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PROPERTIES OF REGULARITY FOR MULTISUBMEASURES

BY

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Abstract. In the present paper we study different types of regularity for multisubmeasures. We also present certain regularity properties of multisubmeasures, connected with their variation, and, taking as starting point the work of DREWNOWSKI [3], we establish results concerning o-continuity and exhaustivity for multisubmeasures.

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0. Basic notions, terminology and notations. Let T be an abstract set, $\mathcal{C}$ a ring of subsets of T, X a real normed space, $\mathcal{P}_0(X)$ the family of all nonempty subsets of X, $\mathcal{P}_f(X)$ the family of closed, nonvoid sets of X, h the Hausdorff pseudometric on $\mathcal{P}_f(X)$. We define $|M| = h(M, \{0\})$, for every $M \in \mathcal{P}_f(X)$, where 0 is the origin of X. If, in addition, X is complete, then so is $\mathcal{P}_f(X)$. It is known that $h(M, N) = \max\{e(M, N), e(N, M)\}$, for every $M, N \subset X$, where $e(M, N) = \sup\{d(x, N); x \in M\}$, $d(x, N)$ being the distance from x to N, with respect to the metric induced by the norm of X.

On $\mathcal{P}_0(X)$ we introduce the Minkowski addition, denoted by the symbol $\stackrel{\bullet}{+}$, defined by:

$$M \stackrel{\bullet}{+} N = \overline{M + N}, \tag{1}$$

for every $M, N \in \mathcal{P}_0(X)$.

We recall the following classical notions:

I) A set function $m : \mathcal{C} \to \overline{R}_+$ is said to be a **submeasure (**a **strictly submeasure,** respectively) in Drewnowski's sense [3], if $m(\emptyset) = 0$, $m(A \cup B) \leq m(A) + m(B)$, for every $A, B \in \mathcal{C}$, $A \cap B = \emptyset$ ($m(A \cup B) \lneqq m(A) + m(B)$, for every $A, B \in \mathcal{C}$, $A \cap B = \emptyset$, $A, B \neq \emptyset$, respectively), and $m(A) \leq m(B)$, for every $A, B \in \mathcal{C}$, with $A \subseteq B$.

In the sequel we shall use multivalued set functions $\mu : \mathcal{C} \to \mathcal{P}_f(X)$.

II) μ is said to be **(finite) additive** if $\mu(\emptyset) = \{0\}$ and $\mu(A \cup B) = \mu(A) \overset{\bullet}{+} \mu(B)$, for every $A, B \in \mathcal{C}$, $A \cap B = \emptyset$.

III) We call the **total variation** of μ, the real valued set function $\overline{\mu}$ defined by:

$$\overline{\mu}(A) = \sup\{\sum |\mu(A_i)|\}, \tag{2}$$

where supremum is extended over all finite families $(A_i)_{i=1,n}$ of disjoint subsets of $\mathcal{C}$, which are contained in A, for every $A \subset T$ (see [1], p. 33-37).

IV) We say that μ has **finite variation** if $\overline{\mu}(A) < \infty$, for every $A \subset T$.

V) We call the **semi-variation** of μ, the real valued set function $\widehat{\mu}$ defined by:

$$\widehat{\mu}(A) = \sup\{|\mu(B)|, B \subset A, B \in \mathcal{C}\}, \text{ for every } A \subset T. \tag{2'}$$

VI) A sequence $(A_n)_{n \in N^*} \subset \mathcal{C}$ is said to be **order-convergent** (in $\mathcal{C}$) to $A \in \mathcal{C}$, denoted $A_n \xrightarrow{o} A$, if there are two sequences $(B_n)_{n \in N^*}$, $(C_n)_{n \in N^*} \subset \mathcal{C}$ such that $B_n \subset A_n \subset C_n$ and $B_n \nearrow A$(that is, $B_n \subset B_{n+1,}$ for every $n \in N^*$ and $\overset{\infty}{\underset{n=1}{\cup}} B_n = A$), $C_n \searrow A$(that is, $C_n \supset C_{n+1,}$ for every $n \in N^*$ and $\overset{\infty}{\underset{n=1}{\cap}} C_n = A$).

VII) $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is said to be **increasing** (respectively, **decreasing) convergent** if for every sequence $(A_n)_{n \in N^*} \subset \mathcal{C}$, with $A_n \nearrow A \in \mathcal{C}$ (respectively, $A_n \searrow A \in \mathcal{C}$), we have $\lim\limits_{n \to \infty} h(\mu(A_n), \mu(A)) = 0$.

1. Properties of regularity for multisubmeasures. In the sequel we shall use a special type of multivalued set function, which will be called a multisubmeasure. Taking as starting point the classical definitions for different types of regularity of additive set functions with values in a Banach space (see [1]) we introduce then here correspondent definitions of regularity for multisubmeasures, with respect to the Hausdorff pseudometric h.

Definition 1.1. I) A multivalued set function μ from $\mathcal{C}$ to $\mathcal{P}_f(X)$ is said to be a **multisubmeasure** if:

$$\begin{cases} a)\ \mu(\emptyset) = \{0\}, \\ b)\ \mu(A \cup B) \subseteq \mu(A) \overset{\bullet}{+} \mu(B), \text{for every } A, B \in \mathcal{C}, \text{ with } A \cap B = \emptyset, \\ c)\ \mu(A) \subseteq \mu(B), \text{for every } A, B \in \mathcal{C}, \text{ with } A \subseteq B. \end{cases}$$

II) A multivalued set function μ from $\mathcal{C}$ to $\mathcal{P}_f(X)$ is said to be a **strictly multisubmeasure** if μ satisfies the above conditions a) and c) and instead of b) we have b'): $\mu(A \cup B) \subsetneq \mu(A) \overset{\bullet}{+} \mu(B)$, for every $A, B \in \mathcal{C}$, $A \cap B = \emptyset$, $A, B \neq \emptyset$.

Remark 1.2. The condition b) from definition 1.1.I) is equivalent to the condition $\mu(A \cup B) \subseteq \mu(A) \overset{\bullet}{+} \mu(B)$, for every $A, B \in \mathcal{C}$.

Example 1.3. (of a strictly multisubmeasure)

Let T be an abstract set, $\mathcal{C}$ a ring of subsets of T, $\nu : \mathcal{C} \to R_+$ a strictly submeasure on $\mathcal{C}$. We define $\mu : \mathcal{C} \to \mathcal{P}_f([0, \infty))$ by $\mu(A) = [0, \nu(A)]$, for every $A \in \mathcal{C}$. Then μ is a strictly multisubmeasure because $\mu(\emptyset) = \{0\}, \mu(A) = [0, \nu(A)] \subseteq [0, \nu(B)] = \mu(B)$, for every $A, B \in \mathcal{C}$, $A \subset B$ and $\mu(A \cup B) \subsetneq \mu(A) \overset{\bullet}{+} \mu(B)$, for every $A, B \in \mathcal{C}, A \cap B = \emptyset$, $A, B \neq \emptyset$.

