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ABOUT THE LOWER SEMICONTINUOUS REGULARIZATION FOR VECTOR FUNCTIONS

BY

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Abstract. This paper extend the lower semicontinuous regularization for vector function from the case of Hilbert valued maps to the case of Banach complete lattices and locally convex complete order lattices valued maps. The main result is applied for obtain a local convex, continuous decomposition for local D.C. continuous functions valued in a Banach lattice.

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1. Introduction. Since 1950 when ALEXANDROFF introduced in [1] the D.C. functions (functions expressed as difference of convex functions), these was intensively studied making an important class of nonconvex and nondifferentiable functions. Between the properties of these functions, the characterization of real D.C. locally Lipschitz functions using the cyclically maximal monotone multifunctions was recently established by ELHILALI ALAOUI in [3] using the decomposition of a continuous D.C. function as a difference of lower semicontinuous convex functions. This study was continued for vector valued functions in [2] and obviously, the first step was the introduction of the lower semicontinuous regularization of a vector function.

It is wellknown that all real function have a lower semicontinuous (l.s.c.) regularization given by

$$\bar{f}(x) = \liminf_{y \to x} f(y).$$

For vector functions defined on a locally convex space X with values in a Hilbert lattice Y ordered by the cone Y_+, the l.s.c. regularization was given in [2] by

$$\bar{f}(x) = \sup A_x^f$$

where

$$A_x^f = \{y \mid \forall V \in \mathcal{V}(0),\ \exists U \in \mathcal{V}(0) \text{ such that } f(U+x) \subset y + V + Y_+\}.$$

The natural question is how can we define a such l.s.c. regularization for functions with values in an order complete locally convex lattice. Following the idea from [2] we may adapt the proofs given for Hilbert lattice for obtaining a weak l.s.c. regularization in this case. Some applications for the continuous D.C. functions are also given.

2. Lower semicontinuous regularization for Banach lattice valued maps. Throughout this paper, without other mentions, X,Y will be locally convex spaces, Y ordered by a closed convex pointed cone Y_+. If A and B are two subsets of Y we will denote by $A+B = \{a+b \mid a \in A, b \in B\}$; if we add to Y a smallest and a biggest element denoted $-\infty$ respectively $+\infty$, we agree that $a + (+\infty) = +\infty$ and $a + (-\infty) = -\infty$ for all $a \in Y$. As usually, $\mathcal{V}_X(0)$, $\mathcal{V}_Y(0)$ denote a fundamental system of neighborhoods of 0 for the topology of X, respectively Y. If the interior of the cone Y_+ denoted Int Y_+ is nonempty, we may consider for $-\infty$ a fundamental system of neighborhoods given of the sets $(y - Y_+) \cup \{-\infty\}, y \in \text{Int } Y_+$, respectively for $+\infty$ a fundamental system of neighborhoods given of the sets $(y+Y_+)\cup\{+\infty\}, y \in \text{Int } Y_+$. The cone Y_+ will be called *normal* if there exists a fundamental system of neighborhoods of 0 such that $V = (V + Y_+) \cap (V - Y_+)$ for all neighborhood V of the system. If $f : X \to Y \cup \{-\infty, +\infty\}$ is a function, we use the standard notations: dom $f = \{x \in X \mid f(x) < +\infty\}$ for the domain of f, epi $f = \{(x, y) \mid f(x) \leq y\}$ for the epigraph of f and $N^\lambda f = \{x \in X \mid f(x) \leq \lambda\}$ for the λ-level set of f . We denote by $f \mid_M$ the restriction of f at the set $M \subset X$.

The function $f : X \to Y \cup \{-\infty, +\infty\}$ will be called *lower semicontinuous* (shortly l.s.c. following the definition given in [4]) at $\bar{x} \in X$ if for all $V \in \mathcal{V}_Y(0)$ and for all $y \in Y$, $y \leq f(\bar{x})$ there exists $U \in \mathcal{V}_X(0)$ such that $f(\bar{x} + U) \subset y + V + Y_+ \cup \{+\infty\}$. We will say that Y is an *order complete* vector space if every majorized subset A of Y has a *supremum* denoted sup A in the sense that sup $A \geq a$ for all $a \in A$ and if $b \geq a$ for all $a \in A$ then $b \geq$ sup A.

The *dual space* of Y is denoted Y^* and Y_+^* means the *dual cone* of Y_+ i.e. $Y_+^* = \{y^* \in Y^* \mid y^*(y) \geq 0,\ \forall y \in Y_+\}$. We agree that $\sup \emptyset = -\infty$,

$y^*(\emptyset) = \emptyset$, $y^*(A) = \{y^*(a),\ a \in a\}$ for a nonempty set A of Y and $y^*(\pm\infty) = \pm\infty$ for all $y^* \in Y^*_+$.

