

EINSTEIN QUASI-ANTI-HERMITIAN STRUCTURES ON THE TANGENT BUNDLE*

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Abstract. We study some geometric properties of the quasi-anti-Kählerian structures and the conformally anti-Kählerian structures (G, J) of natural type on the tangent bundle TM of a Riemannian manifold (M, g) . Characterization of these structures have been obtained in [12]. We obtain the necessary and sufficient conditions for these structures to be Einstein.

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1. Introduction. Let (M, g) be an n -dimensional Riemannian manifold and denote by $\tau : TM \rightarrow M$ its tangent bundle. There are several Riemannian and semi-Riemannian metrics induced on TM from the Riemannian metric g on M . Among them, we may quote the Sasaki metric and the complete lift of the metric g . On the other hand, there are the natural lifts of g to TM , leading to several new geometric structures with many nice geometric properties (see [3], [4], [15]).

In [12] we have considered an almost anti-Hermitian structure (G, J) , defined on TM by using some natural lifts of the Riemannian metric g . The vertical distribution VTM and the horizontal distribution HTM are interchanged by the considered almost complex structure J and the metric G is a natural lift of g of Sasaki type having the property to be anti-Hermitian with respect to J . We have studied the conditions under which

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the considered almost anti-Hermitian structure (TM, G, J) belongs to one from eight classes obtained in [2].

In this paper we consider the class of quasi-anti-Kählerian and the class of conformally anti-Kählerian structures (G, J) on the tangent bundle TM of a Riemannian manifold (M, g) , characterized in [12], and we obtain necessary and sufficient conditions under which these structures are Einstein.

The manifolds, tensor fields and other geometric objects we consider in this paper are assumed to be differentiable of class C^∞ (i.e. smooth). We use the computations in local coordinates in a fixed local chart but many results may be expressed in an invariant form by using the vertical and horizontal lifts. The well known summation convention is used throughout this paper, the range of the indices h, i, j, k, l being always $\{1, \dots, n\}$.

2. An almost anti-Hermitian structure on TM . Let (M, g) be a smooth n -dimensional Riemannian manifold and denote its tangent bundle by $\tau : TM \rightarrow M$. Recall that TM has a structure of a $2n$ -dimensional smooth manifold, induced from the structure of smooth n -dimensional manifold of M . From every local chart $(U, \varphi) = (U, x^1, \dots, x^n)$ on M it is induced a local chart $(\tau^{-1}(U), \Phi) = (\tau^{-1}(U), x^1, \dots, x^n, y^1, \dots, y^n)$, on TM , as follows. For a tangent vector $y \in \tau^{-1}(U) \subset TM$, the first local coordinates x^1, \dots, x^n are the local coordinates x^1, \dots, x^n of its base point $x = \tau(y)$ in the local chart (U, φ) (in fact we made an abuse of notation, identifying x^i with $\tau^*x^i = x^i \circ \tau$, $i = 1, \dots, n$). The last n local coordinates y^1, \dots, y^n of $y \in \tau^{-1}(U)$ are the vector space coordinates of y with respect to the natural basis $((\frac{\partial}{\partial x^1})_{\tau(y)}, \dots, (\frac{\partial}{\partial x^n})_{\tau(y)})$, defined by the local chart (U, φ) , i.e. $y = y^i (\frac{\partial}{\partial x^i})_{\tau(y)}$. Due to this special structure of differentiable manifold for TM , it is possible to introduce the concept of M -tensor field on it. The M -tensor fields are defined by their components with respect to the induced local charts on TM (hence they are defined locally), but they can be interpreted as some (partial) usual tensor fields on TM . However, the essential quality of an M -tensor field on TM is that the local coordinate change rule of its components with respect to the change of induced local charts is the same with the local coordinate change rule of the components of a usual tensor field on M with respect to the change of local charts on M . More precisely, an M -tensor field of type (p, q) on TM is defined by sets of n^{p+q} components (functions depending on x^i and y^i), with p upper indices and q lower indices, assigned to induced local charts $(\tau^{-1}(U), \Phi)$ on TM , such that the local coordinate change rule of these components

(with respect to induced local charts on TM) is that of the local coordinate components of a tensor field of type (p, q) on the base manifold M (with respect to usual local charts on M), when a change of local charts on M (and hence on TM) is performed (see [5], [10], [12] for further details).