Indeed, let $x \in \mu(A \cup B) = [0, \nu(A \cup B)]$; then $0 \leq x \leq \nu(A \cup B) \lneqq \nu(A) + \nu(B)$. Therefore, there exist y, z, $0 \leq y \lneqq \nu(A)$, $0 \leq z \lneqq \nu(B)$ such that $x = y + z$. This implies that $x \in [0, \nu(A)] + [0, \nu(B)] = [0, \nu(A)] \overset{\bullet}{+} [0, \nu(B)] = \mu(A) \overset{\bullet}{+} \mu(B)$. On the other hand, because $\nu(A \cup B) \lneqq \nu(A) + \nu(B)$, there exists $x_0 \in R$ such that $\nu(A \cup B) \lneqq x_0 \leq \nu(A) + \nu(B)$. Hence, there exists $x_0 \in [0, \nu(A)] \overset{\bullet}{+} [0, \nu(B)] = \mu(A) \overset{\bullet}{+} \mu(B)$ such that $x_0 \notin [0, \nu(A \cup B)] = \mu(A \cup B)$, so the inclusion is strictly.

Remark 1.4. It is easy to see that if μ is a multisubmeasure, then $\widehat{\mu}$ defined by (2') is a submeasure in Drewnowski's sense on $\mathcal{C}$.

We also observe that, although μ is "subadditive", $\overline{\mu}$ defined by (2) is still a non-negative, finite additive set function on $\mathcal{C}$. Clearly, $\overline{\mu}(A) \geq |\mu(A)|$, for every $A \in \mathcal{C}$.

In the following we shall present different definitions for the regularity of multisubmeasures. All of them require a topological structure on T. It was seen that a proper framework for the study of regularity is that T is a Hausdorff locally compact space. Let $\mathcal{B}_0$(respectively, $\mathcal{B}'_0$) be the Baire δ-ring (respectively, σ-ring) generated by the compact sets which are G_δ(that is, countable intersection of open sets) and $\mathcal{B}$ (respectively, $\mathcal{B}'$) the borelian δ-ring (respectively, σ-ring) generated by the compact sets of T.

Definition 1.5. I) A set $A \in \mathcal{C}$ is said to be R **- regular** with respect to the multisubmeasure μ if for every $\varepsilon >0$, there exist a compact set $K \subset A$, $K \in \mathcal{C}$ and an open set $D \supset A$, $D \in \mathcal{C}$ such that $h(\mu(A), \mu(B)) < \varepsilon$, for every $B \in \mathcal{C}$, $K \subset B \subset D$.

II) A set $A \in \mathcal{C}$ is said to be R_l-**regular** with respect to the multisubmeasure μ if for every $\varepsilon >0$, there exists a compact set $K \subset A, K \in \mathcal{C}$ such that $h(\mu(A), \mu(B)) < \varepsilon$, for every $B \in \mathcal{C}, K \subset B \subset A$.

III) A set $A \in \mathcal{C}$ is said to be R_r- **regular** with respect to the multisubmeasure μ if for every $\varepsilon > 0$, there exists an open set $D \supset A$, $D \in \mathcal{C}$ such that $h(\mu(A), \mu(B)) < \varepsilon$, for every $B \in \mathcal{C}, A \subset B \subset D$.

Definition 1.6. A multisubmeasure $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is said to be R **- regular** (R_l**- regular,** R_r-**regular)** if every $A \in \mathcal{C}$ is a R - regular (R_l - regular, R_r - regular, respectively) set with respect to μ.

It is easy to see that every compact set $K \in \mathcal{C}$ is R$_l$ - regular (with respect to μ) and every open set $D \in \mathcal{C}$ is R$_r$ - regular (with respect to μ).

Definition 1.7. I) A set $A \in \mathcal{C}$ is said to be R' **- regular** with respect to the multisubmeasure μ if for every $\varepsilon >0$, there are a compact set $K \subset A$, $K \in \mathcal{C}$ and an open set $D \supset A, D \in \mathcal{C}$ such that $|\mu(B)| < \varepsilon$, for every $B \in \mathcal{C}$, $B \subset D \backslash K$.

II) A set $A \in \mathcal{C}$ is said to be R'_l **- regular** with respect to the multisubmeasure μ if for every $\varepsilon >0$, there exists a compact set $K \subset A, K \in \mathcal{C}$ such that $|\mu(B)| < \varepsilon$, for every $B \in \mathcal{C}$, $B \subset A \backslash K$.

III) A set $A \in \mathcal{C}$ is said to be R'_r **- regular** with respect to the multisubmeasure μ if for every $\varepsilon > 0$, there exists an open set $D \supset A, D \in \mathcal{C}$ such that $|\mu(B)| < \varepsilon$, for every $B \in \mathcal{C}$, $B \subset D \backslash A$.

It is easy to see that every compact set $K \in \mathcal{C}$ is R$'_l$ - regular and every open set $D \in \mathcal{C}$ is R'_r - regular.

Definition 1.8. A multisubmeasure $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is said to be R' **- regular** (R'_l **- regular,** R'_r **- regular)** if every $A \in \mathcal{C}$ is a R'- regular (R'_l - regular, R'_r - regular, respectively) set.

We present in the sequel the relationships between the different types of regularity, which have been introduced above.

Theorem 1.9. *A multisubmeasure $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is R - regular if and only if it is R_l - regular and R_r- regular.*

Proof. The "If part" is trivial. To prove the "only if part", assume $A \in \mathcal{C}$ simultaneously R_l - regular and R_r - regular with respect to μ. Let $\varepsilon >$0. There are a compact set $K \subset A, K \in \mathcal{C}$ and an open set $D \supset A, D \in \mathcal{C}$ such that

$$\begin{aligned} &h(\mu(A), \mu(B)) < \frac{\varepsilon}{3}, \\ &\text{for every } B \in \mathcal{C}, K \subset B \subset A \text{ or for every } B \in \mathcal{C}, A \subset B \subset D. \end{aligned} \tag{3}$$

Take $B \in \mathcal{C}$ such that $K \subset B \subset D$. Since $K \subset A \subset D$ we have $K \subset A \cap B \subset A$ and $A \subset A \cup B \subset D$. From here and from (3) we have

$$h(\mu(A),\ \mu(A \cap B)) < \frac{\varepsilon}{3} \text{ and } h(\mu(A),\ \mu(A \cup B)) < \frac{\varepsilon}{3}. \tag{4}$$

On the other hand,

$$h(\mu(A),\ \mu(B)) \le h(\mu(A),\ \mu(A \cap B)) + h(\mu(A \cap B),\ \mu(B)). \tag{5}$$

Taking into account the definition of h, we get that

$$\begin{aligned} h(\mu(B), \mu(A \cap B)) &\le h(\mu(A \cup B), \mu(A \cap B)) \le \\ &\le h(\mu(A \cup B), \mu(A)) + h(\mu(A), \mu(A \cap B)) < \frac{2\varepsilon}{3} \end{aligned}$$

which implies $h(\mu(A), \mu(B)) < \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon$. This completes the proof. □

Corollary 1.10. *a) A compact set $K \in \mathcal{C}$ is R - regular if and only if it is R_r-regular.*

b) An open set $D \in \mathcal{C}$ is R - regular if and only if it is R_l - regular.

