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ABSOLUTELY CONTINUOUS SEMI–GROUPS

BY

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Abstract. In several papers [6], [7], [8], the role of absolutely continuous semi–groups or resolvents was used, for example in the construction of a tensor product of harmonic structure.

From the seminal papers of Boboc – Bucur – Cornea [2] the role of absolutely continuous resolvents in the theory of standard H–cones became well known. It was proved [1] that for each standard H–cone of functions, there exists a standard semi–group (i.e. with absolutely continuous associated resolvent) $\mathcal{P}$, such that the given H–cone coincide with the cone of $\mathcal{P}$–excessive functions.

The main result of this paper gives a sufficient (and "almost" necessary) condition on the resolvent, such that the associated semi–group be absolutely continuous.

Several results centered on the properties of a semi–group to be absolutely continuous, are also collected.

Let us mention that Mokobodzki [5] gives a different characterization for absolutely continuous semi–groups.

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1. Preliminaries. Let $\mathcal{V} = (V_\alpha)_{\alpha\geq 0}$, $\mathcal{V}^* = (V^*_\alpha)_{\alpha\geq 0}$ be sub–markovian resolvents of kernels on a measurable space $(X, \mathcal{X})$, absolutely continuous and in duality with respect to a positive, σ–finite measure μ. A theorem of Kunita–Watanabe [4] proves the existence of family $(v_\alpha)_{\alpha\geq 0}$ with the properties:

- v_α is a positive, numerical measurable function on $(X \times X, \mathcal{X} \otimes \mathcal{X})$, which is $\mathcal{V}$–excessive in the first variable and is $\mathcal{V}^*$–excessive in the second variable;
- $V_\alpha f(x) = \int v_\alpha(x, y) f(y)\mu(dy)$

- $V_\alpha^* f(x) = \int v_\alpha(y,x) f(y)\mu(dy)$, $\forall x \in X$, $\forall f \in \mathcal{F}$.

These properties determine uniquely the family v_α. Moreover, the following relation holds:

$$(*)\ \ v_\alpha(x,y) = v_\beta(x,y) + (\beta-\alpha)\int v_\alpha(x,z)v_\beta(z,y)\mu(dz),\ \forall \alpha < \beta$$

Let us denote by $A := \{(x,y)|\ v_0(x,y) = +\infty\}$; $A_x := \{y|\ (x,y) \in A\}$; $A^y := \{x|\ (x,y) \in A\}$. The sets A_x resp. A^x are (co–)polar, $\mathcal{V}$ and μ–negligible.

By recurrence, from the relation $(*)$ one obtains, for each $\alpha > 0$ and $(x,y) \notin A$:

$$v_\alpha(x,y) = v_{\alpha_0}(x,y) + \sum_{n=1}^{\infty}(-1)^n(\alpha-\alpha_0)a_n(x,y)$$

where:

$$a_n(x,y) = \underbrace{\int \dots \int}_{\text{k times}} v_{\alpha_0}(x,y_1)v_{\alpha_0}(y_1,y_2)\dots v_{\alpha_0}(y_n,y)\mu(dy_1)\dots\mu(dy_n) =$$

$$= V_{\alpha_0}^n\left[v_{\alpha_0}(\cdot,y)\right](x) = V_{\alpha_0}^{*n}\left[v_{\alpha_0}(x,\cdot)\right](y)$$

the convergence being granted for $|\alpha-\alpha_0| < \alpha_0$.

Definition. The semi–group $\mathcal{P} = (P_t)_{t>0}$ is called *absolutely continuous* if there exists a positive, σ–finite measure μ on X, such that

$$\mu(f) = 0 \Longrightarrow P_t f \equiv 0, \forall t > 0$$

The semi–group $\mathcal{P}$ is called *standard* if the associated resolvent is absolutely continuous.