We shall use the horizontal distribution HTM , defined by the Levi Civita connection $\dot{\nabla}$ of g , in order to define some natural lifts to TM of the Riemannian metric g on M . Denote by $VTM = \text{Ker } \tau_* \subset TTM$ the vertical distribution on TM . Then we have the direct sum decomposition

$$(1) \quad TTM = VTM \oplus HTM.$$

If $(\tau^{-1}(U), \Phi) = (\tau^{-1}(U), x^1, \dots, x^n, y^1, \dots, y^n)$ is a local chart on TM , induced from the local chart $(U, \varphi) = (U, x^1, \dots, x^n)$, the local vector fields $\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}$ on $\tau^{-1}(U)$ define a local frame for VTM over $\tau^{-1}(U)$ and the local vector fields $\frac{\delta}{\delta x^1}, \dots, \frac{\delta}{\delta x^n}$ define a local frame for HTM over $\tau^{-1}(U)$, where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma_{0i}^h \frac{\partial}{\partial y^h}, \quad \Gamma_{0i}^h = y^k \Gamma_{ki}^h$$

and $\Gamma_{ki}^h(x)$ are the Christoffel symbols of g .

The set $(\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}, \frac{\delta}{\delta x^1}, \dots, \frac{\delta}{\delta x^n})$ defines a local frame on TM , adapted to the direct sum decomposition (1). Remark that

$$\frac{\partial}{\partial y^i} = (\frac{\partial}{\partial x^i})^V, \quad \frac{\delta}{\delta x^i} = (\frac{\partial}{\partial x^i})^H,$$

where X^V and X^H denote the vertical and horizontal lifts of the vector field X on M .

Let $C = y^i \frac{\partial}{\partial y^i}$ be the Liouville vector field on TM and consider the horizontal vector field $\tilde{C} = y^i \frac{\delta}{\delta x^i}$ on TM , defined in a similar way.

Since we work in a fixed local chart (U, φ) on M and in the corresponding induced local chart $(\tau^{-1}(U), \Phi)$ on TM , we shall use the following simpler notations

$$\frac{\partial}{\partial y^i} = \partial_i, \quad \frac{\delta}{\delta x^i} = \delta_i.$$

Denote by

$$(2) \quad t = \frac{1}{2} \|y\|^2 = \frac{1}{2} g_{\tau(y)}(y, y) = \frac{1}{2} g_{ik}(x) y^i y^k, \quad y \in \tau^{-1}(U)$$

the energy density defined by g in the tangent vector y . We have $t \in [0, \infty)$ for all $y \in TM$. Consider the real valued smooth functions a_1, a_2, b_1, b_2 defined on $[0, \infty) \subset \mathbf{R}$ and define a natural almost complex structure J on TM , by using these coefficients and the Riemannian metric g , just like the 1-st order natural lifts of g to TM are obtained in [3]. The expression of J is given by (see [6], [16])

$$(3) \quad \begin{cases} JX_y^H = a_1(t)X_y^V + b_1(t)g_{\tau(y)}(y, X)C_y, \\ JX_y^V = -a_2(t)X_y^H - b_2(t)g_{\tau(y)}(y, X)\tilde{C}_y. \end{cases}$$

The expression of J in adapted local frames is given by

$$J\delta_i = a_1(t)\partial_i + b_1(t)g_{0i}C,$$

$$J\partial_i = -a_2(t)\delta_i - b_2(t)g_{0i}\tilde{C}.$$

Proposition 1. *The operator J defines an almost complex structure on TM if and only if*

$$(4) \quad a_1a_2 = 1, \quad (a_1 + 2tb_1)(a_2 + 2tb_2) = 1.$$

Proof. The relations are obtained easily from the property $J^2 = -I$ of J and Lemma 1.

Remark. From the conditions (4) we have that the coefficients $a_1, a_2, a_1 + 2tb_1, a_2 + 2tb_2$ cannot vanish and have the same sign. We assume that $a_1 > 0, a_2 > 0, a_1 + 2tb_1 > 0, a_2 + 2tb_2 > 0$ for all $t \geq 0$.

The integrability problem for the almost complex structure J has been studied in [6], [7], [8] by finding the conditions under which the Nijenhuis tensor field N_J of J does vanish.

Theorem 2. *Let (M, g) be an $n(> 2)$ -dimensional connected Riemannian manifold. The almost complex structure J defined by (3) on TM is integrable if and only if (M, g) has constant sectional curvature c and the function b_1 is given by*

$$(5) \quad b_1 = \frac{a_1a_1' - c}{a_1 - 2ta_1'}.$$

Remark. The relations (4) allow us to express two of the coefficients a_1, a_2, b_1, b_2 as functions of the other two; e.g. we have

$$(6) \quad a_2 = \frac{1}{a_1}, \quad b_2 = \frac{-a_2 b_1}{a_1 + 2tb_1} = \frac{-b_1}{a_1(a_1 + 2tb_1)}.$$

Remark. In the case where the almost complex structure J is integrable, we can get the expression of b_2 as a function of a_1 and its first order derivative

$$b_2 = \frac{c - a_1 a_1'}{a_1(a_1^2 - 2ct)}$$

(compare with the corresponding expressions from [6] and [16]).