Theorem 1.11. *If a multisubmeasure $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is R_l' - regular, then μ is R_l - regular.*

Proof. Let $A \in \mathcal{C}$ and $\varepsilon > 0$. Since $A \in \mathcal{C}$ is R_l' - regular, there exists a compact set $K \in \mathcal{C}$, $K \subset A$ such that

$$|\mu(B)| < \varepsilon, \text{for every } B \in \mathcal{C}, B \subset A \backslash K. \tag{6}$$

Take $B \in \mathcal{C}$ such that $K \subset B \subset A$. From the definition of h, we obtain that

$$(7) \qquad \begin{aligned} h(\mu(A), \mu(B)) &\leq h(\mu(A \backslash B) \overset{\bullet}{+} \mu(B), \mu(B)) \leq \\ &\leq |\mu(A \backslash B)|. \end{aligned}$$

On the other hand, applying (6) for $A \backslash B \subset A \backslash K$, we get that $|\mu(A \backslash B)| < \varepsilon$. So, $h(\mu(A), \mu(B)) < \varepsilon$, as claimed. □

In the same way, we can prove:

Theorem 1.12. *If a multisubmeasure $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is R'_r - regular, then μ is R_r - regular.*

Theorem 1.13. *A multisubmeasure $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is R' - regular if and only if it is R'_l-regular and R'_r-regular.*

Proof. The "If part" is trivial.

To prove the "only if part", let $\varepsilon > 0$ and $A \in \mathcal{C}$. Since A is simultaneously R'_l - regular and R'_r - regular, there exist a compact set $K \in \mathcal{C}$ and an open set $D \in \mathcal{C}$, $K \subset A \subset D$ such that

$$(8) \qquad \begin{aligned} &|\mu(B)| < \frac{\varepsilon}{2}, \\ &\text{for every } B \in \mathcal{C} \text{ with either } B \subset A \backslash K, \text{ or } B \subset D \backslash A. \end{aligned}$$

Let $B \in \mathcal{C}$, $B \subset D \backslash K$. Since $K \subset A \subset D$ we obtain $B \backslash A \subset D \backslash A$ and $A \cap B \subset A \backslash K$. From (8) we have

$$(9) \qquad |\mu(B \backslash A)| < \frac{\varepsilon}{2} \text{ and } |\mu(A \cap B)| < \frac{\varepsilon}{2}.$$

From (8) and (9) we get: $|\mu(B)| \leq |\mu(B \backslash A)| + |\mu(A \cap B)| < \varepsilon$, as claimed.□

Corollary 1.14. *a) A compact set $K \in \mathcal{C}$ is R' - regular if and only if it is R'_r - regular.*

b) An open set $D \in \mathcal{C}$ is R' - regular if and only if it is R'_l - regular.

c) If a multisubmeasure $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is R' - regular, then μ is R - regular.

In the following it is natural to ask ourselves if there exists any relationship between R'_r - regularity and R'_l - regularity. The answer is given by the following results, which can easily be obtained as in [1] (see p. 201-204) (for additive set functions taking values in a Banach space).

Theorem 1.15. *If the ring $\mathcal{C}$ has the supplementary property:*

$$\text{(10)} \qquad \text{for every } A \in \mathcal{C}, \text{there exists } A' \in \mathcal{C} \text{ so that } A \subset \overset{\circ}{A'},$$

(where $\overset{\circ}{A'}$ denotes the interior of A') and $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is a R'_l - regular multisubmeasure, then μ is R'_r - regular.

Corollary 1.16. *If $\mathcal{C}$ is the ring (or the δ-ring) generated by the compact sets which are G_δ, or simply by the compact sets, and μ is a R'_l - regular multisubmeasure, then μ is R' - regular.*

Theorem 1.17. *If the ring $\mathcal{C}$ has the two supplementary properties: a) $\mathcal{C}$ consists of relatively compact sets,*

b) $\mathcal{C}$ is dense in $\mathcal{P}(T)$ for the topology τ on $\mathcal{P}(T)$, which has as a base, the family $\mathcal{B} = \{[K, D], K \text{ a compact set}, D \text{ an open set}\}$, $[K, D] = \{A \subset T, K \subset A \subset D\}$ and if the multisubmeasure $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is R'_r - regular, then μ is R'_l - regular.

Corollary 1.18. *If $\mathcal{C}$ is the ring (or the δ-ring) generated by the compact sets which are G_δ, or by the compact sets, we get:*

a) If μ: $\mathcal{C} \to \mathcal{P}_f(X)$ is R'_r - regular, then μ is R' - regular.

b) $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is R'_r - regular if and only if μ is R'_l - regular.

c) $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is R'_l - regular if and only if μ is R' - regular.

2. Properties of regularity and exhaustivity for the variation and semivariation of a multisubmeasure. Let T be a Hausdorff locally compact space, $\mathcal{C}$ a ring of subsets of T and $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ a multisubmeasure. In the following we shall suppose that μ has finite variation.

We present here some linking results between the regularity of a multisubmeasure and the regularity (in the sense of Dinculeanu, see [1]) of its variation.

Theorem 2.1. *A multisubmeasure $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is R_l - regular if and only if $\overline{\mu}$ is R_l - regular on $\mathcal{C}$.*

Proof. The "only if part": It is known from proposition 2, p. 200 [1], that $\overline{\mu} : \mathcal{C} \to R_+$, which is a finite additive set function is R_l- regular if and only if $\overline{\mu}$ is R_l' - regular. So, $\overline{\mu}$ is R_l'- regular. But

$$|\mu(B)| \leq \overline{\mu}(B), \text{for every} B \in \mathcal{C}. \tag{11}$$

This implies that μ is R_l'- regular, hence μ is R_l-regular.

