ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" IAŞI
Tomul LI, s.I, Matematică, 2005, f.2

SOME RESULTS ON BOUNDEDNESS AND STABILITY OF A THIRD ORDER DIFFERENTIAL EQUATION WITH DELAY

BY

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Abstract. Sufficient conditions are established for the uniform asymptotic stability of the zero solution of equation (1) and boundedness of a solution of equation (2).
Mathematics Subject Classification 2000: 34K20.
Key words: Delay differential equation, uniform asymptotic stability, boundedness.

1. Introduction. In this paper we consider third order delay differential equations of the form

$$\dddot{x}(t)+a(t)f(x(t),\dot{x}(t))\ddot{x}(t)+b(t)g(x(t-\tau),\dot{x}(t-\tau))+c(t)h(x(t-\tau))=0 \tag{1}$$

and

$$\begin{aligned}\dddot{x}(t)+a(t)f(x(t),\dot{x}(t))\ddot{x}(t)+b(t)g(x(t-\tau),\dot{x}(t-\tau))\\ +c(t)h(x(t-\tau))=p(t)\end{aligned} \tag{2}$$

where $a(t)$, $b(t)$ and $c(t)$ are continuously differentiable functions in $[0,\infty)$, $p(t)$ is a continuous function in $[0,\infty)$, $\tau>0$ is a constant; f,g and h are continuous functions for all x and y with $g(x,0)=h(0)=0$. Also, f_x, g_x, g_y and $h'(x)$ exist and are continuous for all x and y. All coefficients and functions that are given in the above equations and as well as the solutions of (1) and (2) are assumed to be real valued.

Equations of these types are encountered in the control of a flying apparatus in cosmic space. Particularly, the linear case of equation (1) without delay states the motion $x(t)$ of a steam supply control slide valve.

Many papers have been published on the boundedness, stability, oscillation and periodicity of solutions of (1) and (2) without delay. The book of Reissing, Sansone and Conti is a good source for these kind of studies of third order nonlinear ordinary differential equations [1, see Chapters 4 and 5]. HARA [2] considered the ordinary differential equations

$$\dddot{x} + a(t)f(x,\dot{x})\ddot{x} + b(t)g(x,\dot{x}) + c(t)h(x) = p(t) \tag{3}$$

and

$$\dddot{x} + a(t)f(x,\dot{x})\ddot{x} + b(t)g(x,\dot{x}) + c(t)h(x) = p(t,x,\dot{x},\ddot{x}) \tag{4}$$

He studied the boundedness of solutions of (3) and (4). CHUKWU [3] considered the third order delay differential equation of the form

$$\begin{aligned}\dddot{x}(t) + f(x(t),\dot{x}(t),\ddot{x}(t))\ddot{x}(t) + g(x(t-h),\dot{x}(t-h))\\ +i(x(t-h) = p(t,x(t),\dot{x}(t),x(t-h),\dot{x}(t-h),\ddot{x}(t))\end{aligned} \tag{5}$$

and investigated the existing of a periodic solution of (5). Recently, the first author [4] showed that the zero solution of the fourth order delay differential equation

$$\begin{aligned}x^{(4)}(t) + e\left(t,x(t),\dot{x}(t),\ddot{x}(t),\dddot{x}(t)\right)\dddot{x}(t) + f\left(t,\ddot{x}(t-\tau)\right)\\ +g\left(t,\dot{x}(t-\tau)\right) + h\left(x(t-\tau)\right) = 0\end{aligned} \tag{6}$$

is uniformly asymptotic stable.

As we know, there are no results about boundedness and uniform asymptotic stability for third order delay differential equations. So, our aim in this paper is to establish sufficient conditions for the uniform asymptotic stability of the zero solution of equation (1) and boundedness of a solution of equation (2).

The paper is organized as follows. First, we recall some basic notations and results. Then, in Section 3, we stated our main results and we indicated that the Routh-Hurwitz conditions are satisfied for linear case. Moreover, we introduced a Liapunov functional. Section 4 deals with the proofs of main results. Finally, we presented some remarks and examples.