Some sufficient conditions (formulated on the associated resolvent) for the fact that a standard semi–group is absolutely continuous are proposed in [8]. Unfortunately, it seems that there are no examples of semi–groups satisfying such conditions. We consider next some conditions, fulfilled by the usual semi–groups. See also the conditions formulated in [7].

Theorem 1. *Let us suppose that:*

(C) $\forall (x,y) \notin A \exists C(x,y) \in [0,+\infty)$, *such that the function* $\alpha \mapsto \dfrac{C(x,y)}{\alpha} - v_\alpha(x,y)$ *is completely monotone on* $(0,+\infty)$

There exists then a map $p_t(x,y)$, *defined a.e. on* $(0,+\infty)$ *for each* $(x,y) \notin A$, *such that*

$$v_\alpha(x,y) = \int_0^\infty e^{-\alpha t} p_t(x,y) dt$$

for each $\alpha > 0$ *and* $(x,y) \notin A$

Proof. Since the function $f(\alpha) := v_\alpha(x,y)$ is completely monotone on $(0,+\infty)$, as $f^{(k)}(\alpha) = (-1)^k k! a_n(x,y)$. Using Bernstein's theorem [9], there exists, for each $(x,y) \notin A$, a positive measure on $[0,+\infty]$, denoted by μ_{xy}, for which

$$v_\alpha(x,y) = \int_0^{+\infty} e^{\alpha t} \mu_{xy}(dt)$$

From condition (C) we conclude that, for each $(x,y) \notin A$ there exists a positive measure ν_{xy} pe $[0,+\infty]$ such that

$$\frac{C(x,y)}{\alpha} - v_\alpha(x,y) = \int_0^\infty e^{-\alpha t} d\nu_{xy}(t)$$

We prove now that μ_{xy} is absolutely continuous with respect to the Lebesgue measure on $(0,+\infty)$. More precisely: $\mu_{xy} \le C(x,y).dt$. In this way, the density, denoted by $p_t(x,y)$ (and defined for each $(x,y) \notin A$, a. e. on $(0,+\infty)$), is bounded from above by $C(x,y)$.

Indeed, from the positivity of the function $\dfrac{C(x,y)}{\alpha} - v_\alpha(x,y)$ we obtain that

$$\int_0^\infty e^{-\alpha t}.\mu_{xy}(dt) \le \int_0^\infty e^{-\alpha t}.C(x,y).dt$$

Let now denote

$$f(t) := \sum_{i=1}^n c_i.e^{-\alpha_i t} - \sum_{j=1}^m d_j.e^{-\beta_j t}, \forall t \in [0,+\infty]$$

with $c_i, d_j \ge 0$, $\alpha_i, \beta_j > 0$. The desired inequality is:

$$\int f(t).\mu_{xy}(dt) \le \int_0^\infty f(t).C(x,y).dt$$

and can be transformed as:

$$\sum_{i=1}^{n} c_i \left[\frac{C(x,y)}{\alpha_i} - v_{\alpha_i}(x,y) \right] \geq \sum_{j=1}^{m} d_j \left[\frac{C(x,y)}{\beta_j} - v_{\beta_j}(x,y) \right]$$

Using now the condition (C) we can write:

$$\sum_{i=1}^{n} c_i \int_0^\infty e^{-\alpha_i t} \nu_{xy}(dt) \geq \sum_{j=1}^{m} d_j \int_0^\infty e^{-\beta_j t} \nu_{xy}(dt)$$

Let us suppose that $f(t) \geq 0$, $\forall t \in [0, +\infty]$; then the inequality is true.

