ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" IAŞI
Tomul LI, s.I, Matematică, 2005, f.2

A NOTE TO THE PAPER OF D. RICHESON AND J. WISEMAN*

BY

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Abstract. The aim of paper [12] was to prove a fixed point theorem for dynamical systems with windows (see Th. 1). In this note, we document that much more general results than Th. 1 are already well-known for many years.

Mathematics Subject Classification 2000: 37C25, 47H10, 54H25.

Key words: Systems with windows, fixed points, ANR-spaces, compact attractors, compact absorbing contractions, condensing maps.

1. Recalling the formulation of Th. 1 ([12]). This paper concerns only various versions of asymptotic fixed point theorems. Their application to dissipative systems of ordinary differential equations and inclusions for obtaining periodic solutions is only indicated in Remark D.2 below. The extended version of this paper [2] which involves a more detailed exposition with applications was recently submitted for publication.

In the entire text, we shall assume that all considered topological spaces are metric and all mappings between such spaces are continuous.

Let $f : X \to X$ be a continuous map and let $x \in X$. Then the set

$$O(x) = \{x, f(x), f^2(x), \dots, f^n(x), \dots\}$$

is called the *orbit* of x under f.

A (compact) set $W \subset X$ is called a *window* for f if, for every $x \in X$, we have

$$O(x) \cap W \neq \emptyset.$$

*Workshop on *Viability and Invariance*, August 22-25, 2005, Gălăneşti, Romania.

†The first author is supported by the Council of Czech Government (MSM 6198959214)

In [12], it is proved the following main theorem:

Theorem 1. *If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous map which possesses a compact window, then*

$$\mathrm{Fix}(f) = \{x \in \mathbb{R}^n \mid f(x) = x\} \neq \emptyset,$$

where $\mathbb{R}^n$ denotes the n-dimensional Euclidean space.

2. Possible generalizations of Theorem 1

A. Mappings with compact attractors. Following R. D. Nussbaum (11; see also: [1], [6], [7], [8], [9]), we say that a (compact) set $A \subset X$ is an *attractor* for $f : X \to X$ if, for every $x \in X$, we have

$$\overline{O(x)} \cap A \neq \emptyset,$$

where $\overline{O(x)}$ denotes the closure of $O(x)$ in X.

Remark A.1. Every window for $f : X \to X$ is apparently an attractor for f. Moreover, let us observe that, for example, any contraction $f : \mathbb{R}^n \to \mathbb{R}^n$ (or, more generally, the one $f : X \to X$, where X is a complete metric space) admits an attractor, but not necessarily a window.

We recall that a map $f : X \to X$ is *locally compact* if, for every $x \in X$, there exists an open neighbourhood U_x of x in X such that $\overline{f(U_x)}$ is compact.

Remark A.2. Obviously, if X is a locally compact space (in particular, if $X = \mathbb{R}^n$), then any continuous map $f : X \to X$ is locally compact.

Let us still recall two notions introduced by K. Borsuk (see: [1], [7], [9]).

A space X is called an *absolute neighbourhood retract* (ANR, for short) if there exists an open set U of a normed space E which r-dominates X, i.e., if there are continuous mappings $r : U \to X$ and $s : X \to U$ such that $r \circ s = \mathrm{id}_X$. If $U = E$ is a normed space which r-dominates X, then X is called an *absolute retract* (AR).

Remark A.3. Evidently, AR $\subset$ ANR, and every normed space is an absolute retract.

In 1975, Fournier [6] proved:

Theorem A.4. *If X is an* ANR*-space and $f : X \to X$ is a locally compact map with compact attractor, then*

(i) *the (generalized) Lefschetz number* $\Lambda(f)$ *of* f *is well-defined and*
(ii) $\Lambda(f) \neq 0$ *implies that* $\mathrm{Fix}(f) \neq \emptyset$.

As an immediate consequence of A.4, we obtain

Corollary A.5. *If* X *is a locally compact* ANR *and* $f : X \to X$ *is a map with compact attractor, then*
(i) *the generalized Lefschetz number* $\Lambda(f)$ *of* f *is well-defined and*
(ii) $\Lambda(f) \neq 0$ *implies that* $\mathrm{Fix}(f) \neq \emptyset$.

Since every AR-space X is contractible, we infer that $\Lambda(f) = 1$, for an arbitrary $f : X \to X$, and so from A.4 (or A.5), we obtain

Corollary A.6. *If* $X \in \mathrm{AR}$ (*X is a locally compact AR-space*), *then every locally compact map with compact attractor* (*every map with compact attractor*) $f : X \to X$ *has a fixed point.*

Remark A.7. Observe that Corollary A.6 is a far generalization of Theorem 1 in [12]. Let us also note that the idea of A.6 is, in fact, already present in Theorem 2 in [4] published in 1959.