2. Basic notations and result. The following notations will be used: For $x \in R^n$, $|x|$ denotes the Euclidean norm in R^n. For a given $\tau > 0$, C denotes the space of continuous functions mapping the interval

$[-\tau, 0]$ into R^n, and, for $\varphi \in C$, $\|\varphi\| = \sup_{-\tau \le \theta \le 0} |\varphi(\theta)|$. Also, C_H will denote the set of $\varphi \in C$ such that $\|\varphi\| \le H$ where $H > 0$. If x is a continuous function of u defined on $-\tau \le u < A$, $A > 0$, and if t is a fixed number satisfying $0 \le t < A$, then the symbol x_t denotes the restriction of x to the interval $[t-\tau, t]$, i.e., x_t is an element of C defined by $x_t(\theta) = x(t+\theta)$ for $-\tau \le \theta \le 0$.

Let $\dot{x}(t)$ denote the right-hand derivative of $x(u)$ at $u = t$ and consider the functional-differential equation

$$\dot{x}(t) = F(t, x_t) \tag{7}$$

where $F(t, \varphi) \in R^n$ is defined and continuous on $[0, \infty) \times C_H, F(t, 0) \equiv 0$ for all $t \ge 0$.

Definition 1. *A function $x(t_0, \varphi)$ is said to be a solution of* (7) *with initial condition $\varphi \in C_H$ at $t = t_0$, $t_0 \ge 0$, if there is an $A > 0$ such that $x(t_0, \varphi)$ is a function from $[t_0 - \tau, t_0 + A)$ into R^n with the properties*

(i) $x_t(t_0, \varphi) \in C_H$ *for* $t_0 \le t < t_0 + A$,

(ii) $x_{t_0}(t_0, \varphi) = \varphi$,

(iii) $x(t_0, \varphi)$ *satisfies (7) for* $t_0 \le t < t_0 + A$.

We denote by $x(t; t_0, \varphi)$ the value of $x(t_0, \varphi)$ at t.

Definition 2. *A Liapunov functional is a continuous $V(t, \varphi) : [0, \infty) \times C_H \to [0, \infty)$ whose derivative along a solution of* (7) *will be denote by $\dot{V}_{(7)}$ and is defined by*

$$\dot{V}_{(7)}(t, \varphi) = \limsup_{\delta \to 0^+} \frac{1}{\delta} \left\{ V\left(t + \delta, x_{t+\delta}(t, \varphi)\right) - V(t, \varphi) \right\} \tag{8}$$

where $x(t_0, \varphi)$ is the solution of (7) *with $x_{t_0}(t_0, \varphi) = \varphi$.*

Definition 3. *The solution $x = 0$ of equation* (7) *is said to be stable if for any $t_0 \ge 0$ and $\varepsilon > 0$, there is a positive $\delta = \delta(\varepsilon, t_0)$ such that $\varphi \in C_\delta$ implies $|x(t; t_0, \varphi)| < \varepsilon$ for $t \ge t_0$. The solution $x = 0$ is said to be asymptotically stable if it is stable and if for each $t_0 \ge 0$ there is a $\delta > 0$ such that $\varphi \in C_\delta$ implies $x(t; t_0, \varphi) \to 0$ as $t \to \infty$. The solution $x = 0$ is said to be uniformly stable if the number δ in the definition is independent of t_0. The solution $x = 0$ is said to be uniformly asymptotically stable if it is uniformly stable and if there is a $\delta_0 > 0$ and for every $\eta > 0$ there exists* *a* $T(\eta) > 0$ *such that* $\varphi \in C_{\delta_0}$ *implies* $|x(t; t_0, \varphi)| < \eta$ *for* $t \geq t_0 + T(\eta)$ *and for every* $t_0 \geq 0$ [5].

Now, let us state a well known result due to BURTON [6], important in our development.

Theorem 1. *Suppose there is a continuous functional* $V(t, \varphi)$ *locally Lipschitz in* φ, *defined on* $0 \leq t < \infty$, $\|\varphi\| < H$ *which satisfies following conditions:*

(H1) $W(|\varphi(0)|) \leq V(t, \varphi) \leq W_1(|\varphi(0)|) + W_2(|||\varphi|||)$,

(H2) $\dot{V}_{(7)}(t, \varphi) \leq -W_3(|\varphi(0)|)$,

where W, W_1, W_2 *and* W_3 *will denote continuous functions from* $[0, \infty)$ *into* $[0, \infty)$ *with* $W_i(0) = 0$, $W_i(r) > 0$ *if* $r > 0$, W_i *nondecreasing* $(i = 1, 2, 3)$ *and*

$$|||\varphi||| = \left(\int_{-\tau}^{0} \left[\sum_{i=1}^{n} \varphi_i^2(s) \right] ds \right)^{1/2}.$$

Then the zero solution of (7) *is uniformly asymptotically stable.*

3. Main results.

The following two theorems are our main results.