Now, each function $g \in \mathcal{C}^+ ([0, +\infty])$ is a uniform limit of such functions, hence the existence of a density $p_t(x,y) \leq C(x,y)$. □

Remarks. (i) Since the k–th order derivative of this function is:

$$(-1)^k \left[\frac{C(x,y)k!}{\alpha^{k+1}} - k! \underbrace{\int \dots \int}_{\text{k times}} v_\alpha(x,y_1) \dots v_\alpha(y_k,y) \mu(dy_1) \dots \mu(dy_k) \right]$$

the condition (C) is equivalent with:

$$\forall (x,y) \notin A \exists C(x,y) \in [0, +\infty) \text{ such that } \forall n = 0, 1, \dots$$

$$\alpha^{k+1} \underbrace{\int \dots \int}_{\text{k times}} v_\alpha(x,y_1) \dots v_\alpha(y_k,y) \mu(dy_1) \dots \mu(dy_k) \leq C(x,y)$$

(ii) The condition (C) is necessary (cf. [9] th. VII 16a, pg. 315). Indeed, if $v_\alpha(x,y) = \displaystyle\int_0^\infty e^{-\alpha t} p_t(x,y) dt$, with $t \mapsto p_t(x,y)$ defined a. e. and bounded by $p_t(x,y) \leq C(x,y)$, then denoting $f(\alpha) := v_\alpha(x,y)$, the k–th order derivative is $f^{(k)}(\alpha) = (-1)^k k! a_k(x,y)$ hence $\alpha^{k+1} a_k(x,y) = (-1)^k \dfrac{\alpha^{k+1} f^{(k)}(\alpha)}{k!}$. As we are allowed to take the derivative under the integral sign, we get:

$$f^{(k)}(\alpha) = (-1)^k \int_0^\infty e^{-\alpha t} t^k p_t(x,y) dt$$

hence:

$$\alpha^{k+1} a_k(x,y) = \frac{\alpha^{k+1}}{k!} \int_0^\infty e^{-\alpha t} t^k p_t(x,y) dt \leq$$

$$\leq C(x,y)\frac{\alpha^{k+1}}{k!}\int_0^\infty e^{-\alpha t}t^k dt = C(x,y)$$

(iii) $p_t(x,y)$ is the derivative (a. e.) of the (absolutely continuous) function $\nu_{xy}([0,t]) - t.C(x,y)$. Indeed:

$$\int_0^\infty e^{-\alpha t}\nu_{xy}([0,t])dt = \int_0^\infty \nu_{xy}e^{-\alpha t}\int_0^t \nu_{xy}(ds)dt =$$

$$= \int_0^\infty \left(\int_s^\infty e^{-\alpha t}dt\right)\nu_{xy}(ds) = \frac{1}{\alpha}\int_0^\infty e^{-\alpha s}\nu_{xy}(ds) = \frac{1}{\alpha}\left(\frac{C(x,y)}{\alpha} - v_\alpha(x,y)\right)$$

which may be written as:

$$\frac{1}{\alpha}\int_0^\infty e^{-\alpha t}p_t(x,y)dt = \int_0^\infty e^{-\alpha t}\left(\int_0^t p_s(x,y)ds\right)dt =$$

$$= \int_0^\infty e^{-\alpha t}\left[\nu_{xy}([0,t]) - t.C(x,y)\right]dt$$

(iv) We apply the above result to the function $f(\alpha) := \alpha.v_\alpha(x,y)$ (see also [9], th. 16a, pg. 315.). The condition is $\left|f^{(k)}(\alpha)\right| \leq \frac{M.k!}{\alpha^{k+1}}$, which may be written as:

$$\alpha^{k+1}\left|\alpha.k.V_\alpha^k\left[v_\alpha(\cdot,y)\right](x) - V_\alpha^{k-1}\left[v_\alpha(\cdot,y)\right](x)\right| \leq M(x,y)$$

or

$$\alpha^{k+1}\left|V_\alpha^{k-1}\left[v_\alpha(x,y) - \alpha.k.\int v_\alpha(x,y_1).v_\alpha(y_1,y).\mu(dy_1)\right]\right| \leq M(x,y)$$