Let recall from [2] the definitions of the *superior* and the *inferior* set for a function $f : X \to Y \cup \{+\infty\}$ at $\bar{x} \in \text{dom } f$:

$$A^f_{\bar{x}} = \{y \in Y \mid \ \forall V \in \mathcal{V}_Y(0), \exists U \in \mathcal{V}_X(0) \text{ such that } f(\bar{x} + U) \subset y + V + Y_+ \cup \{+\infty\}\}$$

$$B^f_{\bar{x}} = \{y \in Y \mid \ \forall V \in \mathcal{V}_Y(0), \exists U \in \mathcal{V}_X(0) \text{ such that } f(\bar{x} + U) \subset y + V - Y_+ \cup \{+\infty\}\}.$$

Among other properties of these sets given in [2] we mention the following:

1) If Y is a Banach lattice, A^f_x is a majorized and upper directed set and B^f_x is a minorated and lower directed set.

2) If $Y = \mathbb{R}^n$ and $f = (f_1, f_2, ...f_n)$ we have

$$cl(A^f_{\bar{x}}) = \prod_{i=1;n} (-\infty, \liminf_{x \to \bar{x}} f_i(x)]$$

$$cl(B^f_{\bar{x}}) = \prod_{i=1;n} [\limsup_{x \to \bar{x}} f_i(x), +\infty).$$

3) If X and Y are metric spaces, then:

$$A_{\bar{x}} = \{y \in Y \mid \forall x_n \to \bar{x},\ \exists b_n \to y \text{ such that } b_n \leq f(x_n), \forall n \in \mathbb{N}\}$$

$$B_{\bar{x}} = \{y \in Y \mid \forall x_n \to \bar{x},\ \exists b_n \to y \text{ such that } b_n \geq f(x_n), \forall n \in \mathbb{N}\}.$$

The following properties are given in [2] for the case of metrizable spaces but they rest valid in locally convex spaces, too.

4) If $h : X \to Y \cup \{+\infty\}$ and $\bar{x} \in \text{dom } h \cap \text{dom } f$ then, $A^f_{\bar{x}} + A^h_{\bar{x}} \subseteq A^{f+h}_{\bar{x}}$; if f or h is continuous at $\bar{x}$, we have equality.

5) f is lower semicontinuous at $\bar{x} \in \text{dom } f$ if and only if $A^f_{\bar{x}} = f(\bar{x}) - Y_+$
The main lemma from [2] which unable us to define the l.s.c. regularization of f is the following:

Lemma 2.1. [2] *Let Y be a Hilbert lattice ordered by a closed, convex, pointed, with nonempty interior cone Y_+ and let $(e_n)_{n \in \mathbb{N}}$ be a Hilbert base* *for Y. If $f : X \to Y \cup \{+\infty\}$ is a map, $\bar{x} \in \mathrm{dom}\, f$ and $A_{\bar{x}}^{f} \neq \emptyset$ then for all $p \in \mathbb{N}$ we have*

$$e_p \circ A_{\bar{x}}^{f} = A_{\bar{x}}^{<e_p, f>}$$

where $e_p \circ A_{\bar{x}}^{f} = \{< e_p, y >,\ y \in A_{\bar{x}}^{f}\}$.

Now we can prove this lemma for the case when Y is a locally convex space ordered by a convex, pointed cone Y_+.

Lemma 2.2. *Let Y be a locally convex space ordered by a closed, pointed, convex cone Y_+, $f : X \to Y \cup \{+\infty\}$, and $\bar{x} \in \mathrm{dom}\, f$ such that $A_{\bar{x}}^{f} \neq \emptyset$. Then, for all $y^* \in Y_+^* \setminus \{0\}$ we have*

$$y^* \circ A_{\bar{x}}^{f} = A_{\bar{x}}^{y^* \circ f}.$$

Proof. Let $y^* \in Y_+^* \setminus \{0\}$ and $y \in A_{\bar{x}}^{f}$. Following the definition of the inferior set we have that for all $V \in \mathcal{V}_Y(0)$ there exists $U \in \mathcal{V}_X(0)$ such that

$$f(U + \bar{x}) \subset y + V + Y_+ \cup \{+\infty\}.$$

Following the continuity of y^* we get that $y^*(y) \in A_{\bar{x}}^{y^* \circ f}$ and thus $y^* \circ A_{\bar{x}}^{f} \subseteq A_{\bar{x}}^{y^* \circ f}$ for all $y^* \in Y_+^* \setminus \{0\}$.
For the converse inclusion, let $y^* \in Y_+^* \setminus \{0\}$ and $b \in A_{\bar{x}}^{y^* \circ f}$. Following the definition of the inferior set we have that for all $\varepsilon > 0$ there exists $U \in \mathcal{V}(0)$ such that