To prove the "if part", let $\varepsilon > 0$ and $A \in \mathcal{C}$. By the definition of $\overline{\mu}$, we find a finite partition $(A_i)_{i=1,n} \subset \mathcal{C}$ of A, for which

$$\overline{\mu}(A) < \sum_{i=1}^{n} |\mu(A_i)| + \frac{\varepsilon}{2}. \tag{12}$$

But every A_i is R_l - regular and then, for every i, there exists a compact set $K_i \subset A_i,\ i = \overline{1,n}$ such that

$$h((\mu(A_i), \mu(B)) < \frac{\varepsilon}{2n}, \text{ for every } B \in \mathcal{C}\ , K_i \subset B \subset A_i. \tag{13}$$

Let us denote $K = \bigcup_{i=1}^{n} K_i$. It is clear that $K \subset A$ and K is a compact set. Let $B \in \mathcal{C}$ such that $K \subset B \subset A$ and $B_i = B \cap A_i, i = \overline{1,n}$. Then $B_i \in \mathcal{C}$, $K_i \subset B_i \subset A_i$ and $B_i \cap B_j = \emptyset$, for $i \neq j$.

Using (13) we obtain:

$$\begin{aligned} \overline{\mu}(A) - \overline{\mu}(B) &< \sum_{i=1}^{n} |\mu(A_i)| + \frac{\varepsilon}{2} - \sum_{i=1}^{n} |\mu(B_i)| = \\ &= \frac{\varepsilon}{2} + \sum_{i=1}^{n} [|\mu(A_i)| - |\mu(B_i)|] \leq \\ &\leq \frac{\varepsilon}{2} + \sum_{i=1}^{n} h((\mu(A_i), \mu(B_i)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2n} \cdot n = \varepsilon. \end{aligned} \tag{14}$$

This implies that $\overline{\mu}$ is R_l - regular, as claimed. □

Corollary 2.2. *The multisubmeasure $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is R_l' - regular if and only if $\overline{\mu}$ is R_l' - regular on $\mathcal{C}$.*

Proof. The "only if part" is trivial.

The "if part": The R'_l - regularity of μ implies its R_l - regularity. By theorem 2.1, we get that $\overline{\mu}$ is R_l - regular, which is equivalent to the fact that $\overline{\mu}$ is R'_l - regular. $\square$

Theorem 2.3. *If $\overline{\mu}$ is R' - regular (R'_l - regular or R'_r - regular, respectively) on $\mathcal{C}$, then μ is R'- regular (R'_l - regular or R'_r - regular, respectively).*

Using similar arguments as in proposition 22, p. 210 [1] (for additive set functions taking values in a Banach space) we can easily prove:

Theorem 2.4. *If μ: $\mathcal{B} \to \mathcal{P}_f(X)$ is a R_l- regular multisubmeasure, then: $\overline{\mu}(A) = \sup\{\sum |\mu(K_i)|\}$, where supremum is extended over all finite families $(K_i)_{\overline{i=1,n}}$ of disjoint, compact subsets of A, for every $A \subset T$.*

For R_r-regularity we have:

Theorem 2.5. *If $\overline{\mu} : \mathcal{C} \to R_+$ is R_r - regular, then μ is R_r - regular.*

Proof. The R_r- regularity of $\overline{\mu}$ is equivalent to the R'_r- regularity of $\overline{\mu}$, hence μ is R'_r- regular. Then μ is R_r- regular. $\square$

Remark 2.6. Conversely, it is not true in general.

We introduce now two important types of multisubmeasures, namely exhaustive and semiexhaustive and we establish results concerning exhaustivity of a multisubmeasure μ and boundedness of $\hat{\mu}$ (respectively, $\overline{\mu}$) on $\mathcal{C}$.

Definition 2.7. I) A multivalued set function $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is said to be **exhaustive** with respect to h, if $\lim\limits_{n\to\infty} |\mu(A_n)| = 0$, for every disjoint sequence $(A_n)_{n\in N^*} \subset \mathcal{C}$.

II) A multivalued set function $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is said to be **semiexhaustive** with respect to h, if $\lim\limits_{n\to\infty} | \mu(A_n)| = 0$, for every disjoint sequence $(A_n)_{n\in N^*} \subset \mathcal{C}$, with $\overset{\infty}{\underset{n=1}{\cup}} A_n \in \mathcal{C}$.

It is easy to prove the following:

Theorem 2.8. *A multisubmeasure $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is exhaustive if and only if every monotone sequence $(A_n)_{n\in N} \subset \mathcal{C}$ is Cauchy with respect to μ, that is,* $\lim\limits_{\substack{n\to\infty \\ m\to\infty}} |\mu(A_n \Delta A_m)| = 0$.

Remarks 2.9. **I)** If μ is exhaustive and X is a Banach space then for every monotone sequence $(A_n)_{n\in N} \subset \mathcal{C}$, $(\mu(A_n))_n$ is convergent (with respect to h).

II) It is obvious that if $(A_n)_{n\in N} \subset \mathcal{C}$ is an increasing sequence and μ is exhaustive then

$$\lim_{n\to\infty} |\mu(A_{n+1} \backslash A_n)| = 0.$$

Theorem 2.10. *If $\mu : C \to P_f(X)$ is an exhaustive multisubmeasure, then $\hat{\mu}$ restricted to C is bounded (hence, finite) on C (this means that μ is bounded in $|\cdot|$).*

Proof. We have observed that $\hat{\mu} : \mathcal{C} \to R_+$ is a submeasure in Drewnowski's sense. First, we prove that $\hat{\mu}$ is bounded if and only if for every increasing sequence $(A_n)_{n\in N} \subset \mathcal{C}$, $(\hat{\mu}(A_n))_{n\in N}$ is convergent in R_+.

Indeed, if $(A_n)_{n\in N}$ is an increasing sequence and $\hat{\mu}$ is bounded, then $(\hat{\mu}(A_n))_{n\in N}$ is also increasing and bounded, hence there exists $\lim \hat{\mu}(A_n) = l \in R_+$.

For the other implication, we suppose that, on the contrary, $\hat{\mu}$ is not bounded, which means that, for every $n \in N$ there exists $A_n \in \mathcal{C}$ such that $\hat{\mu}(A_n) > n+1$. Let $B_n = \bigcup\limits_{i=1}^{n} A_i$. Then $(B_n)_{n\in N} \subset \mathcal{C}$ is increasing and, consequently, $(\hat{\mu}(B_n)_{n\in N}$ converges in R_+. But $A_n \subset B_n$, for every $n \in N$, hence $\hat{\mu}(B_n) > n+1$, contradiction.