If and only if such a condition holds, we obtain the existence, for each $(x,y)\notin A$, of a measurable and bounded function φ_{xy}, such that:

$$\alpha v_\alpha(x,y) = \int_0^\infty e^{-\alpha t}\varphi_{xy}(t).dt$$

Let us denote by $F_{xy}(t) := \int_0^t \varphi_{xy}(s)ds$; we obtain an absolutely continuous function, such that

$$\int_0^\infty e^{-\alpha t}F_{xy}(t)dt = \int_0^\infty e^{-\alpha t}\left(\int_0^t \varphi_{xy}(s)ds\right)dt =$$

$$= \int_0^\infty \varphi_{xy}(s)\left(\int_s^\infty e^{-\alpha t}dt\right)ds = \int_0^\infty \frac{1}{\alpha}e^{-\alpha s}\varphi_{xy}ds = v_\alpha(x,y)$$

In the same way ([9] th. 15a, pg. 313 and th. 17a, pg. 318), one may consider conditions on $\varphi_{xy} \in L^p$ for $p > 1$, resp. $p = 1$, obtaining thus situations when the density p_t has the derivative a.e. in L^p.

Example. Let $\mathcal{V}$ be the resolvent associated with the translations semi–group on $\mathbb{R}$. For this case, the property (C) does not holds. Indeed, there should exists C such that the function $\dfrac{C}{\alpha} - e^{-\alpha(x-y)}$ be completely monotone, for each $x \geq y$. However, since the k–th order derivative is equal to:

$$(-1)^k\left(\frac{Ck!}{\alpha^{k+1}} - (x-y)^k e^{\alpha(x-y)}\right)$$

the condition becomes $C \geq \dfrac{(x-y)^k}{k!}\alpha^{k+1}e^{-\alpha(x-y)}$, $\forall \alpha > 0$. The left hand side function has an absolute maximum for $\alpha = \dfrac{k+1}{x-y}$. Now, the condition (C) is: $C \geq \dfrac{(k+1)^{k+1}}{k!e^{k+1}(x-y)}$. Using Stirling's formula, we get:

$$C \geq \frac{1}{e}\frac{1}{x_k}\frac{1}{\sqrt{2\pi}}(1+\frac{1}{k})^k\frac{k+1}{\sqrt{k}}$$

(where $x_k \to 1$); such a constant does not exists! Of course, the associated semi–group is **not** absolutely continuous with respect to any σ–finite measure.

Next, we look for conditions under which the measures μ_{xy} are absolutely continuous with respect to the Lebesgue measure on $[0, +\infty)$. The corresponding density will serve to the construction of the absolutely continuous associated semi–group.

Theorem 2. *Let us suppose that:*

(D) the function $\alpha \mapsto \alpha v_\alpha(x,y)$ is a difference of completely monotone functions on $(0, +\infty)$

Then there exist the semi–groups of kernels $(P_t)_{t>0}$ and $(P_t^*)_{t>0}$ on X, which are in duality, absolutely continuous, right continuous and associated with $\mathcal{V}$ resp. $\mathcal{V}^*$.

Proof. From (D) we obtain

$$\alpha v_\alpha(x,y) = \int_0^\infty e^{-\alpha t} d\nu_{xy}^1(t) - \int_0^\infty e^{-\alpha t} d\nu_{xy}^2(t)$$

with ν_{xy}^1 and ν_{xy}^2 positive measures on $[0,+\infty]$.

Denoting $\nu_{xy} := \nu_{xy}^1 - \nu_{xy}^2$, we have:

$$v_\alpha(x,y) = \frac{1}{\alpha}\int_0^\infty e^{-\alpha t} d\nu_{xy}(t) = \int_0^\infty \left(-\frac{1}{\alpha} e^{-\alpha s}\right)|_t^\infty d\nu_{xy}(t) =$$

$$= \int_0^\infty \left(\int_t^\infty e^{-\alpha s} ds\right) d\nu_{xy}(t) = \int_0^\infty e^{-\alpha s}\left[\int_0^s d\nu_{xy}(t)\right] ds =$$

$$= \int_0^\infty e^{-\alpha s} p_s(x,y) ds$$

Hence, we can choose $p_t(x,y) := \nu_{xy}([0,t])$ and all the properties hold true.

