mathcal{C}$, $B \subset A_p \backslash K'_p = (A_p \backslash K_1) \cup ...(A_p \backslash K_p) \subset (A_1 \backslash K_1) \cup ...(A_p \backslash K_p)$. Then $B = T_1 \cup ... T_p$, $T_t \subset A_i \backslash K_i$, for every $i = \overline{1,p}$ and from (12) we get that $|\mu(B)| \leq \sum\limits_{i=1}^{p} |\mu(T_i)| < \sum\limits_{i=1}^{p} \frac{\varepsilon}{2^i} < \varepsilon$. The construction of the sequence $(K'_n)_{n \in N}$ is thus finished. Because $K'_n \subset A_n$, for every $n \in N$

and $\bigcap_{n=1}^{\infty} A_n = \emptyset$, we get that $\bigcap_{n=1}^{\infty} K'_n = \emptyset$. But, on the other hand, $K'_n \subset K'_1$, for every $n \geq 1$, where K'_1 is a compact set in $\mathcal{C}$. From here we draw the conclusion that there exists $n_0(\varepsilon) \in N^*$ such that $\bigcap_{n=1}^{n_0} K'_n = K'_{n_0} = \emptyset$. We have then $A_n \subset A_{n_0} = A_{n_0} \setminus K'_{n_0}$, for every $n \geq n_0$, from which we obtain that $|\mu(A_n)| < \varepsilon$, for every $n \geq n_0$. This completes the proof. □

Corollary 2.8. *Let $\mu : \mathcal{B}_0 \longrightarrow \mathcal{P}_f(X)$ be a multisubmeasure. Then:*
i) μ is o-continuous on $\mathcal{B}_0$ if and only if μ is R'- regular on $\mathcal{B}_0$.
ii) μ is o-continuous on $\mathcal{B}_0$ if and only if μ is R'_l(R'_r) - regular on $\mathcal{B}_0$.

Corollary 2.9. *If $\mu : \mathcal{B}$ (or $\mathcal{B}'$, respectively)$\longrightarrow \mathcal{P}_f(X)$ is a R'_l - regular multisubmeasure on $\mathcal{B}$ (or $\mathcal{B}'$, respectively), then μ is o-continuous.*

We present now the relationship between regularity and exhaustivity.

Theorem 2.10. *If $\mu :\ \mathcal{C} \longrightarrow \mathcal{P}_f(X)$ is a R'_l - regular multisubmeasure, then μ is exhaustive.*

Proof. Let $\varepsilon > 0$ and $(A_n)_{n \in N} \subset \mathcal{C}$, $A_n \cap A_m = \emptyset$, $m \neq n$. Since every A_n is R'_l - regular, we obtain that for every $n \in N$, there exists a compact set $K_n \in \mathcal{C}, K_n \subset A_n$, such that $|\mu(B)| < \frac{\varepsilon}{2^{n+1}}$, for every $B \in \mathcal{C}, B \subset A_n \setminus K_n$. Then also, since $|\mu((A_n \setminus K_n) \setminus B)| < \frac{\varepsilon}{2^{n+1}}$, for a fixed set $B \in \mathcal{C}$, we get that

$$|\mu(A_n \setminus K_n)| < \frac{\varepsilon}{2^n}, \text{ for every } n \in N. \tag{13}$$

But $K_n \cap K_m = \emptyset$, $m \neq n$, and, therefore, $A_n \subset \bigcup_{i=1}^{n} (A_i \setminus K_i)$, for every $n \in N$. From (13) and the definition of h we obtain that $|\mu(A_n)| < \varepsilon$, for every $n \in N$. Hence μ is exhaustive. □

Remark 2.11. The converse of the theorem 2.10. does not hold in general: there exist multisubmeasures which are exhaustive and not regular in a certain sense (see, for instance, example 2.1.).

Remark 2.12. It is known from [8] that if $\mathcal{C}$ is a σ-ring and $\mu : \mathcal{C} \longrightarrow \mathcal{P}_f(X)$ is o-continuous then μ is exhaustive. In this situation, theorem 2.10. appears as a consequence of theorem 2.7. Then o-continuity stands between regularity and exhaustivity, for the case of a σ-ring.

We also can obtain an analog of theorem 2.3. on a $\sigma-$ring:

Theorem 2.13. *If $\mathcal{C}$ is the $\sigma-$ ring generated by a class $\mathcal{D}$ of subsets of T and if $\mu : \mathcal{C} \longrightarrow \mathcal{P}_f(X)$ is an o-continuous multisubmeasure which is R' - regular on $\mathcal{D}$, then μ is R' - regular on $\mathcal{C}$.*

Proof. We use the same notations as in theorem 2.3. We have to establish that $\mathcal{A}$ is a $\sigma-$ring, that is, without loss of generality, for every sequence $(A_n)_{n\in N} \subset \mathcal{A}$, $A_n \nearrow A$ involves $A \in \mathcal{A}$. Let $\varepsilon > 0$. For every $n \in N$, A_n is R'- regular, so, for every $n \in N$, there can be found a compact set $K_n \in \mathcal{C}$ and an open set $D_n \in \mathcal{C}$, $K_n \subset A_n \subset D_n$, such that $|\mu(B)| < \frac{\varepsilon}{2^{n+1}}$, for every $B \in \mathcal{C}$, $B \subset D_n \backslash K_n$. Let us denote $C_n = \bigcap\limits_{i=1}^{n} K_i$, $E_n = \bigcup\limits_{i=1}^{n} D_i$, $D = \bigcup\limits_{i=1}^{\infty} E_n$. Since $D\backslash E_n \searrow \emptyset$, from lemma 2.2., we obtain that there exists $n_0 \in N$ such that

$$(14) \qquad |\mu(B)| < \tfrac{\varepsilon}{2}, \text{ for every } B \in \mathcal{C}, B \subset D\backslash E_{n_0}.$$