We are able now to demonstrate the theorem. We suppose that, on the contrary, there exists an increasing sequence $(A_n)_{n\in N} \subset \mathcal{C}$ such that $(\hat{\mu}(A_n))_{n\in N}$ does not converge in R_+. This means that it is not Cauchy. Then there exists $\varepsilon > 0$ such that for every $k \in N^*$, there exists $n_k > k$, $n_k \in N^*$ so that $\hat{\mu}(A_{n_k}) - \hat{\mu}(A_k) \geq \varepsilon$. But $\hat{\mu}(A_{n_k}) \leq \hat{\mu}(A_k) + \hat{\mu}(A_{n_k} \backslash A_k)$, for every $n_k > k$, hence $\hat{\mu}(A_{n_k} \backslash A_k) \geq \varepsilon$. Particularly, for $k = 1$, there exists $n_1 \in N^*$ such that $\hat{\mu}(A_{n_1} \backslash A_1) \geq \varepsilon$. For $k = n_1$, there exists $n_2 > n_1$ such that $\hat{\mu}(A_{n_2} \backslash A_2) \geq \varepsilon$. Continuing this way, we get a sequence $(A_{n_k})_{k\in N} \subset \mathcal{C}$ such that $\hat{\mu}(A_{n_{k+1}} \backslash A_{n_k}) = |\mu(A_{n_{k+1}} \backslash A_{n_k})| \geq \varepsilon$, for every $k \in N^*$, contradiction because the sets $A_{n_{k+1}} \backslash A_{n_k}$ are mutual disjoint and μ is exhaustive. □

Theorem 2.11. *If $\mu : C \to P_f(X)$ is a multisubmeasure such that $\overline{\mu}$ is bounded on C, then μ is exhaustive.*

Proof. Let $(A_n)_{n\in N} \subset \mathcal{C}$ be such that $A_n \cap A_m = \emptyset$, $m \neq n$. We denote $B_n = \overset{n}{\underset{i=1}{\cup}} A_i$. Then $(B_n)_{n\in N}$ is an increasing sequence, hence $(\overline{\mu}(B_n))_n$ is increasing and bounded in R_+, which means that there exists $\lim \overline{\mu}(B_n) = l \in R_+$. But $\overline{\mu}(B_n) = \overline{\mu}(A_n) + \overline{\mu}(B_{n-1})$ and, consequently, there exists $\lim \overline{\mu}(A_n) = 0$. Because $\overline{\mu}(A_n) \geq |\mu(A_n)|$, we get that $\lim |\mu(A_n)| = 0$, as claimed. □

3. O-continuous multisubmeasures. In this section we introduce the notion of an o-continuous multisubmeasure. Taking as starting point the works [2], [3] and [4] for submeasures taking values in $\overline{R}_+$ or in a topological group, we obtain here certain important results concerning h-σ-subadditivity, o-continuity, exhaustivity and increasing (respectively, decreasing) convergence for multisubmeasures.

Let T be an abstract set, $\mathcal{C}$ a ring of subsets of T and $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ a multivalued set function.

Definition 3.1. $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is said to be **order-continuous** (briefly, **o-continuous**) with respect to h, if $\underset{n\to\infty}{lim} |\mu(A_n)| = 0$, for every sequence $(A_n)_{n\in N^*} \subset \mathcal{C}$ such that $A_n \searrow \emptyset$ (that is, $A_n \supset A_{n+1}$, for every $n \in N^*$ and $\overset{\infty}{\underset{n=1}{\cap}} A_n = \emptyset$).

In the following let $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ be a multisubmeasure.

We easily get the following:

Theorem 3.2. *A multisubmeasure $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is o-continuous if and only if for every sequence $(A_n)_{n\in N} \subset \mathcal{C}$, $A_n \overset{o}{\to} A \in \mathcal{C}$ implies* $\lim |\mu(A \Delta A_n)| = 0$.

Theorem 3.3. *If $\mathcal{C}$ is a ring and the multisubmeasure $\mu : C \to P_f(X)$ is o-continuous, then μ is increasing and decreasing convergent.*

Remark 3.4. It is evident that if a multisubmeasure $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is decreasing convergent, then μ is o-continuous. So, μ is o-continuous if and only if it is decreasing convergent.

In the sequel we study the relationships between o-continuity and exhaustivity. The following example shows that there exist multisubmeasures which are exhaustive and o-continuous.

Example 3.5. Let $\mathcal{C} = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ which is a ring in $T = R$, $\nu : \mathcal{C} \to R_+$, defined by:

$$\nu(A) = \begin{cases} 0, & A = \emptyset, \\ 1, & A = \{1\} \text{ or } A = \{2\}, \\ \frac{3}{2}, & A = \{1,2\}. \end{cases} \text{ for every } A \in \mathcal{C}.$$

It is easy to see that ν is a strictly submeasure. Let $\mu : \mathcal{C} \to \mathcal{P}_f([0,\infty))$, $\mu(A) = [0, \nu(A)]$, for every $A \in \mathcal{C}$. Then μ is a strictly multisubmeasure which is o-continuous and exhaustive. On the other hand, it is obvious that μ is $R'_l(R_l)$-regular but it is not $R'_r(R_r)$-regular.

Remark 3.6. The following example shows that there exist strictly multisubmeasures, which are o-continuous and not exhaustive.

Example 3.7. Let $\mathcal{C}$ =$\{A \subset R,\ A \text{ finite}\}$ which is a ring in $T = R$ and $\nu : \mathcal{C} \to R_+$, defined by:

$$\nu(A) = \begin{cases} 0, \ A = \emptyset, \\ 1 + cardA, \ A \neq \emptyset, \ A \subset R, \ A \text{ finite} \end{cases}$$

(where $cardA$ represents the number of elements of A), for every $A \in \mathcal{C}$.

Let $\mu : \mathcal{C} \to \mathcal{P}_f([0,\infty))$, $\mu(A) = [0, \nu(A)]$, for every $A \in \mathcal{C}$. We observe that ν is a strictly submeasure, hence μ is a strictly multisubmeasure. Let $(A_n)_n \subset \mathcal{C}$, $A_n \searrow \emptyset$. Since A_n is finite, for every $n \in N$, we get that there exists $n_0 \in N$ such that $A_n = \emptyset$, for every $n \geq n_0$, which yields $\mu(A_n) = \{0\}$, for every $n \geq n_0$, so μ is o-continuous.

We prove now that μ is not exhaustive. Suppose that μ is exhaustive and let $(A_n)_n \subset \mathcal{C}$, $A_n \cap A_m = \emptyset$, $m \neq n$. Then $\lim |\mu(A_n)| = 0 = \lim |[0, \nu(A_n)]|$, which implies that ν is exhaustive. This is a contradiction because we can find a sequence $(A_n)_n \subset \mathcal{C}$, $A_n \cap A_m = \emptyset$, $m \neq n$, A_n finite, nonvoid, for every $n \in N$ (for instance, $A_n = \{n\}$, for every $n \in N$). Then $\nu(A_n) = card(A_n) + 1 > 1$, for every $n \in N$.

Remark 3.8. In the sequel we indicate an example of an exhaustive multisubmeasure, which is not o-continuous.