We still have to prove that the function $(x,y) \mapsto p_t(x,y)$ is measurable on $X \times X \setminus A$. We use a reasoning similar to that of Meyer [3]. We consider the (measurable) functions $\varphi : [0,+\infty] \to \mathbb{R}$ for which the map $(x,y) \mapsto \int_0^\infty \varphi(t) p_t(x,y) dt$ is measurable. This is true for each $\varphi(x) = e^{-\alpha t}$ (with $\alpha \in [0,+\infty]$). Hence it is true for all linear combinations of such functions. As these functions form an algebra in $\mathcal{C}([0,+\infty])$, which contains the constants and separates the points, it follows that any function $\varphi \in \mathcal{C}([0,+\infty])$ is good. Now, using monotone sequences, we conclude that, for each $t_0 \in (0,+\infty)$ and any $n \in \mathbb{N}$, the map $(x,y) \mapsto \int_0^\infty n\chi_{[t_0,t_0+\frac{1}{n})} p_t(x,y) dt$ is measurable. Since $t \mapsto p_t(x,y)$ is right continuous, it follows that

$$\int_0^\infty n\chi_{[t_0,t_0+\frac{1}{n})} p_t(x,y) dt \to p_{t_0}(x,y)$$

and thus the measurability is proved. □

2. Complements to absolutely continuous semi–groups. Let us recall that we denoted by F, G the functions defined on $X \times \mathbb{R}$ by:

$$F(x) := \begin{cases} 0 & \text{if } \ t \le 0 \\ P_t f(x) & \text{if } \ t > 0 \end{cases}$$

$$G(x) := \begin{cases} 0 & \text{if } \ t \le 0 \\ 1 - P_t f(x) & \text{if } \ t > 0 \end{cases}$$

Also, we denote by $\mathcal{T}$ the translation semi–group on $\mathbb{R}$: $\mathcal{T}_t f(x) := f(x+t)$.

From [6] the following characterizations are obtained:

Proposition 3. *$\mathcal{P}$ is a standard semi–group iff there exists a separable (and semi–metrisable) topology on X such that $x \mapsto V_0 f(x)$ is continuous, for all measurable and bounded functions f.*

Proposition 4. *$\mathcal{P}$ is an absolutely continuous semi–group iff there exists a separable (and semi–metrisable) topology on $X \times (0, +\infty)$ such that $(x, t) \mapsto P_t f(x)$ is continuous, for all measurable and bounded functions f.*

Proof. Let $\mathcal{P}$ be an absolutely continuous semi–group. From [7] it follows that $\mathcal{E}_{\mathcal{P}\otimes\mathcal{T}}$ is a standard H–cone. By definition [2], there exists a countable, increasingly dense part $D \subset \mathcal{E}_{\mathcal{P}\otimes\mathcal{T}}$. Let us consider the coarsest topology on $X \times \mathbb{R}$, in which all functions from D are continuous. Especially, the functions F and G being $\mathcal{P} \otimes \mathcal{T}$–excessive (see [7]), we get the continuity of the map $(x, t) \mapsto P_t f(x)$.