Theorem 2. *Suppose that for every* t, x, y, z *in*

$$\Omega := \{(t, x, y, z) \in R^4 : 0 \leq t < \infty, |x| < H_1, |y| < H_1, \\ |z| < H_1, H_1 \leq H\}$$

(i) *there exist positive constants* a_0, b_0, c_0, A, B, C *such that*

$$0 < a_0 \leq a(t) \leq A, \quad 0 < b_0 \leq b(t) \leq B, \quad 0 < c_0 \leq c(t) \leq C,$$

(ii) f_0, $\overline{f}$, g_0 *and* $\overline{g}$ *are constants such that*

$$\begin{aligned} 0 &< f_0 \leq f(x, y) \leq \overline{f} \\ 0 &< g_0 \leq \frac{g(x, y)}{y} \leq \overline{g} \text{ for } y \neq 0, \end{aligned}$$

(iii) $yf_x(x, y) \leq 0$, $-L \leq g_x(x, y) \leq 0, g_y(x, y) \leq L'$ *where* L *and* L' *are positive constants,*

(iv) $h'(x) \leq h_1 \leq \dfrac{a_0 b_0 f_0 g_0}{C}$,

(v) *δ is a positive constant such that for $x \neq 0$,*

$$\frac{\mu_2}{4c_0}\left\{A(\overline{f}-f_0)+\frac{B}{\mu_1}(\overline{g}-g_0)\right\}<\delta \leq h(x)/x$$

where μ_1 and μ_2 are arbitrary constants satisfying

$$(Ch_1/b_0g_0)<\mu_1<a_0f_0 \quad , \quad 0<\mu_2<(a_0b_0f_0g_0-Ch_1)/Af_0,$$

(vi) $(|a'(t)|+|b'(t)|+|c'(t)|) \leq k_4/k_5$; *where k_4 is a certain nonnegative constant and k_5 is a certain positive constant.*

Then the zero solution of (1) *is uniformly asymptotically stable.*

Theorem 3. *If hypotheses* (*i*)-(*vi*) *of Theorem 2 hold and if further for all $t \geq t_0$*

(vii) $\int_{t_0}^{t}|p(s)|\,ds \leq \overline{p}<\infty$,

then there exists a finite positive constant D such that the solution $x(t)$ of (2) *defined by the initial conditions*

$$x(t)=\phi(t),\ \dot{x}(t)=\dot{\phi}(t),\ \ddot{x}(t)=\ddot{\phi}(t), \qquad t_0-\tau \leq t \leq t_0\ , \tag{9}$$

satisfies the inequalities

$$|x(t)| \leq D,\ \left|\dot{x}(t)\right| \leq D,\ \left|\ddot{x}(t)\right| \leq D \tag{10}$$

for all $t \geq t_0$, where $\overline{p}$ is a finite positive number and $\phi \in C^2([t_0-\tau,t_0],R)$.

The following special case should be noted:

$$a(t)f(x(t),\dot{x}(t)) \equiv a,\ b(t)g(x(t-\tau),\dot{x}(t-\tau)) \equiv b\dot{x}(t),\ c(t)h(x(t-\tau)) \equiv cx(t),$$

where a, b, and c are real constants. Then the hypotheses (i), (ii), and (iv) reduce to the Routh-Hurwitz criteria (that is, $a>0, b>0, c>0$, and $ab-c>0$) for the global asymptotic stability of the trivial solution of equation

$$\dddot{x}(t)+a\ddot{x}(t)+b\dot{x}(t)+cx(t)=0.$$

The hypotheses (iii), (v), (vi) are trivally fulfilled.

It is note that we assume the existence of solutions for both equations. But, the uniqueness of solutions is just needed for our second result. So, we suppose that the solution of equation (2) with initial conditions (9) is unique.

The proofs of Theorem 2 and 3 are based on a Liapunov functional constructed as below.