Example 3.9. Let T be an abstract set, $\mathcal{C}$ a ring of subsets of T and $\nu_1, ...\nu_p : \mathcal{C} \to R_+$ finite additive set functions. We suppose that ν_i are exhaustive, for every $i = \overline{1,p}$ (equivalently, bounded) and ν_1 is not o-continuous (equivalently, not σ-additive). First, we define $\mu : \mathcal{C} \to \mathcal{P}_f(R_+)$, $\mu(A) = \{\nu_1(A), ...\nu_p(A)\}$, for every $A \in \mathcal{C}$.

Then $\mu(\emptyset) = \{0\}$ and $\mu(A \cup B) \subseteq \mu(A) \overset{\bullet}{+} \mu(B)$, for every $A, B \in \mathcal{C}$, $A \cap B = \emptyset$. We observe that μ does not have the property c) from definition 1.1.I), but μ is exhaustive (from the exhaustivity of all $\nu_i, i = \overline{1,p}$) and it is not o-continuous (if we suppose that, on the contrary, μ is o-continuous, then ν_1 is o-continuous, contradiction). Now we shall introduce with the aid of μ, another multivalued set function $\mu^\vee : \mathcal{C} \to \mathcal{P}_f(R_+)$, defined by:

$\mu^\vee(A) = \overline{\underset{B \subset A,\, B \in \mathcal{C}}{\cup} \mu(B)}$, for every $A \in \mathcal{C}$,

which will be the desired multisubmeasure. Indeed,

$\mu^\vee(\emptyset) = \{0\}$, $\mu^\vee(A_1) = \overline{\underset{B \subset A_1,\, B \in \mathcal{C}}{\cup} \mu(B)} \subseteq \overline{\underset{B \subset A_2,\, B \in \mathcal{C}}{\cup} \mu(B)} = \mu^\vee(A_2)$,

for every $A_1, A_2 \in \mathcal{C}$, $A_1 \subset A_2$ and $\mu^\vee(A_1 \cup A_2) \subseteq \mu^\vee(A_1) \overset{\bullet}{+} \mu^\vee(A_2)$, for every $A_1, A_2 \in \mathcal{C}$, $A_1 \cap A_2 = \emptyset$ because if we consider $x \in \underset{B \subset A_1 \cup A_2,\, B \in \mathcal{C}}{\cup} \mu(B)$, then there exists $B \in \mathcal{C}$, $B \subset A_1 \cup A_2$ such that x$\in \mu(B)$. Since $B \subset A_1 \cup A_2$, there exist $B_1, B_2 \in \mathcal{C}$, $B_1 \subset A_1, B_2 \subset A_2$, $B_1 \cap B_2 = \emptyset$, $B = B_1 \cup B_2$. Then $x \in \mu(B) \subseteq \mu(B_1) \overset{\bullet}{+} \mu(B_2) \subseteq \underset{B \subset A_1,\, B \in \mathcal{C}}{\cup} \mu(B) \overset{\bullet}{+} \underset{B \subset A_2,\, B \in \mathcal{C}}{\cup} \mu(B) \subseteq \mu^\vee(A_1) \overset{\bullet}{+}$ $\mu^\vee(A_2)$, which yields $\overline{\underset{B \subset A_1 \cup A_2,\, B \in \mathcal{C}}{\cup} \mu(B)} = \mu^\vee(A_1 \cup A_2) \subseteq \mu^\vee(A_1) \overset{\bullet}{+} \mu^\vee(A_2)$.

We prove now that $\mu^\vee$ is not o-continuous. If, on the contrary, $\mu^\vee$ is o-continuous, from the fact that $\mu(A) = \overline{\mu(A)} \subseteq \mu^\vee(A)$, we deduce that μ is o-continuous, contradiction.

Also, $\mu^\vee$ is exhaustive. Indeed, because μ is exhaustive, from proposition 3.8 [5] we know that the multivalued set function $A \in \mathcal{C} \to \underset{B \subset A,\, B \in \mathcal{C}}{\cup} \mu(B)$ is exhaustive. It is then quite obvious then also $\mu^\vee$ is exhaustive.

For instance, let particularly, $T = \{x_n\}_{n \in N}$ (where $x_n \in R$, for every $n \in N$) be a countable set, the algebra of subsets of T, $\mathcal{C}$ =$\{A \subset T, A$ finite or cA finite$\}$ and $\nu : \mathcal{C} \to R_+$ defined by:

$$\nu(A) = \begin{cases} 0, & A \text{ finite} \\ 1, & cA \text{ finite} \end{cases}, \text{for every } A \in \mathcal{C}.$$

Then ν is finite additive, exhaustive and it is not o-continuous. Let us define as before, the multivalued set function $\mu : \mathcal{C} \to \mathcal{P}_f(R_+)$,

$$\mu(A) = \begin{cases} \{0\}, & A \text{ finite} \\ \{1\}, & cA \text{ finite} \end{cases}, \text{for every } A \in \mathcal{C}.$$

We observe that the associated multivalued set function $\mu^\vee : \mathcal{C} \to \mathcal{P}_f(R_+)$ is defined by:

$$\mu^\vee(A) = \begin{cases} \{0\}, & A \text{ finite} \\ \{0,1\}, & cA \text{ finite} \end{cases}, \text{for every } A \in \mathcal{C}.$$

It is easy to prove that $\mu^\vee$ is a multisubmeasure. We show that $\mu^\vee$ is not o-continuous. Indeed, let us consider the sequence $(A_n)_n \subset \mathcal{C}$, $A_n = T\backslash\{x_1, ...x_n\}$. Then for every $n \in N$, $A_n \searrow \emptyset$ and $\mu^\vee(A_n) = \{0, 1\}$. Then $\lim |\mu^\vee(A_n)| = |\{0, 1\}| = 1 \neq 0$.

We prove now that $\mu^\vee$ is exhaustive. Let $(B_n)_n \subset \mathcal{C}$, $B_n \cap B_m = \emptyset$, $n \neq m$. We observe that there can not exist more than one set B_{n_0} in $\mathcal{C}$, so that cB_{n_0} is finite. Then B_n is finite for every $n > n_0$, which implies that $\lim |\mu^\vee(B_n)| = 0$.

In the following, taking as starting point the definition of σ-subadditivity for submeasures taking values in $\overline{R}_+$, we introduce here, for the case of multisubmeasures, a special type of σ-subadditivity with respect to the Hausdorff pseudometric h, which will be called h-σ-subadditivity. We prove that the h-σ-subadditivity of a multisubmeasure μ is equivalent to the σ-additivity of its variation $\bar{\mu}$. Then, using this result and some ideas from [2], [3] and [6] (for real valued submeasures), we establish the relationships between o-continuity, h-σ-subadditivity, exhaustivity and increasing convergence.