Conversely, let us fix a countable, dense part $\{(x_n, t_n) \| \, n \in \mathbf{N}\}$ of $X \times \mathbb{R}$. We define the following measures on $(X, \mathcal{X})$:

$$\nu := \sum_{n=1}^{\infty} 2^{-n}.\varepsilon_{x_n}; \; \nu_m := \nu.P_{t_m}; \; \mu := \sum_{m=1}^{\infty} 2^{-m}.\nu_m$$

Now the semi–group $\mathcal{P}$ is absolutely continuous with respect to the measure μ. Indeed, $\mu(f) = 0$ means $\nu_m(f) = 0$, $\forall m \in \mathbb{N}$, hence $\nu\left(P_{t_m} f\right)(x_n) = 0$, $\forall m, n \in \mathbb{N}$. So, the continuous function $(x, t) \mapsto P_t f(x)$ is 0 on a dense set. We obtain: $P_t f(x) = 0$, $\forall t > 0$, $\forall x \in X$. □

Remarks. (i) An equivalent condition for "*$\mathcal{P}$ is an absolutely continuous semi–group*" is:

there exists a separable topology on X, such that , for each bounded and measurable function f on $(X, \mathcal{X})$ and any $t_0 > 0$, the map $x \mapsto P_{t_0} f(x)$ is continuous.

Indeed, it suffices to add to the family D, considered in the proof, the countable set of functions of the form $(x, t) \mapsto s\left(x, \frac{1}{n}\right)$, with $s \in D$, $n \in \mathbb{N}$. In this way, the continuity of each map $x \mapsto P_{\frac{1}{n}} f(x)$ is granted, in the finest topology on X, for which the canonical projection $X \times (0, +\infty) \to X$

is continuous (this topology is separable). As for each $t_0 > 0$ there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < t_0$, writing $P_{t_0} = P_{\frac{1}{n}}\left[P_{t_0-\frac{1}{n}}\right]$ we get the asserted continuity.

Conversely, let us choose a countable part, dense in X and construct the measure μ as in the proof.

(ii) One can assure the continuity of the function $(x,t) \mapsto P_t f(x)$ in a separable topology, for functions f which are differences of functions $\mathcal{V}$–excessive and 1–continuous.

Indeed, let us consider on X the coarsest topology, for which all the functions $V_0 f_n$ are continuous: this topology is separable (and semi–metrisable), and the conclusion follows from:

$$\begin{aligned}|P_t(V_0f)(x) \; -P_{t_0}(V_0f)(x_0)| &\le |P_t(V_0f)(x) - P_{t_0}(V_0f)(x)| + \\ &+|P_{t_0}(V_0f)(x) - P_{t_0}(V_0f)(x_0)| = \left|\int_t^{t_0} P_\tau f(x)d\tau\right| + \\ &+|V_0(P_{t_0}f)(x) - V_0(P_{t_0}f)(x_0)| \le \\ &\le |t-t_0| + |V_0(P_{t_0}f)(x) - V_0(P_{t_0}f)(x_0)|\end{aligned}$$

(iii) The passage from the continuity in the topology on X to that on $X \times (0,+\infty)$ (as in prop. 2.) can be realized in the presence of the following kind of right–continuity of the semi–group:

Let (X,τ) be a metrisable and separable topological space, such that :

- *for each bounded and measurable function f and for each n, the map $x \mapsto P_{\frac{1}{n}} f(x)$ is continuous;*
- *for each continuous function g on X we have: $P_t g \to P_{t_0} g$ locally uniformly for $t \to t_0 > 0$*

Indeed, in

$$|P_t f(x) - P_{t_0} f_{x_0}| \le |P_t f(x) - P_{t_0} f(x)| + |P_{t_0} f(x) - P_{t_0} f(x_0|$$

we choose $\frac{1}{n} < \min(t, t_0)$ and write the right hand side term as $P_{t-\frac{1}{n}}\left(P_{\frac{1}{n}} f\right)$; the continuity of the function $P_{\frac{1}{n}} f$ guarantees the desired property for the first term. As for the second term, let us remark that the function $P_{t_0} f$ is continuous.