Now, we consider the following particular 1-st order natural lift G of g to TM , defined by four real valued smooth functions $c_1, d_1, c_2, d_2 : [0, \infty) \rightarrow \mathbf{R}$, given by

$$(7) \quad \begin{cases} G_y(X^H, Y^H) = c_1(t)g_{\tau(y)}(X, Y) + d_1(t)g_{\tau(y)}(y, X)g_{\tau(y)}(y, Y) \\ G_y(X^V, Y^V) = c_2(t)g_{\tau(y)}(X, Y) + d_2(t)g_{\tau(y)}(y, X)g_{\tau(y)}(y, Y) \\ G_y(X^H, Y^V) = G_y(Y^V, X^H) = 0. \end{cases}$$

The expression of G in local adapted frames is defined by the expressions

$$G_{ij}^{(1)} = G(\delta_i, \delta_j) = c_1 g_{ij} + d_1 g_{0i} g_{0j}, \quad G_{ij}^{(2)} = G(\partial_i, \partial_j) = c_2 g_{ij} + d_2 g_{0i} g_{0j},$$

$$G(\partial_i, \delta_j) = G(\delta_i, \partial_j) = 0$$

The conditions for G to be nondegenerate are assured if

$$(8) \quad c_1 c_2 \neq 0, \quad (c_1 + 2td_1)(c_2 + 2td_2) \neq 0.$$

The conditions for G to be anti-Hermitian with respect to J , are obtained from the relation

$$(9) \quad G(JX, JY) = -G(X, Y),$$

for all vector fields X, Y on TM (see [2]). Such a metric is called, sometimes, a Norden metric.

The property (9) of the semi-Riemannian metric G , implies that it has the signature (n, n) . Using the expressions in the adapted local frames we

can obtain the explicit conditions for G to be almost anti-Hermitian. We have

Proposition 3. *The semi-Riemannian metric G is anti-Hermitian with respect to the almost complex structure J on (TM) , i.e. (TM, G, J) is an almost anti-Hermitian manifold, if and only if the coefficients c_1, c_2, d_1, d_2 satisfy the following relations*

$$(10) \quad a_2c_1 + a_1c_2 = 0, \quad (a_1 + 2tb_1)(c_2 + 2td_2) + (a_2 + 2tb_2)(c_1 + 2td_1) = 0.$$

Remark. From the conditions for (TM, G, J) to be almost anti - Hermitian manifold, we get the following relations

$$(11) \quad \begin{cases} a_2 = \frac{1}{a_1}, & b_2 = -\frac{b_1}{a_1(a_1+2tb_1)} \\ c_2 = -\frac{c_1}{a_1^2}, & d_2 = \frac{2a_1b_1c_1+2tb_1^2c_1-a_1^2d_1}{a_1^2(a_1+2tb_1)^2}. \end{cases}$$

It follows that only the coefficients a_1, b_1, c_1, d_1 can be considered as being essential.

Next the Levi Civita connection of G can be expressed in the considered adapted local frame.

Proposition 4. *The Levi Civita connection ∇ of the pseudo - Riemannian metric G on TM has the following expression in the local adapted frame $(\partial_i, \dots, \partial_n, \delta_i, \dots, \delta_n)$*

$$\nabla_{\partial_i}\partial_j = Q_{ij}^h\partial_h, \quad \nabla_{\delta_i}\partial_j = \Gamma_{ij}^h\partial_h + P_{ji}^h\delta_h,$$

$$\nabla_{\partial_i}\delta_j = P_{ij}^h\delta_h, \quad \nabla_{\delta_i}\delta_j = \Gamma_{ij}^h\delta_h + S_{ij}^h\partial_h,$$

where the M -tensor fields $P_{ij}^h, Q_{ij}^h, S_{ij}^h$ are given by

$$\begin{aligned} P_{ji}^h &= \frac{c'_1}{2c_1}g_{0j}\delta_i^h + \frac{d_1}{2c_1}g_{0i}\delta_j^h - \frac{c'_1d_1 + (d_1)^2 - c_1d'_1}{2c_1(c_1 + 2td_1)}g_{0i}g_{0j}y^h + \\ &\quad \frac{d_1}{2(c_1 + 2td_1)}g_{ij}y^h - \frac{c_2}{2c_1}R_{ij0}^h - \frac{c_2d_1}{2c_1(c_1 + 2td_1)}R_{j0i0}y^h, \\ Q_{ij}^h &= \frac{c'_2}{2c_2}(g_{0i}\delta_j^h + g_{0j}\delta_i^h) + \frac{2d_2 - c'_2}{2(c_2 + 2td_2)}g_{ij}y^h + \frac{c_2d'_2 - 2d_2c'_2}{2c_2(c_2 + 2td_2)}g_{0i}g_{0j}y^h, \end{aligned}$$