B. Compact absorbing contractions. Theorem A.4 is not the most general known result. We recall (see: [1], [7], [9]) that a continuous map $f : X \to X$ is called a *compact absorbing contraction* (written, $f \in \mathrm{CAC}(X)$) if there exists an open subset $U \subset X$ such that the following conditions are satisfied:
(i) $O(x) \cap U \neq \emptyset$, for every $x \in X$,
(ii) $f(U) \subset U$,
(iii) the map $\widetilde{f} : U \to U$, $\widetilde{f}(x) = f(x)|_{x \in U}$, is compact.

We let

$$\begin{aligned} \mathrm{CA}(X) &= \{f : X \to X \mid f \text{ is continuous with compact attractor}\}, \\ \mathrm{CA}_0(X) &= \{f : X \to X \mid f \text{ is continuous and locally compact with compact attractor}\}. \end{aligned}$$

It is well-known (see: [1], [2], [9]) that

$$\mathrm{CA}_0(X) \subset \mathrm{CAC}(X) \subset \mathrm{CA}(X).$$

and both of the above inclusions are proper.

Remark B.1. We would like to point out that A.4, A.5 and A.6 can be reformulated for CAC-mappings (see again [1], [8], [9]).

Let us recall the following old open problem:

Open Problem. Is it possible to prove A.4 (or A.5 or A.6) for CA-mappings?

C. Condensing mappings. Some further results being a far generalization of Theorem 1 in [12] will be still mentioned here.

Let E be a Banach space and let $B(E) = \{A \subset E \mid A \text{ is a bounded subset of } E\}$. By $\alpha : B(E) \to [0, \infty)$, we denote a *measure of noncompactness* (see: [1], [7], [8] or 11). For the sake of simplicity, we can assume that α is the Kuratowski measure of noncompactness. Let $X \subset E$ and $f : X \to X$ be a continuous map. We say that f is a *condensing* map if, for every bounded $A \subset X$ with $\alpha(A) > 0$, we have:

$$\alpha(f(A)) < \alpha(A).$$

Nussbaum [11] proved:

Theorem C.1. *Let X be an open subset of E and let $f : X \to X$ be a condensing map with compact attractor. Then*

(i) *the (generalized) Lefschetz number $\Lambda(f)$ of f is well-defined and*

(ii) *$\Lambda(f) \neq 0$ implies that $\mathrm{Fix}(f) \neq \emptyset$.*

We say that a closed bounded subset X of E is a *special ANR* (see: [8] or [1]) if there exists an open $U \subset E$ and a continuous map $r : U \to X$ such that:

(i) $X \subset U$

(ii) $r(x) = x$, for every $x \in X$,

(iii) for every $A \subset U$, we have $\alpha(r(A)) \leq \alpha(A)$.

In [8], the following result was proved:

Theorem C.2. *Let X be a special* ANR *and let $f : X \to X$ be a condensing map. Then*

(i) *the (generalized) Lefschetz number $\Lambda(f)$ of f is well-defined and*

(ii) *$\Lambda(f) \neq 0$ implies that $\mathrm{Fix}(f) \neq \emptyset$.*

D. Concluding remarks. The above arguments demonstrate that, in the paper of Richeson and Wieseman [12], there are neither new results, nor original ideas. We would also like to supply the following three remarks.

Remark D.1. As we have shown in [1], all results presented in sections A–B can be extended to locally convex (not necessarily metric) spaces and those in Section C to Fréchet spaces. On the other hand, to guarantee the existence of a compact attractor in an infinite dimensional space like in Theorem C.2 is a delicate problem (cf. e.g. [5], [10], [14]).

Remark D.2. All the above results remain true for admissible multivalued mappings with an appropriately modified notion of a compact attractor; see: [1], [7], and the references therein. As a direct consequence of this fact, uniformly dissipative in the sense of N. Levinson (upper) Carathéodory systems of equations (inclusions) in $\mathbb{R}^n$: $x' = F(t,x)$ ($x' \in F(t,x)$), where $F(t,x) \equiv F(t+\omega,x)$, i.e. those with uniformly ultimately bounded solutions $x(\cdot) \in \mathrm{AC}_{loc}([0,\infty),\mathbb{R}^n)$, admit ω-periodic solutions (see [1]). In the case of uniqueness, the same is true for dissipative systems, i.e. when $\limsup_{t\to\infty} |x(t)| \le D$, where D is a common constant for all solutions $x(\cdot) \in \mathrm{AC}_{loc}([0,\infty),\mathbb{R}^n)$, (cf. [5], [10]).

Remark D.3. Since, according to [13], the Nielsen number $N(f)$ for a single-valued continuous map $f : X \to X$ is well-defined, provided

(i) X is an ANR,
(ii) $\mathrm{Fix}(f)$ is compact,
(iii) $\Lambda(f)$ is well-defined,

the conclusions of A.4, its analogy for CAC-maps (see B.1), C.1 and C.2 can be completed by the cardinality $\#\mathrm{Fix}(f) \ge \mathrm{N}(f)$. For further information about multiple fixed and periodic points of CAC-maps and their localization obtained on the basis of the relative Nielsen numbers, see [3].

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Received: 27.VI.2005

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