Definition 3.10. A multisubmeasure $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is said to be ***h*-*σ*-subadditive** if

$$\text{(15)} \quad \begin{array}{c} \text{for every disjoint sequence } (A_n)_{n\in N^*} \subset \mathcal{C} \text{ , with } \bigcup\limits_{n=1}^{\infty} A_n \in \mathcal{C} \text{ ,} \\ \text{we have } |\mu(\bigcup\limits_{n=1}^{\infty} A_n)| \leq \sum\limits_{n=1}^{\infty} |\mu(A_n)|. \end{array}$$

It is easy to prove the following statement:

Theorem 3.11. *If a multisubmeasure* $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ *is o-continuous, then* μ *is* h-σ*-subadditive.*

The following theorem will be very useful in the sequel:

Theorem 3.12. *Let* $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ *be a multisubmeasure. Then* μ is h-σ*-subadditive if and only if* $\bar{\mu}$ *is* σ*-additive on* $\mathcal{C}$ *.*

Proof. The "only if part": Since $|\mu(A)| \leq \bar{\mu}(A)$, for every $A \in C$ and $\bar{\mu}$ is σ-additive (equivalently, o-continuous) on C, we get that μ is o-continuous. Then, from theorem 3.11., μ is h-σ-subadditive.

The "if part": Suppose that μ is h-σ-subadditive. This means that $|\mu(\overset{\infty}{\underset{n=1}{\cup}} A_n)| \leq \sum\limits_{n=1}^{\infty} |\mu(A_n)|$, for every disjoint sequence $(A_n)_n \subset C$, with $\overset{\infty}{\underset{n=1}{\cup}} A_n \in C$. We prove that $\bar{\mu}$ is σ-subadditive and σ-superadditive. Indeed, it is well-known (see, for instance, [1]) that any real valued, finite additive set function is σ-superadditive. Consequently, $\bar{\mu}$ is σ-superadditive. It remains to demonstrate the σ-subadditivity of $\bar{\mu}$. For this, let $(B_j)_{j=\overline{1,m}}$ be a finite family of disjoint sets of C, which is contained in $\overset{\infty}{\underset{n=1}{\cup}} A_n$. Then, since $B_j = \overset{\infty}{\underset{n=1}{\cup}} (A_n \cap B_j)$ and $(A_n \cap B_j)_n$ are disjoint sets of C, for every $j = \overline{1,m}$, we have that

$$\sum_{j=1}^{m} |\mu(B_j)| = \sum_{j=1}^{m} |\mu(\overset{\infty}{\underset{n=1}{\cup}} (A_n \cap B_j))| \leq \sum_{j=1}^{m} \sum_{n=1}^{\infty} |\mu(A_n \cap B_j)| =$$

$$= \sum_{n=1}^{\infty} \sum_{j=1}^{m} |\mu(A_n \cap B_j)| \leq \sum_{n=1}^{\infty} \overline{\mu}(A_n).$$

We obtain that $\overline{\mu}(\overset{\infty}{\underset{n=1}{\cup}} A_n) \leq \sum\limits_{n=1}^{\infty} \overline{\mu}(A_n)$, as claimed. □

Corollary 3.13. *Let* $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ *be a multisubmeasure. Then* μ *is o-continuous if and only if it is* h-σ*-subadditive.*

Proof. For the "if part", we use theorem 3.11. It remains to prove the "only if part": If μ is h-σ-subadditive, then $\overline{\mu}$ is σ-additive, equivalently, o-continuous on $\mathcal{C}$. This yields the o-continuity of μ, as claimed. □

We also easily get the following:

Theorem 3.14. *Suppose that $\mathcal{C}$ is a σ- ring. If the multisubmeasure $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is o-continuous, then μ is exhaustive.*

Theorem 3.15. *Let $\mathcal{C}$ be a ring and $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ a multisubmeasure. If μ is exhaustive and increasing convergent, then μ is o-continuous.*

Proof. Let $A_n \searrow \emptyset$. From the exhaustivity of μ we get that for every $\varepsilon > 0$, there exists $n_0(\varepsilon) \in N$ such that $|\mu(A_n \Delta A_m)| < \frac{\varepsilon}{2}$, for every $m, n \geq n_0$. Particularly, $|\mu(A_{n_0} \backslash A_n)| < \frac{\varepsilon}{2}$, for every $n \geq n_0$. But $A_{n_0} \backslash A_n \nearrow A_{n_0}$, μ is increasing convergent and, therefore, there exists $n_1(\varepsilon) \in N$ such that $h(\mu(A_{n_0} \backslash A_n), \mu(A_{n_0})) < \frac{\varepsilon}{2}$, for every $n \geq n_1$. Then $|\mu(A_{n_0})| < \varepsilon$. But $A_n \subset A_{n_0}$, for every $n \geq n_0$, which yields $|\mu(A_n)| < \varepsilon$, as claimed. $\square$

Corollary 3.16. *Let $\mathcal{C}$ be a σ-ring and $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ a multisubmeasure. Then μ is o-continuous if and only if μ is exhaustive and increasing convergent.*

In fact, using some arguments of Gaina [6] for submeasures taking values in R_+, we may obtain for the particular case when $\mathcal{C}$ is the σ-ring generated by a δ-ring $\mathcal{C}_1 \subset \mathcal{C}$, the following important result.

Theorem 3.17. *Suppose C is the σ-ring generated by a δ-ring $C_1 \subset C$ and $\mu : C \to P_f(X)$ is an increasing convergent multisubmeasure. Then μ is h-σ-subadditive.*

Proof. Let $\varepsilon > 0$ and $(A_n)_n \subset \mathcal{C}$, $A = \bigcup\limits_{n=1}^{\infty} A_n \in \mathcal{C}$. We prove that $|\mu(A)| \leq \sum\limits_{n=1}^{\infty} |\mu(A_n)|$. Since $(A_n)_n \subset \mathcal{C}$ which is the σ-ring generated by a δ-ring, we get that for every $n \geq 1$, there exists a sequence $(B_k^n) \subset \mathcal{C}_1$, $B_k^n \nearrow A_n$. So, $A_n = \bigcup\limits_{k=1}^{\infty} B_k^n$, for every $n \geq 1$. Let us denote $C_k = \bigcup\limits_{n=1}^{\infty} B_k^n \in \mathcal{C}$, for every $k \in N$. Obviously, $C_k \nearrow A$. From this we get that there exists $k_0(\varepsilon) \in N$ such that $h(\mu(C_k), \mu(A)) < \frac{\varepsilon}{2}$, for every $k \geq k_0$. On the other hand, we have $|\mu(\bigcup\limits_{i=1}^{n} B_k^i)| \leq \sum\limits_{i=1}^{n} |\mu(B_k^i)| \leq \sum\limits_{n=1}^{\infty} |\mu(B_k^n)| \leq \sum\limits_{n=1}^{\infty} |\mu(A_n)|$, for every $k \in N$. But $\bigcup\limits_{i=1}^{n} B_k^i \nearrow C_k$, for every $k \in N$, so, there exists $n_1 \in$