$$S_{ij}^h = -\frac{d_1}{2c_2}(g_{0i}\delta_j^h + g_{0j}\delta_i^h) - \frac{c_1'}{2(c_2 + 2td_2)}g_{ij}y^h + \\ + \frac{2d_1d_2 - c_2d_1'}{2c_2(c_2 + 2td_2)}g_{0i}g_{0j}y^h - \frac{1}{2}R_{0ij}^h,$$

R_{likj} denoting the local coordinate components of the Riemann-Christoffel tensor of $\dot{\nabla}$ on M and $R_{0ikj} = R_{likj}y^l$, $R_{0ij0} = R_{likj}y^l y^k$.

3. A quasi-anti-Kählerian Einstein structure on TM . The conditions under which the almost anti-Hermitian structure (G, J) on TM , considered in the previous section, belongs to each from the eight classes of anti-Hermitian manifolds obtained in the classification in [2] have been obtained by the present authors in [12]. We recall here three of the classes of anti-Hermitian manifolds obtained in [2], [12]. For this aim, we consider the tensor field F of type (0,3) defined by

$$F(X, Y, Z) = G((\nabla_X J)Y, Z), \quad X, Y, Z \in \Gamma(TM),$$

and introduce the 1-form ϕ , associated with F defined by

$$\phi(X) = G^{ij}F(E_i, E_j, X), \quad X \in \Gamma(TM), \quad i, j = 1, \dots, 2n,$$

where (E_1, \dots, E_{2n}) is a local frame in TTM and G^{ij} are the entries of the inverse of the matrix (G_{ij}) associated to G in the local frame (E_1, \dots, E_{2n}) .

Then the considered almost anti-Hermitian structure (G, J) on TM is called:

An *anti-Kählerian structure* if

$$F(X, Y, Z) = 0,$$

or, equivalently, $\nabla J = 0$.

A *quasi-anti-Kählerian structure* if

$$F(X, Y, Z) + F(Y, Z, X) + F(Z, X, Y) = 0.$$

A *conformally anti-Kählerian structure* if

$$2nF(X, Y, Z) = G(X, Y)\phi(Z) + G(X, Z)\phi(Y) + \\ + G(X, JY)\phi(JZ) + G(X, JZ)\phi(JY).$$

In [12] we have obtained the necessary and sufficient conditions under which the anti-Hermitian structure (G, J) defined in previous section is in one of the eight classes given in classification from [2].

We recall here the obtained results concerning the three structures defined above. We have (see Theorems 7, 8 and 14 in [12]).

Theorem 5. (i) *The anti-Hermitian manifold (TM, G, J) is an anti-Kählerian manifold if and only if the base manifold (M, g) has constant sectional curvature c and the functions $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2$ are given by*

$$(12) \quad \begin{cases} a_1 = \sqrt{B + 2ct}, & b_1 = 0, & c_1 = A(B + 2ct), & d_1 = -cA \\ a_2 = \frac{1}{\sqrt{B+2ct}}, & b_2 = 0, & c_2 = -A, & d_2 = \frac{cA}{B+2ct}, \end{cases}$$

where A is a nonzero real constant and B is a positive constant.

(ii) *The anti-Hermitian manifold (TM, G, J) is a quasi-anti-Kählerian manifold if and only if (M, g) has constant sectional curvature c and the essential functions a_1, b_1, c_1, d_1 are solutions of the following differential system*

$$(13) \quad \begin{cases} a_1' = \frac{a_1 b_1 c_1 - c c_1 - 2a_1^2 d_1}{c_1 (a_1 + 2tb_1)} \\ c_1' = -\frac{2a_1 d_1}{a_1 + 2tb_1} \\ d_1' = -\frac{2b_1 d_1}{a_1 + 2tb_1}. \end{cases}$$

(iii) *The anti-Hermitian manifold (TM, G, J) is a conformally anti-Kählerian manifold if and only if (M, g) has constant sectional curvature c and the essential functions a_1, b_1, c_1, d_1 satisfy the relations*

$$(14) \quad \begin{cases} d_1 = -\frac{c_1 c}{a_1^2} \\ (a_1 + 2tb_1)a_1' = a_1 b_1 + c. \end{cases}$$

Remark that in the cases of anti-Kählerian and conformally anti-Kählerian manifolds the almost complex structure J is integrable (i.e. the relation (5) holds), but in the case of quasi-anti-Kählerian manifolds the almost complex structure J is, generally, not integrable.