$$y^* \circ f(U + \bar{x}) \geq -\varepsilon + b \tag{1}$$

Let $z_0 \in A_{\bar{x}}^{f}$ and $z = z_0 + (b - y^*(z_0))u$ where u is an element of Y such that $y^*(u) = 1$ and $v^*(u) = 0$ for all $v^* \in Y^* \setminus \{\lambda y^*;\ \lambda > 0\}$. Thus $y^*(z) = b$ and $v^*(z) = v^*(z_0)$. Since $z_0 \in A_{\bar{x}}^{f}$ we have that for all $V \in \mathcal{V}(0)$ there exists $U' \in \mathcal{V}(0)$ such that

$$f(U' + \bar{x}) \subset z_0 + V + Y_+ \cup \{+\infty\} \tag{2}$$

Let consider V an arbitrary symmetric neighborhood of 0. If $\inf y^*(V) = -\infty$ then obviously from (2) there exists $U' \in \mathcal{V}(0)$ such that

$$y^* \circ f(U' + \bar{x}) \subset y^*(z_0) + y^*(V) + R_+ \cup \{+\infty\} \subset y^*(z) + y^*(V) + R_+ \cup \{+\infty\}.$$

If $\inf y^*(V) = -\varepsilon$, then we from (1) will exists a neighborhood U of 0 such that

$$y^* \circ f(U+\bar{x}) \subset y^*(z) - \varepsilon + \mu + R_+ \cup \{+\infty\} \subset y^*(z) + y^*(V) + R_+ \cup \{+\infty\}$$

(where $\mu > 0$). Thus for $\tilde{U} = U' \cap U$ we have for all $\lambda > 0$ that

$$\lambda y^* \circ f(\tilde{U}+\bar{x}) \subset \lambda y^*(z) + \lambda y^*(V) + R_+ \cup \{+\infty\}.$$

Now, for all $v^* \in Y_+^* \setminus \{\lambda y^*;\ \lambda > 0\}$ we have from (2) that

$$v^* \circ f(\tilde{U}+\bar{x}) \subset v^*(z) + v^*(V) + R_+ \cup \{+\infty\}.$$

Thus $f(\tilde{U}+\bar{x}) \subset z + V + Y_+ \cup \{+\infty\}$ and hence $A_{\bar{x}}^{y^* \circ f} \subseteq y^* \circ A_{\bar{x}}^f$ and finally the equality from the lemma follows. □

Remark 2.1. If $y^* = 0$, the equality from the precedent lemma is false. Indeed, the left term will be 0 and the right term will be the interval $(-\infty, 0]$.

We denote $A_f = \{x \in X \mid A_x^f \neq \emptyset\}$. It is not difficult to see that A_f is an open set.

Using Lemma 2.2, we may consider a weak lower semicontinuous regularization of a map valued in a Banach lattice as we can see from the following theorem.

Theorem 2.1. *Let Y be an order complete locally convex lattice ordered by a closed, convex, pointed cone Y_+ with nonempty interior. If $f : X \to Y \cup \{+\infty\}$ is a map, the application $\bar{f} : X \to Y \cup \{-\infty\}$ given by*

$$\bar{f}(x) = \begin{cases} \sup A_x^f & if \quad x \in A_f \\ -\infty & if \quad x \notin A_f. \end{cases}$$

is a weak l.s.c. map.

Proof. Let $y^* \in Y_+^* \setminus \{0\}$. Firstly, let remind that in a topological lattice, if M is a majorized upper directed set then $\sup y^* \circ M = y^*(\sup M)$ (see [8] for more details). Thus, following the properties of A_x^f we have that for $x \in A_f$

$$y^* \circ \bar{f}(x) = y^* \circ (\sup A_x^f) = \sup(y^* \circ A_{\bar{x}}^f).$$

Using the precedent lemma, and the s.c.i. regularization for a real function we have for $x \in A_f$ that

$$y^* \circ \bar{f}(x) = \sup A_x^{y^* \circ f} = \liminf_{y \to x} y^* \circ f(y) \tag{3}$$

Using the convention made, we have for $x \notin A_f$ that $y^* \circ \bar{f}(x) = -\infty$. Since the map $x \to \liminf_{y \to x} y^* \circ f(y)$ is a l.s.c. map then $y^* \circ \bar{f}$ is a l.s.c. map on A_f for all $y^* \in Y_+^* \setminus \{0\}$. If $y^* = 0$, then $y^* \circ \bar{f}(x) = 0$ if $x \in A_f$ and $y^* \circ \bar{f}(x) = -\infty$ if $x \notin A_f$ and so, $y^* \circ \bar{f}$ is a l.s.c. function on A_f for all $y^* \in Y_+^*$. Since Y_+^* is a generating cone for Y^* and A_f is an open set we conclude that $\bar{f}$ is a weak l.s.c. map. □

A natural question arises now: which properties for f rest valid for $\bar{f}$? One of them is the local convexity and we shall prove that in the following proposition.