So, we get that $C_{n_0} = \bigcap\limits_{i=1}^{n_0} K_i$ is a compact set, $D = \bigcup\limits_{n=1}^{\infty} E_n$ is an open set, $C_{n_0} \subset A \subset D$ and $C_{n_0} \subset A_{n_0} \subset E_{n_0}$. Exactly as in the proof of theorem 2.3., we can show that

$$(15) \qquad |\mu(B)| < \frac{\varepsilon}{2}, \text{ for every } B \in \mathcal{C}, B \subset E_{n_0}\backslash C_{n_0}.$$

It remains to prove that $|\mu(B)| < \varepsilon$, for every $B \in \mathcal{C}$, $B \subset D\backslash C_{n_0}$. Indeed, since $B \subset D\backslash C_{n_0} = (D\backslash E_{n_0}) \cup (E_{n_0}\backslash C_{n_0})$, we obtain that $B = B_1 \cup B_2$, $B_1 \subset D\backslash E_{n_0}$, $B_2 \subset E_{n_0}\backslash C_{n_0}$, $B_1 \cap B_2 = \emptyset$. Thus, from (14) and (15), $|\mu(B)| \leq |\mu(B_1)| + |\mu(B_2)| < \varepsilon$. This completes the proof.

We observe that it also can be indicated another proof of this theorem, based on the fact that o-continuity is equivalent to h-σ-subadditivity (see [8]). Indeed, if we consider the same K_n and D_n, then $K = \bigcap\limits_{n=1}^{\infty} K_n \in \mathcal{C}$ is a compact set, $D = \bigcup\limits_{n=1}^{\infty} D_n \in \mathcal{C}$ is an open set and $K \subset A \subset D$. If $B \in \mathcal{C}$, $B \subset D\backslash K = \bigcup\limits_{n=1}^{\infty} (D_n\backslash K_n)$, then since μ is h-σ-subadditive, we get that

$$|\mu(B)| \leq \sum_{n=1}^{\infty} |\mu(B_n)| < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}} < \frac{\varepsilon}{2} < \varepsilon,$$

that is, $A \in \mathcal{A}$. □

Corollary 2.14. *If $\mu : \mathcal{B}'_0 \longrightarrow \mathcal{P}_f(X)$ is an o-continuous multisubmeasure on $\mathcal{B}'_0$, then μ is R' - regular (on $\mathcal{B}'_0$).*

We also have from theorem 2.7. the following theorem, which represents the converse of corollary 2.14.

Theorem 2.15. *If $\mu : \mathcal{B}'_0 \longrightarrow \mathcal{P}_f(X)$ is a R'_l- regular multisubmeasure on $\mathcal{B}'_0$, then μ is o-continuous on $\mathcal{B}'_0$.*

Corollary 2.16. *Let $\mu : \mathcal{B}'_0 \longrightarrow \mathcal{P}_f(X)$ be a multisubmeasure. Then μ is R' - regular on $\mathcal{B}'_0$ if and only if μ is o-continuous on $\mathcal{B}'_0$.*

In the following we shall demonstrate that $R-$regularity also implies o-continuity.

Theorem 2.17. *Let $\mathcal{C}$ be a ring and $\mu : \mathcal{C} \longrightarrow \mathcal{P}_f(X)$ a R - regular multisubmeasure. Then μ is o-continuous.*

Proof. Let $(A_n)_n \subset \mathcal{C}$ such that $A = \bigcup\limits_{n=1}^{\infty} A_n \in \mathcal{C}$ and $\varepsilon > 0$. Since μ is R - regular, then A is R_l- regular, that is, there exists a compact set $K \in \mathcal{C}$, $K \subset A$ such that $h(\mu(A), \mu(B)) < \frac{\varepsilon}{2}$, for every $B \in \mathcal{C}$, $K \subset B \subset A$. Particularly, $h(\mu(A), \mu(K)) < \frac{\varepsilon}{2}$. From the R - regularity of μ it also results that every A_n is R_r - regular, that is, for every $n \in N$ there exists an open set $D_n \in \mathcal{C}$, $A_n \subset D_n$, so that $h(\mu(A_n), \mu(D_n)) < \frac{\varepsilon}{2^{n+1}}$. But, on the other hand, $K \subset \bigcup\limits_{n=1}^{\infty} D_n$ and, consequently, there exists $n_0 \in N$ such that $K \subset \bigcup\limits_{i=1}^{n} D_i$, for every $n \geq n_0$. We obtain then that

$$\begin{aligned} |\mu(A)| &\leq h(\mu(A), \mu(K)) + |\mu(K)| < \frac{\varepsilon}{2} + |\mu(K)| \leq \\ &\leq \frac{\varepsilon}{2} + |\mu(\bigcup_{i=1}^{n} D_i)| \leq \frac{\varepsilon}{2} + \sum_{i=1}^{n} |\mu(D_i)| \leq \\ &\leq \frac{\varepsilon}{2} + \sum_{n=1}^{\infty} h(\mu(A_i), \mu(D_i)) + \sum_{i=1}^{n} |\mu(A_i)| < \frac{\varepsilon}{2} + \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}} + \\ &+ \sum_{n=1}^{\infty} |\mu(A_n)| < \varepsilon + \sum_{n=1}^{\infty} |\mu(A_n)|, \end{aligned}$$

for every $\varepsilon > 0$, which means that $|\mu(A)| \leq \sum_{n=1}^{\infty} |\mu(A_n)|$, that is, μ is h-σ-subadditive, equivalently, o-continuous, as claimed. $\square$

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