For the anti-Kählerian manifolds given by assertion (i) from Theorem 5, we have proved that these manifolds are Einstein (see Theorem 6 in [13]).

In the following we study the necessary and sufficient conditions under which the quasi-anti-Kählerian manifolds defined above on TM are Einstein manifolds. For this aim, we assume that the assertion (ii) from Theorem 5 holds and we remark that from (13) we get easily $(c_1 + 2td_1)' = 0$. Hence we must have $c_1 + 2td_1 = B$, where B is a nonzero real constant, i.e. $c_1 + 2td_1$ is a prime integral of the system (13). Taking into account (13), we obtain by a direct computation that the expressions of the M -tensor fields P_{ij}^h , Q_{ij}^h , S_{ij}^h which appear in Proposition 4 become

$$\begin{aligned}
P_{ij}^h &= -\frac{a_1 d_1}{c_1(a_1 + 2tb_1)} g_{0i} \delta_j^h + \frac{cc_1 + a_1^2 d_1}{2a_1^2 c_1} g_{0j} \delta_i^h + \frac{a_1^2 d_1 - cc_1}{2a_1^2 (c_1 + 2td_1)} g_{ij} y^h + \\
&+ \frac{a_1^3 d_1^2 - 2a_1^2 b_1 c_1 d_1 - a_1 cc_1 d_1 - 2tb_1 cc_1 d_1 - 2ta_1^2 b_1 d_1^2}{2a_1^2 c_1 (a_1 + 2tb_1)(c_1 + 2td_1)} g_{0i} g_{0j} y^h, \\
Q_{ij}^h &= \frac{cc_1 + a_1^2 d_1 - a_1 b_1 c_1}{a_1 c_1 (a_1 + 2tb_1)} (g_{0i} \delta_j^h + g_{0j} \delta_i^h) - \frac{a_1 cc_1 + a_1^2 b_1 c_1 + 2tb_1 cc_1 + 2ta_1^2 b_1 d_1}{a_1^3 (c_1 + 2td_1)} g_{ij} y^h + \\
&+ \left[\frac{2a_1^3 b_1^2 c_1^2 - a_1^4 b_1' c_1^2 + a_1^2 b_1 cc_1^2 - 2a_1^4 b_1 c_1 d_1 - a_1^3 cc_1 d_1 + 2ta_1^2 b_1^3 c_1^2 - 2ta_1^3 b_1 b_1' c_1^2 + 6ta_1 b_1^2 cc_1^2}{a_1^3 c_1 (a_1 + 2tb_1)^2 (c_1 + 2td_1)} + \right. \\
&+ \frac{4ta_1^3 b_1^2 c_1 d_1 - 2ta_1^4 b_1' c_1 d_1 - 4ta_1^2 b_1 cc_1 d_1 - 4ta_1^4 b_1 d_1^2 + 4t^2 b_1^3 cc_1^2}{a_1^3 c_1 (a_1 + 2tb_1)^2 (c_1 + 2td_1)} + \\
&+ \left. \frac{4t^2 a_1^2 b_1^3 c_1 d_1 - 4t^2 a_1^3 b_1 b_1' c_1 d_1}{a_1^3 c_1 (a_1 + 2tb_1)^2 (c_1 + 2td_1)} \right] g_{0i} g_{0j} y^h, \\
S_{ij}^h &= \frac{cc_1 + a_1^2 d_1}{2c_1} g_{0i} \delta_j^h - \frac{cc_1 - a_1^2 d_1}{2c_1} g_{0j} \delta_i^h - \frac{(a_1 + 2tb_1)^2 (d_1 + td_1')}{c_1 + 2td_1} g_{ij} y^h + \\
&+ \frac{a_1 d_1 (b_1 c_1 - a_1 d_1)}{c_1 (c_1 + 2td_1)} g_{0i} g_{0j} y^h.
\end{aligned}$$

In order to obtain the necessary and sufficient conditions under which the considered quasi-anti-Kählerian manifold (TM, G, J) is an Einstein manifold, we use the curvature tensor field K of the Levi Civita connection ∇ of G , defined by

$$K(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z, \quad X, Y, Z \in \Gamma(TM).$$

Taking into account that the base manifold (M, g) has constant sectional

curvature c , we obtain by a straightforward computation

$$\left\{ \begin{array}{l} K(\delta_i, \delta_j)\delta_k = XX X_{kij}^h \delta_h, \quad K(\delta_i, \delta_j)\partial_k = XXY_{kij}^h \partial_h, \\ K(\partial_i, \partial_j)\delta_k = YY X_{kij}^h \delta_h, \quad K(\partial_i, \partial_j)\partial_k = YYY_{kij}^h \partial_h, \\ K(\partial_i, \delta_j)\delta_k = YX X_{kij}^h \partial_h, \quad K(\partial_i, \delta_j)\partial_k = YXY_{kij}^h \delta_h, \end{array} \right.$$

where $XX X_{kij}^h$, XXY_{kij}^h , $YY X_{kij}^h$, YYY_{kij}^h , $YX X_{kij}^h$, YXY_{kij}^h are components defining M -tensor fields of type $(1,3)$ on TM . Their explicit expressions are quite involved.