N such that $h(\mu(C_k), \mu(\bigcup_{i=1}^{n} B_k^i)) < \frac{\varepsilon}{2}$, for every $n \geq n_1$ and $k \in N$. We obtain that $|\mu(C_k)| < \sum_{n=1}^{\infty} |\mu(A_n)| + \frac{\varepsilon}{2}$, for every $k \in N$, and, therefore, $|\mu(A)| < \sum_{n=1}^{\infty} |\mu(A_n)| + \varepsilon$, for every $\varepsilon > 0$. This means $|\mu(A)| \leq \sum_{n=1}^{\infty} |\mu(A_n)|$, as claimed. □

Corollary 3.18. *Let $\mathcal{C}$ be the σ-ring generated by a δ-ring $\mathcal{C}_1 \subset \mathcal{C}$ and $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ a multisubmeasure. Then μ is o-continuous if and only if it is increasing convergent.*

Corollary 3.19. *Let $\mathcal{C}$ be the σ-ring generated by a δ-ring $\mathcal{C}_1 \subset \mathcal{C}$ and $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ a multisubmeasure. The following statements are equivalent:*

i) μ is o-continuous.

ii) μ is decreasing convergent.

iii) μ is increasing convergent.

iv) μ is h-σ-subadditive.

It is also easy to remark the following o-continuity and exhaustivity relationships between μ and $\overline{\mu}$ (respectively, $\hat{\mu}$):

Theorem 3.20. *Let $\mathcal{C}$ be a ring and $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ a multisubmeasure. Then:*

i) *$\bar{\mu}$ is o-continuous on $\mathcal{C}$ if and only if μ is o-continuous.*

ii) *If $\overline{\mu}$ is exhaustive on $\mathcal{C}$, then μ is exhaustive.*

Theorem 3.21. *Let $\mathcal{C}$ be a ring and $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ a multisubmeasure. Then:*

i) *μ is exhaustive if and only if $\hat{\mu}$ is exhaustive on $\mathcal{C}$.*

ii) *μ is o-continuous if and only if $\hat{\mu}$ is o-continuous on $\mathcal{C}$.*

Remark 3.22. We have observed in remarks 2.9.I) that if $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is an exhaustive multisubmeasure and X a Banach space, then $(\mu(A_n))_n$ is convergent with respect to h, for every monotone sequence $(A_n)_n \subset \mathcal{C}$. Although, we can not say that μ is increasing convergent because its limit is not $\mu(A)$. More precisely, we observed that the multisubmeasure $\mu^{\vee}$ from example 3.9. is exhaustive but one can easily get that it is not increasing convergent. Indeed, there exists the sequence $A_n = \{x_1, x_2, ...x_n\} \subset \mathcal{C}$ such that $\mathrm{A}_n \nearrow T = \{x_1, x_2, ...x_n...\} \in \mathcal{C}$, but $\mu^{\vee}(A_n) = \{0\}$, for every $n \in N$, $\mu^{\vee}(T) = \{0,1\}$, hence $h(\mu^{\vee}(A_n), \mu^{\vee}(T)) = |\{0,1\}| = 1 \neq 0$. Consequently, there exist multisubmeasures which are exhaustive but not increasing convergent. We also observe from example 3.7. that the multisubmeasure μ introduced there is o-continuous, hence increasing convergent but not exhaustive, (even if X is a Banach space).

In fact, using some ideas of Pap in [7] for semigroup valued additive set functions we prove that exhaustivity implies increasing convergence on a special subset of a σ-algebra $\mathcal{C}$.

Theorem 3.23. *Let $\mathcal{C}$ be a σ-algebra of subsets of T and $\mu : \mathcal{C} \to P_f(X)$ an exhaustive multisubmeasure. Then μ has the following property:*

(16) *every disjoint sequence $(A_n)_{n\in N} \subset \mathcal{C}$ contains a subsequence (A_{k_n}) such that μ is increasing convergent on the $\sigma-$ algebra $\mathcal{C}_0$ generated by (A_{k_n}).*

Proof. Let $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ be an exhaustive multisubmeasure and $(A_n)_{n\in N} \subset \mathcal{C}$ a disjoint sequence. From theorem 3.21., we know that $\widehat{\mu}$ is an exhaustive and subadditive set function on $\mathcal{C}$. From Drewnowski's lemma [4] and Pap's theorem 1 [7], there exists a subsequence (A_{k_n}) of (A_n) such that $\widehat{\mu}$ is o- continuous on the σ- algebra $\mathcal{C}_0$ generated by (A_{k_n}). Let $(C_n)_{n\in N} \subset \mathcal{C}_0, C_n \nearrow C = \overset{\infty}{\underset{n=1}{\cup}} C_n$. We prove that $\lim h(\mu(C_n), \mu(C)) = 0$.

Indeed, we have $e(\mu(C_n), \mu(C)) = 0$, and

$$e(\mu(C), \mu(C_n)) \leq e(\mu(C), \mu(\bigcup_{k=1}^{n} C_k) \overset{\bullet}{+} \mu(\bigcup_{k=n+1}^{\infty} C_k))+$$

$$+e(\mu(\bigcup_{k=1}^{n} C_k) \overset{\bullet}{+} \mu(\bigcup_{k=n+1}^{\infty} C_k), \mu(C_n)) =$$

$$= e(\mu(C_n) \overset{\bullet}{+} \mu(\bigcup_{k=n+1}^{\infty} C_k), \mu(C_n)) \leq |\mu(\bigcup_{k=n+1}^{\infty} C_k)| = \widehat{\mu}(\bigcup_{k=n+1}^{\infty} C_k).$$

From the o-continuity of $\widehat{\mu}$ we get that $\lim h(\mu(C_n), \mu(C)) = 0$, as claimed. □

Corollary 3.24. *Let $\mathcal{C}$ be a $\sigma-$ algebra of subsets of T and $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ a multisubmeasure. We suppose that μ is exhaustive on $\mathcal{C}$. Then μ is o-continuous on $\mathcal{C}_0$.*

Proof. μ is exhaustive on $\mathcal{C}_0 \subset \mathcal{C}$ and, from theorem 3.23., it is also increasing convergent on $\mathcal{C}_0$. From theorem 3.15., μ is o-continuous on $\mathcal{C}_0$.□

Remark 3.25. The exhaustivity of μ on $\mathcal{C}$ implies the o-continuity of μ on $\mathcal{C}_0$, which in turn implies the exhaustivity of μ on $\mathcal{C}_0$.

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