As a comparison, let as recall the following characterization:

Proposition 5. *$\mathcal{P}$ is a strongly Feller semi–group iff there exists a separable and metrisable topology X such that $(x,t) \mapsto P_t f(x)$ be continuous in the product topology, for any measurable and bounded function f*

3. Tensor products of standard and absolutely continuous semigroups.

Lemma 6. *Let $\mathcal{P}$ be an absolutely continuous semi–group with respect to the measure μ. If the associated resolvent is absolutely continuous with respect to the measure ν, then $\mathcal{P}$ is also absolutely continuous with respect to the measure ν.*

Proof. $\nu(f) = 0 \Longrightarrow V_0 f = 0 \Longrightarrow \int_0^\infty P_t f(x) = 0,\ \forall x \in X \Longrightarrow$ ($\forall x \in X : t \mapsto P_t f(x)$ is negligible with respect to the Lebesgue m) $\Longrightarrow$ $\{(x,t)|\ P_t f(x) \neq 0\}$ is $\mu \times m$–negligible (any other measure instead of μ will do) $\Longrightarrow \exists B \subset (0,+\infty)$, m–negligible, such that $\forall t \notin B$: $x \mapsto P_t f(x)$ is μ–negligible. Hence $\forall t \notin B$: $\mu(P_t f) = 0$. It follows that $P_s(P_t f) \equiv 0$, $\forall s > 0$, hence $P_\tau f \equiv 0$, $\forall \tau > 0$. □

Let us recall that, for a resolvent $\mathcal{V}$ (resp. a semi–group $\mathcal{P}$) we denote by $\mathcal{S}_\mathcal{V}$ resp. $\mathcal{S}_\mathcal{P}$ the set of supramedian functions (i.e. $\alpha V_\alpha s \leq s$, $\forall \alpha > 0$; resp. $P_t s \leq s$, $\forall t > 0$). For each supramedian function s, we denote its regularization $\hat{s}$ defined as $\hat{s} = \sup_{\alpha>0} \alpha V_\alpha s$ resp. $\hat{s} = \sup_{t>0} P_t s$).

Proposition 7. *The following properties are equivalent:*
(i) $\mathcal{V}$–negligible $\Longrightarrow$ $\mathcal{P}$–negligible.
(ii) $\mathcal{S}_\mathcal{V} = \mathcal{S}_\mathcal{P}$

Proof. (i) $\Longrightarrow$ (ii) Let $s \in \mathcal{S}_\mathcal{V}$ be finite, hence $\hat{s} = s$, $\mathcal{V}$–a. e. But $\hat{s} \in \mathcal{E}_\mathcal{V}$ $\Longrightarrow s \in \mathcal{E}_\mathcal{P}$. Hence $\hat{s} = s$ $\mathcal{P}$–a. e. by hypothesis, hence $P_t s = P_t \hat{s} \leq \hat{s} \leq s$ everywhere.

Let now $s \in \mathcal{S}_\mathcal{V}$ be arbitrary. We consider $s_n := \min(s,n) \in \mathcal{S}_\mathcal{V}$ (it suffices the existence of a $s_0 \in \mathcal{S}_\mathcal{V}$ such that $0 < s_0 < +\infty$). Now $s_n \in \mathcal{S}_\mathcal{P}$, $s_n \nearrow s$, hence $s \in \mathcal{S}_\mathcal{P}$.

(ii) $\Longrightarrow$ (i). If $V_\alpha(\chi_A) \equiv 0$ then $\alpha V_\alpha(\chi_A) \leq \chi_A$, hence $\chi_A \in \mathcal{S}_\mathcal{V}$, hence $\chi_A \in \mathcal{S}_\mathcal{P}$. Since the regularized functions do coincide, we get $\hat{\chi_A} \equiv 0$. But $t \mapsto P_t(\chi_A)(x)$ is increasing, while

$$\lim_{t\to 0} P_t(\chi_A)(x) = \lim_{\alpha\to+\infty} \alpha V_\alpha(\chi_A)(x) \equiv 0$$

It follows that each $P_t(\chi_A)(x) = 0$. □

Proposition 8. *The resolvent $\mathcal{V}$ with proper V_0, is absolutely continuous iff there exists a positive and σ–finite measure μ, such that $s \in \mathcal{E}_{\mathcal{V}}$, $\mu(s) = 0 \Longrightarrow s \equiv 0$.*

Proof. (see [3][XII 41]) If $\mathcal{V}$ is absolutely continuous, then from $\mu(s) = 0$ it follows $V_\alpha s \equiv 0$, $\forall \alpha > 0$. But $\alpha V_\alpha s \nearrow s$, hence $s \equiv 0$.