First, we obtain the components $XX X_{kij}^h$ and $YX X_{kij}^h$ in order to get the components of the Ricci tensor $S(\delta_j, \delta_k) = \text{trace}(X \rightarrow K(X, \delta_j)\delta_k)$. We have

$$\begin{aligned} XX X_{kij}^h &= \frac{cc_1^2 + 2tcc_1d_1 + 2ta_1^2d_1^2}{c_1(c_1 + 2td_1)}(g_{jk}\delta_i^h - g_{ik}\delta_j^h) + \\ &+ \frac{5a_1^4c_1d_1^2 - 3c^2c_1^3 - 2a_1^2cc_1^2d_1 - 6te^2c_1^2d_1 - 4ta_1^2cc_1d_1^2 + 2ta_1^4d_1^3}{4a_1^2c_1^2(c_1 + 2td_1)}(g_{0i}g_{0k}\delta_j^h - g_{0j}g_{0k}\delta_i^h) + \\ &+ \frac{3c^2c_1^2 - 2a_1^2cc_1d_1 - 5a_1^4d_1^2}{4a_1^2c_1(c_1 + 2td_1)}(g_{jk}g_{0i}y^h - g_{ik}g_{0j}y^h), \\ YX X_{kij}^h &= \frac{cc_1 + a_1^2d_1}{2c_1}g_{ij}\delta_k^h - \frac{d_1(a_1^2c_1 + 2tcc_1 + 2ta_1^2d_1)}{c_1(c_1 + 2td_1)}g_{jk}\delta_i^h + \\ &+ \frac{(a_1^2d_1 - cc_1)(a_1^2c_1 + tcc_1 + ta_1^2d_1)}{2a_1^2c_1(c_1 + 2td_1)}g_{ik}\delta_j^h + \frac{c^2c_1 - a_1b_1cc_1 - a_1^3b_1d_1 + a_1^2cd_1}{2a_1c_1(a_1 + 2tb_1)}g_{0i}g_{0j}\delta_k^h + \\ &+ \left[\frac{2a_1^2b_1cc_1^3 - a_1c^2c_1^3 - 2a_1^4b_1c_1^2d_1 - 6a_1^3cc_1^2d_1 - a_1^5c_1d_1^2 + 2tb_1c^2c_1^3 - 4ta_1c^2c_1^2d_1 - 2ta_1^4b_1c_1d_1^2}{4a_1^2c_1^2(a_1 + 2tb_1)(c_1 + 2td_1)} - \right. \\ &- \left. \frac{8ta_1^3cc_1d_1^2 + 4ta_1^5d_1^3}{4a_1^2c_1^2(a_1 + 2tb_1)(c_1 + 2td_1)} \right]g_{0i}g_{0k}\delta_j^h + \left[\frac{2a_1^2cc_1^2d_1 - c^2c_1^3 - a_1^4c_1d_1^2 - 2tc^2c_1^2d_1 - 4ta_1^2cc_1d_1^2}{4a_1^2c_1^2(c_1 + 2td_1)} - \right. \\ &- \left. \frac{2ta_1^4d_1^3}{4a_1^2c_1^2(c_1 + 2td_1)} \right]g_{0j}g_{0k}\delta_i^h + \frac{2(cc_1d_1 + a_1^2d_1^2)}{c_1(c_1 + 2td_1)}g_{0i}g_{jk}y^h + \left[\frac{2a_1^4b_1c_1d_1 - 2a_1^2b_1cc_1^2}{4a_1^3c_1(c_1 + 2td_1)} - \right. \\ &- \left. \frac{a_1c^2c_1^2 + 4a_1^3cc_1d_1 + 3a_1^5d_1^2 + 4tb_1c^2c_1^2 + 8ta_1^2b_1cc_1d_1 + 4ta_1^4b_1d_1^2}{4a_1^3c_1(c_1 + 2td_1)} \right]g_{0j}g_{ik}y^h + \\ &+ \left[\frac{a_1^2b_1cc_1^2 + a_1c^2c_1^2}{2a_1^3c_1(c_1 + 2td_1)} + \frac{a_1^4b_1c_1d_1 - a_1^5d_1^2 + 2tb_1c^2c_1^2 + 2ta_1^2b_1cc_1d_1}{2a_1^3c_1(c_1 + 2td_1)} \right]g_{0k}g_{ij}y^h + \end{aligned}$$