The equation (1) is equivalent to the system

$$\begin{cases} \dot{x}(t) = y(t) \\ \dot{y}(t) = z(t) \\ \dot{z}(t) = -a(t)f(x(t),y(t))z(t) - b(t)g(x(t),y(t)) - c(t)h(x(t)) + G(t) \end{cases} \tag{11}$$

where

$$\begin{aligned} G(t) &= b(t)\int_{-\tau}^{0} g_x\left(x(t+\theta),y(t+\theta)\right)y(t+\theta)d\theta \\ &+b(t)\int_{-\tau}^{0} g_y\left(x(t+\theta),y(t+\theta)\right)z(t+\theta)d\theta \\ &+c(t)\int_{-\tau}^{0} h^{'}\left(x(t+\theta)\right)y(t+\theta)d\theta. \end{aligned} \tag{12}$$

Now, we introduce the Liapunov functional $V(t,\varphi)$ defined as

$$V(t,\varphi) = V_1(t) + V_2(t) + V_3(t) + V_4(t,\varphi) \tag{13}$$

where $V_1(t), V_2(t), V_3(t)$ and $V_4(t,\varphi)$ are given by

$$\begin{aligned} 2V_1(t) =& 2\mu_1 c(t)\int_0^{x(t)} h(\xi)d\xi + 2c(t)h(x(t))y(t) + 2b(t)\int_0^{y(t)} g(x(t),\eta)d\eta \\ &+2\mu_1 a(t)\int_0^{y(t)} f(x(t),\eta)\eta d\eta + 2\mu_1 y(t)z(t) + z^2(t), \end{aligned} \tag{14}$$

$$\begin{aligned} 2V_2(t) &= \mu_2 b(t)g_0 x^2(t) + 2a(t)f_0 c(t)\int_0^{x(t)} h(\xi)d\xi + a^2(t)f_0^2 y^2(t) \\ &-\mu_2 y^2(t) + 2b(t)\int_0^{y(t)} g(x(t),\eta)d\eta + z^2(t) + 2\mu_2 a(t)f_0 x(t)y(t) \\ &+2\mu_2 x(t)z(t) + 2a(t)f_0 y(t)z(t) + 2c(t)h(x(t))y(t), \end{aligned} \tag{15}$$

$$2V_3(t) = 2a^2(t)f_0\int_0^{y(t)} f(x(t),\eta)\eta d\eta - a^2(t)f_0^2 y^2(t), \tag{16}$$

$$V_4(t,\varphi)=\frac{2k_4}{\tau}\int_{-\tau}^{0}\left\{\int_{\theta_1}^{0}\left(x^2(t+\theta)+y^2(t+\theta)+z^2(t+\theta)\right)d\theta\right\}d\theta_1 \tag{17}$$

where $3k_4=k_3-\sqrt{k_3^2-4k^2\tau^2}$ is nonnegative, k_3 is a positive constant that is determined in (21) and

$$\begin{aligned} k \;=\; & \max\{(BL+Ch_1)\mu_2,(BL+Ch_1)(\mu_1+Af_0),2(BL+Ch_1),\\ & BL'\mu_2, BL'(\mu_1+Af_0),2BL'\}\,. \end{aligned}$$

4. Proofs of theorems

Proof of Theorem 2. This theorem is proved by using of Theorem 1. First of all, it is clear that $V(t,\varphi)$ defined by (13) satisfies locally Lipschitz condition in φ because it is continuously differentiable with respect to φ for $0\le t<\infty, \|\varphi\|\le H_1$.

Now, we have to show that the functional $V(t,\varphi)$ verifies conditions (H1) and (H2) of Theorem 1. For (H1), we consider the following inequalities estimated by Hara [2]

$$\begin{aligned} k_1(x^2(t)+y^2(t)+z^2(t)) &\le V_1(t)+V_2(t)+V_3(t)\\ &\le k_2(x^2(t)+y^2(t)+z^2(t)) \end{aligned} \tag{18}$$

where k_1 and k_2 are positive numbers. Adding $V_4(t,\varphi)$ to the both sides of (18) gives

$$\begin{aligned} V_4(t,\varphi)+k_1(x^2(t)+y^2(t)+z^2(t)) &\le V(t,\varphi)\\ &\le k_2(x^2(t)+y^2(t)+z^2(t))+V_4(t,\varphi). \end{aligned}$$