We say that a subset D of X is a *local convex set* if for each point $u \in D$ there exists a neighborhood U of u such that $D \cap U$ is a convex set. A map f on a local convex map D will be called *local convex function* if for each point $u \in D$ there exists a neighborhood U of u, such that f is convex on $U \cap D$. A map f will be called L.D.C. if there exists two convex maps g and h such that $f = g - h$, $A_g \cap A_h \neq \emptyset$.

Proposition 2.1. *Let Y be a local convex lattice ordered by a closed, convex, pointed cone with nonempty interior and $f : X \to Y \cup \{+\infty\}$ be a local convex map on a local convex set D. Then $\bar{f} : X \to \bar{Y}$ is a local convex map on $A_f \cap D$ (if it is nonempty).*

Proof. Obviously, the nonempty intersection of two local convex set is a local convex set. Thus, if $D \cap A_f$ is nonempty, then in our hypothesis, it is a local convex set. Let $u \in D \cap A_f$ and U_1 be a convex neighborhood of u such that $D \cap U_1$ is a convex set. Also, we can find a convex neighborhood of u, U_2 such that f is convex on $D \cap U_1 \cap U_2$ and another neighborhood U_3 of u such that $D \cap A_f \cap U_3$ is a convex set, too. If we denote by $U = U_1 \cap U_2 \cap U_3$, it is a convex neighborhood of u, $D \cap U$ is a convex set on which f is convex and $U' = D \cap A_f \cap U$ is a convex set, and obviously, f is convex on U', too. Following the precedent theorem we get that $\bar{f}$ is finite on U' and

$$y^* \circ \bar{f}\,|_{U'}(x) = \liminf_{y \to x} y^* \circ f\,|_{U'}(y)$$

for all $x \in U'$ and $y^* \in Y_+^* \setminus \{0\}$. Thus,

$$\text{epi } y^* \circ \bar{f} \mid_{U'} = \overline{\text{epi } y^* \circ f \mid_{U'}}$$

for all $y^* \in Y_+^* \setminus \{0\}$. Since $f \mid_{U'}$ is a convex map, then $y^* \circ \bar{f} \mid_{U'}$ is a convex map for all $y^* \in Y_+^* \setminus \{0\}$ and thus $\bar{f} \mid_{U'}$ is a convex map. We conclude now that $\bar{f}$ is a local convex map on $D \cap A_f$. □

In the end of this section we prove that all continuous local L.D.C. function on a reflexive Banach lattice admits a lower semicontinuous local L.D.C. decomposition on a nonempty subset D. Let recall from [7] the following theorem which will be useful for our proposition.

Theorem 2.2. [7] *Let $f : X \to Y$ be a vector function and suppose that the ordering cone is normal. If f is a convex and lower semicontinuous map and* Int (dom f) *is nonempty then f is continuous on* Int (dom f).

Proposition 2.2. *Let Y be a reflexive local convex lattice ordered by a closed, convex, pointed, normal cone Y_+ with nonempty interior and $f : X \to Y$ be a continuous local L.D.C. map on D. Then, there exists an open subset D_1 such that f can be written as a difference of continuous local convex functions on D_1.*

Proof. Let $g, h : X \to Y$ be two local convex maps on X such that $f = g - h$ and $A_h \cap A_g \neq \emptyset$. Then, following the property 4 of the inferior set we have that $A_x^f + A_x^h = A_x^g$ which implies that $f(x) + A_x^h = A_x^g$ and thus $A_h = A_g$. Also we obtain by taking the supremum that $f(x) + \bar{h}(x) = \bar{g}(x)$ which gives that $f = \bar{g} - \bar{h}$ on A_h. Let denote $D_1 = A_h \cap D$; it is a nonempty local convex set. Using the precedent proposition we have that $\bar{g}$ and $\bar{h}$ are local convex functions on D_1 and thus f admits a lower semicontinuous local convex decomposition on D_1. Now, following Lemma 2.2 we get that $\bar{h}$ and $\bar{g}$ are continuous on D_1 and the proposition is proved. □

We remark that in [4] which is a recent version of [2], the authors proved that $\bar{f}$ is a lower semicontinuous map if Y is a complete Banach lattice using the Painleve-Kuratowsky limit for lower sets.

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