$$+ \left[\frac{a_1 c^2 c_1^2 d_1 - 4a_1^3 b_1^2 c_1^2 d_1 + 4a_1^4 b_1 c_1 d_1^2 + 4a_1^3 c c_1 d_1^2 + 3a_1^5 d_1^3 + 4ta_1 b_1^2 c c_1^2 d_1 + 2tb_1 c^2 c_1^2 d_1}{4a_1^2 c_1^2 (a_1 + 2tb_1)(c_1 + 2td_1)} + \right. \\ \left. + \frac{4ta_1^3 b_1^2 c_1 d_1^2 + 4ta_1^2 b_1 c c_1 d_1^2 + 2ta_1^4 b_1 d_1^3}{4a_1^2 c_1^2 (a_1 + 2tb_1)(c_1 + 2td_1)} \right] g_{0i} g_{0j} g_{0k} y^h.$$

By using the above expressions, we get

$$S(\delta_j, \delta_k) = Y X X_{kij}^i + X X X_{kij}^i = \left[\frac{a_1^4 c_1 d_1 - a_1^2 c c_1^2 + a_1^2 c c_1^2 n - a_1^4 c_1 d_1 n}{a_1^2 c_1 (c_1 + 2td_1)} + \right. \\ \left. + \frac{tc^2 c_1^2 + 2ta_1^2 c c_1 d_1 + ta_1^4 d_1^2}{a_1^2 c_1 (c_1 + 2td_1)} \right] g_{jk} + \left[\frac{2a_1^4 c_1 d_1^2 - 2c^2 c_1^3 - 4a_1^2 c c_1^2 d_1 + c^2 c_1^3 n}{2a_1^2 c_1^2 (c_1 + 2td_1)} + \right. \\ \left. + \frac{2a_1^2 c c_1^2 d_1 n - 3a_1^4 c_1 d_1^2 n - 2tc^2 c_1^2 d_1 + 2ta_1^4 d_1^3 + 2tc^2 c_1^2 d_1 n - 2ta_1^4 d_1^3 n}{2a_1^2 c_1^2 (c_1 + 2td_1)} \right] g_{0j} g_{0k}.$$

Denoting the coefficient of g_{jk} by α and the coefficient of $g_{0j} g_{0k}$ by β , the condition $S(\delta_j, \delta_k) = \kappa G(\delta_j, \delta_k)$, where κ is a constant, gives

$$(15) \quad \frac{\alpha}{c_1} = \frac{\beta}{d_1} \text{ and } \frac{\alpha}{c_1} = \kappa = \text{constant.}$$

Next, it follows that the first condition (15) is fulfilled if and only if we have

$$(16) \quad (a_1^2 d_1 + c c_1)(a_1^2 d_1 n - c c_1 n + 2c c_1) = 0.$$

From (16) we have that either $d_1 = -\frac{c c_1}{a_1^2}$ or $d_1 = \frac{c c_1 (n-2)}{n a_1^2}$. Assuming that $n > 2$ and $d_1 = \frac{c c_1 (n-2)}{n a_1^2}$, by direct computation we get that the second condition (15) is not satisfied.

Next, by taking into account that $c_1 + 2td_1 = B = \text{constant}$, we get that if

$$(17) \quad d_1 = -\frac{c c_1}{a_1^2},$$

both conditions (15) are fulfilled, and in this case we get $\kappa = \frac{2c(n-1)}{B}$. Therefore, (17) implies

$$(18) \quad S(\delta_j, \delta_k) = \frac{2c(n-1)}{B} G(\delta_j, \delta_k).$$

By taking the value of d_1 obtained in (17) we get the other components of the curvature tensor field K , which become simpler and are expressed by