Since $V_4(t,\varphi)$ is nonnegative, we have

$$\begin{aligned} & k_1\,|\varphi(0)|^2\le V(t,\varphi)\\ & \le k_2\,|\varphi(0)|^2+2k_4\int_{-\tau}^{0}\left[x^2(t+\theta)+y^2(t+\theta)+z^2(t+\theta)\right]d\theta\\ & = k_2\,|\varphi(0)|^2+2k_4\int_{-\tau}^{0}|\varphi(\theta)|^2\,d\theta\\ & = k_2\,|\varphi(0)|^2+2k_4\,|||\varphi|||^2 \end{aligned} \tag{19}$$

From (19), it follows $W(r)=k_1r^2, W_1(r)=k_2r^2, W_2(r)=2k_4r^2$. Therefore, hypothesis (H1) is satisfied.

To see the remain hypothesis (H2), we calculate the derivative of the functional $V(t,\varphi)$ along a solution $(x(t),y(t),z(t))$ of (11) and obtain that

$$\begin{aligned}\dot{V}(t,\varphi)= & -U_1\left(t,x(t),y(t),z(t)\right)+U_2\left(t,x(t),y(t),z(t)\right)\\ & +G(t)\left[\mu_2 x(t)+\left(\mu_1+a(t)f_0\right)y(t)+2z(t)\right]\\ & +\frac{2k_4}{\tau}\int_{-\tau}^{0}\left[x^2(t)+y^2(t)+z^2(t)-x^2(t+\theta)-y^2(t+\theta)-z^2(t+\theta)\right]d\theta\end{aligned}\tag{20}$$

where

$$\begin{aligned}U_1(t,x(t),y(t),z(t)) &= -2c(t)h'(x(t))y^2(t)-\mu_1 z^2(t)\\ &+\mu_1 b(t)g(x(t),y(t))y(t)+2a(t)f(x(t),y(t))z^2(t)-\mu_2 b(t)g_0 x(t)y(t)\\ &-\mu_2 a(t)f_0 y^2(t)-\mu_2 a(t)f_0 x(t)z(t)\\ &+\mu_2 a(t)f(x(t),y(t))x(t)z(t)+\mu_2 b(t)g(x(t),y(t))x(t)\\ &+\mu_2 c(t)h(x(t))x(t)-a(t)f_0 z^2(t)\\ &+a(t)b(t)f_0 g(x(t),y(t))y(t)\end{aligned}$$

and

$$\begin{aligned}U_2(t,x(t),y(t),z(t)) &= \left[\mu_1 c'(t)+a'(t)c(t)f_0+a(t)c'(t)f_0\right]\int_0^{x(t)}h(\xi)d\xi\\ &+2b'(t)\int_0^{y(t)}g(x(t),\eta)d\eta+2b(t)y(t)\int_0^{y(t)}g_x(x(t),\eta)d\eta\\ &+\left[\mu_1 a'(t)+2a(t)a'(t)f_0\right]\int_0^{y(t)}f(x(t),\eta)\eta d\eta\\ &+\left[\mu_1 a(t)y(t)+a^2(t)f_0 y(t)\right]\int_0^{y(t)}f_x(x(t),\eta)\eta d\eta\\ &+2c'(t)h(x(t))y(t)+\frac{1}{2}\mu_2 b'(t)g_0 x^2(t)+\mu_2 a'(t)f_0 x(t)y(t)+a'(t)f_0 y(t)z(t).\end{aligned}$$

Note that the following inequalities are estimated by Hara [2]

$$U_1\geq 4k_3\left(x^2(t)+y^2(t)+z^2(t)\right),$$

$$U_2\leq k_5\left(\left|a'(t)\right|+\left|b'(t)\right|+\left|c'(t)\right|\right)\left(x^2(t)+y^2(t)+z^2(t)\right)\tag{21}$$

where k_3 and k_5 are certain positive constants depending on $\mu_1, \mu_2, a_0, b_0, c_0$, A, B, C, f_0, g_0, δ. Substituting these inequalities into (20), we obtain

$$\begin{aligned}
\dot{V} \quad &\leq -4k_3\left(x^2(t)+y^2(t)+z^2(t)\right)\\
&+k_5\left(\left|a'(t)\right|+\left|b'(t)\right|+\left|c'(t)\right|\right)\left(x^2(t)+y^2(t)+z^2(t)\right)\\
&+\left|G(t)\right|\left[\left|\mu_2 x(t)\right|+\left|(\mu_1+a(t)\,f_0)y(t)\right|+\left|2z(t)\right|\right]\\
&+\frac{2k_4}{\tau}\int_{-\tau}^{0}\left[x^2(t)+y^2(t)+z^2(t)-x^2(t+\theta)-y^2(t+\theta)-z^2(t+\theta)\right]d\theta
\end{aligned}$$

where $G(t)$ is given by (12).