Conversely, let us denote $\nu := \mu V_0$ (if V_0 is not proper, then ν will be only α–excessive, $\forall \alpha > 0$). Let $f \in \mathcal{F}$ be such that $\nu(f) = 0$, hence $\mu(V_0 f) = 0$. Choosing $f_0 \in \mathcal{F}$ with $f_0 > 0$ and $V_0 f_0$ bounded, we denote $f_n := \min(f, n.f_0)$. We have $V_0 f_n \in \mathcal{E}_{\mathcal{V}}$ and of course $\mu(V_0 f_n) = 0$; by hypothesis, $V_0 f_n \equiv 0$, hence $V_0 f \equiv 0$, q. e. d. □

Finally, let us consider the product semi–groups [7].

Proposition 9. *Let $\mathcal{P}$, $\mathcal{Q}$ be standard semigroups. Let us suppose that $\mathcal{P}$ and $\mathcal{P} \otimes \mathcal{Q}$ are absolutely continuous, and $P_t 1 \not\equiv 0$, $\forall t > 0$. Then $\mathcal{Q}$ is also absolutely continuous.*

Proof. Let us denote by μ the measure with respect to which $\mathcal{P}$ is absolutely continuous and by ν the measure with respect to which the resolvent associated with $\mathcal{Q}$ is absolutely continuous. By lemma 6., the resolvent associated with $\mathcal{P} \otimes \mathcal{Q}$ is absolutely continuous with respect to the product measure $\mu \otimes \nu$. Now:

$$\nu(f) = 0 \Longrightarrow (\mu \otimes \nu)(1 \otimes f) = 0 \Longrightarrow (P_t \otimes Q_t)(1 \otimes f) \equiv 0 \Longrightarrow P_t 1.Q_t f \equiv 0$$

hence $\nu(f) = 0 \Longrightarrow Q_t f \equiv 0$. □

Remark. Let V_0 be absolutely continuous with respect to a measure θ on $X \times \mathbb{R}$. It is also absolutely continuous with respect to $\theta.V_0$. But the image measure ν of $\theta.V_0$ on $\mathbb{R}$ is absolutely continuous with respect to the Lebesgue measure:

$$\nu(A) = \theta.V_0(X \times A) = \int V_0\left(\chi_{X\times A}\right) d\theta = \int \left[\int_0^\infty P_s 1(x).T_s(\chi_A)(t) ds\right] d\theta =$$

$$= \int \left[\int_{A-t} P_s 1(x) ds\right] d\theta$$

Now, if $m(A) = 0$, then $m(A - t) = 0$ hence $\int_{A-t} P_s 1(x) ds$ meaning that $\nu(A) = 0$.

More generally, let $\mathcal{P}$ be associated with the resolvent $\mathcal{W}$; and $\mathcal{P}\otimes\mathcal{Q}$ be associated with the resolvent $\mathcal{V}$. We suppose that $\mathcal{V}$ is absolutely continuous with respect to θ on $X\times Y$. Let $\mu(f):=\theta(f\otimes 1)$ be the projection of θ on X. Then $\mathcal{W}$ is absolutely continuous with respect to μ: $\mu(f)=0 \Longleftrightarrow \theta(f\otimes 1)=0 \Longrightarrow V_0(f\otimes 1)=0$. But $V_0(f\otimes 1)=W_0f$ if $Q_t1(x)\equiv 1$.

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