$$\left\{ \begin{aligned} &YYY_{kij}^h = \frac{c}{a_1^2 - 2ct} (g_{ik}\delta_j^h - g_{jk}\delta_i^h) + \frac{2a_1b_1c + c^2 + 2tb_1^2c}{(a_1 + 2tb_1)^2(a_1^2 - 2ct)} (g_{0j}g_{0k}\delta_i^h - g_{0i}g_{0k}\delta_j^h), \\ &YXY_{kij}^h = \frac{c}{a_1^2 - 2ct} (g_{ik}\delta_j^h - g_{jk}\delta_i^h) - \frac{c(2a_1b_1 + c + 2tb_1^2)}{(a_1 + 2tb_1)^2(a_1^2 - 2ct)} g_{0i}g_{0k}\delta_j^h + \\ &\quad + \frac{c(a_1b_1 + c)}{a_1(a_1 + 2tb_1)(a_1^2 - 2ct)} g_{0j}g_{0k}\delta_i^h + \frac{b_1c}{(a_1 + 2tb_1)(a_1^2 - 2ct)} g_{0i}g_{jk}y^h - \\ &\quad - \frac{a_1b_1^2c + b_1c^2}{a_1(a_1 + 2tb_1)^2(a_1^2 - 2ct)} g_{0i}g_{0j}g_{0k}y^h, \\ &XXY_{kij}^h = \frac{a_1^2c}{a_1^2 - 2ct} (g_{jk}\delta_i^h - g_{ik}\delta_j^h) + \frac{a_1c(a_1b_1 + c)}{(a_1 + 2tb_1)(a_1^2 - 2ct)} (g_{0i}g_{0k}\delta_j^h - g_{0j}g_{0k}\delta_i^h) + \\ &\quad + \frac{a_1b_1c}{a_1^2 - 2ct} (g_{jk}g_{0i}y^h - g_{ik}g_{0j}y^h), \\ &YYX_{kij}^h = \frac{c}{a_1^2 - 2ct} (g_{ik}\delta_j^h - g_{jk}\delta_i^h) + \frac{c(a_1b_1 + c)}{a_1(a_1 + 2tb_1)(a_1^2 - 2ct)} (g_{0j}g_{0k}\delta_i^h - g_{0i}g_{0k}\delta_j^h) + \\ &\quad + \frac{b_1c}{(a_1 + 2tb_1)(a_1^2 - 2ct)} (g_{jk}g_{0i}y^h - g_{ik}g_{0j}y^h). \end{aligned} \right.$$

Next, by using (11) and (17) we obtain that the other components of the Ricci tensor S are

$$(19) \quad \left\{ \begin{aligned} &S(\partial_j, \partial_k) = -\frac{2c(n-1)}{a_1^2 - 2ct} g_{jk} + \left[\frac{4a_1b_1cn - 2c^2 - 4a_1b_1c}{(a_1 + 2tb_1)^2(a_1^2 - 2ct)} + \right. \\ &\quad \left. + \frac{2c^2n - 4tb_1^2c + 4tb_1^2cn}{(a_1 + 2tb_1)^2(a_1^2 - 2ct)} \right] g_{0j}g_{0k} = \frac{2c(n-1)}{B} G(\partial_j, \partial_k), \\ &S(\partial_j, \delta_k) = S(\delta_k, \partial_j) = G(\partial_j, \delta_k) = G(\delta_k, \partial_j) = 0. \end{aligned} \right.$$

From (18) and (19) it follows that

$$S(X, Y) = \frac{2c(n-1)}{B} G(X, Y), \quad X, Y \in \Gamma(TM).$$

Hence we have

Theorem 6. *The quasi-anti-Kählerian structure (G, J) on TM defined by (3), (7) is Einstein if and only if the condition (17) is satisfied.*

Remark. By using (17) we get that the differential system (13) is equivalent to

$$(20) \quad \begin{cases} (a_1 + 2tb_1)a'_1 = a_1b_1 + c \\ d_1 = -\frac{cc_1}{a_1^2} \\ c'_1 = \frac{2cc_1}{a_1(a_1+2tb_1)}. \end{cases}$$

Comparing (20) and (14) we obtain

Corollary 7. *The quasi-anti-Kählerian structure (G, J) on TM is Einstein if and only if it is a conformally anti-Kählerian structure too, or equivalently, if and only if the almost complex structure J is integrable.*

4. Some final remarks.

(i) In the particular case where $d_1 = c_1$, we get from (17) $a_1^2 = -c$. Then by using (20) and (11) we get

$$(21) \quad a_1 = b_1 = \sqrt{-c}, \quad c_1 = d_1 = \frac{B}{2t+1}, \quad c_2 = \frac{B}{c(2t+1)}, \quad d_2 = -\frac{B}{c(2t+1)^2}.$$

Hence we have

Corollary 8. *In the particular case when $c_1 = d_1$, the quasi-anti-Kählerian structure (G, J) is Einstein if and only if the base manifold (M, g) has negative constant curvature and the coefficients $a_1, b_1, c_1, d_1, c_2, d_2$ are given by (21).*

(ii) In the particular case when $b_1 = 0$, by using (20) and the assertion (i) from Theorem 5, we obtain

Corollary 9. *In the case when $b_1 = 0$, the quasi-anti-Kählerian structure (G, J) is Einstein if and only if it is an anti-Kählerian structure.*

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