From hypotheses (ii), (iii), (iv) and (vi) of Theorem 2, we have

$$\begin{aligned}
&\dot{V}\leq -4k_3\left(x^2(t)+y^2(t)+z^2(t)\right)+k_4\left(x^2(t)+y^2(t)+z^2(t)\right)\\
&+\left((BL+Ch_1)\int_{-\tau}^{0}\left|y(t+\theta)\right|d\theta+BL'\int_{-\tau}^{0}\left|z(t+\theta)\right|d\theta\right)\\
&\times\left(\mu_2\left|x(t)\right|+(\mu_1+Af_0)\left|y(t)\right|+2\left|z(t)\right|\right)\\
&+\frac{2k_4}{\tau}\int_{-\tau}^{0}\left[x^2(t)+y^2(t)+z^2(t)-x^2(t+\theta)-y^2(t+\theta)-z^2(t+\theta)\right]d\theta\\
&\leq -4k_3(x^2(t)+y^2(t)+z^2(t))+3k_4(x^2(t)+y^2(t)+z^2(t))\\
&+k\int_{-\tau}^{0}\left|x(t)\right|\left|y(t+\theta)\right|d\theta+k\int_{-\tau}^{0}\left|y(t)\right|\left|y(t+\theta)\right|d\theta\\
&+k\int_{-\tau}^{0}\left|z(t)\right|\left|y(t+\theta)\right|d\theta+k\int_{-\tau}^{0}\left|x(t)\right|\left|z(t+\theta)\right|d\theta\\
&+k\int_{-\tau}^{0}\left|y(t)\right|\left|z(t+\theta)\right|d\theta+k\int_{-\tau}^{0}\left|z(t)\right|\left|z(t+\theta)\right|d\theta\\
&-\frac{2k_4}{\tau}\int_{-\tau}^{0}\left[x^2(t+\theta)+y^2(t+\theta)+z^2(t+\theta)\right]d\theta
\end{aligned}\tag{22}$$

Inequality (22) can be rewritten as below

$$\begin{aligned}
&\dot{V}\leq\ -2\,k_3\left(x^2(t)+y^2(t)+z^2(t)\right)\\
&-\frac{1}{\tau}\int_{-\tau}^{0}\left[\frac{1}{2}k_4^{*}x^2(t)-k\tau\left|x(t)\right|\left|y(t+\theta\right|+\frac{1}{2}k_4y^2(t+\theta)\right]d\theta
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{\tau}\int_{-\tau}^{0}\left[\frac{1}{2}k_4^*y^2(t)-k\tau\,|y(t)|\,|y(t+\theta|+\frac{1}{2}k_4y^2(t+\theta)\right]d\theta\\
&-\frac{1}{\tau}\int_{-\tau}^{0}\left[\frac{1}{2}k_4^*z^2(t)-k\tau\,|z(t)|\,|y(t+\theta|+\frac{1}{2}k_4y^2(t+\theta)\right]d\theta\\
&-\frac{1}{\tau}\int_{-\tau}^{0}\left[\frac{1}{2}k_4^*x^2(t)-k\tau\,|x(t)|\,|z(t+\theta|+\frac{1}{2}k_4z^2(t+\theta)\right]d\theta\\
&-\frac{1}{\tau}\int_{-\tau}^{0}\left[\frac{1}{2}k_4^*y^2(t)-k\tau\,|y(t)|\,|z(t+\theta|+\frac{1}{2}k_4z^2(t+\theta)\right]d\theta\\
&-\frac{1}{\tau}\int_{-\tau}^{0}\left[\frac{1}{2}k_4^*z^2(t)-k\tau\,|z(t)|\,|z(t+\theta|+\frac{1}{2}k_4z^2(t+\theta)\right]d\theta\\
&-\frac{3}{2\tau}k_4\int_{-\tau}^{0}x^2(t+\theta)d\theta
\end{aligned}
\tag{23}
$$

where $k_4^* = k_3 + \sqrt{k_3^2 - 4k^2\tau^2}$.

It is clear that every integrand in (23) has the form $A\xi^2 + B\xi\eta + C\eta^2$ which is positive definite. Because $A = \frac{1}{2}k_4^* > 0,\ B^2 - 4AC = -\frac{1}{3}k^2\tau^2 < 0$. Therefore, it follows that

$$\dot V \le -2k_3\left(x^2(t)+y^2(t)+z^2(t)\right) = -2k_3\,|\varphi(0)|^2\,. \tag{24}$$

Thus we obtain that $W_3(r) = 2k_3r^2$ which is continuous, positive and-increasing for $r \neq 0$. This completes the verification of (H2). Hence all conditions of Theorem 1 are fulfilled.

Proof of Theorem 3. Here, we use the following system which is equivalent to equation (2):

$$
\begin{cases}
\dot x(t) = y(t)\\
\dot y(t) = z(t)\\
\dot z(t) = -a(t)f(x(t),y(t))z(t) - b(t)g(x(t),y(t))\\
\qquad\qquad -c(t)h(x(t)) + G(t) + p(t)
\end{cases}
\tag{25}
$$

where $G(t)$ is the same as (12). Let $(x(t), y(t), z(t))$ be the solution of (25) with $x(t) = \phi(t)$, $y(t) = \dot\phi(t)$, $z(t) = \ddot\phi(t)$. Consider the functional $V(t,\varphi)$ defined as in (13). Calculating the derivative of $V(t,\varphi)$ along the solution

$(x(t), y(t), z(t))$, we get

$$\begin{aligned}\dot{V}(t,\varphi) &\leq -2k_3\left(x^2(t)+y^2(t)+z^2(t)\right)+p(t)\left[\mu_2\, x+(\mu_1+a(t)f_0)y+2z\right]\\ &\leq |p(t)|\, D_1\left[|x|+|y|+|z|\right]\end{aligned}$$

where $D_1 = \max\{\mu_2, \mu_1 + Af_0, 2\}$. Since $|x| < 1 + x^2$, $|y| < 1 + y^2$, and $|z| < 1 + z^2$, we have

$$\dot{V}(t,\varphi) \leq 3D_1\,|p(t)| + D_1\,|p(t)|\left[|x|^2+|y|^2+|z|^2\right].$$

By (19),

$$k_1\left(x^2(t)+y^2(t)+z^2(t)\right) \leq V(t,\varphi). \tag{26}$$

So, we get

$$\dot{V}(t,\varphi) - \frac{D_1}{k_1}\,|p(t)|\,V(t,\varphi) \leq 3D_1\,|p(t)|$$

which means that

$$\dot{V}(t,x_t,y_t,z_t) - \frac{D_1}{k_1}\,|p(t)|\,V(t,x_t,y_t,z_t) \leq 3D_1\,|p(t)|\,.$$

Therefore, by (vii), we obtain that

$$\begin{aligned}&V\left(t,x_t,y_t,z_t\right) \leq \left[V(t_0,x_{t_0},y_{t_0},z_{t_0})+3\overline{p}D_1\right]\exp\left(\frac{D_1\overline{p}}{k_1}\right)\\ &\leq \Big[\sup_{-\tau\leq\theta\leq 0} V\left(t_0,x_{t_0},y_{t_0},z_{t_0}\right)+3\overline{p}D_1\Big]\exp\left(\frac{D_1\overline{p}}{k_1}\right)<\infty.\end{aligned} \tag{27}$$

From (26) and (27), we get the inequalities in (10). Hence the proof of the theorem is complete.

5. Some remarks and examples

Remark 1. Suppose that all conditions of Theorem 2 are hold. Therefore, it is clear that $\lim_{r\to+\infty} W(r) = +\infty$. In this case, all hypotheses of the theorem [7, p.27] are satisfied. So, the solutions of equation (1) are uniformly bounded (that is, for any $\alpha > 0$, there is a $\beta = \beta(\alpha) > 0$ such that, for all $\sigma \in R$, $\phi \in C$, $\|\phi\| \leq \alpha$, we have $|x(\sigma,\phi)(t)| < \beta$ for all $t \geq \sigma$).

Remark 2. Under the hypotheses of Chukwu's theorem, it can be guarantee that the zero solution of equation

$$\dddot{x}(t) + f(x(t), \dot{x}(t), \ddot{x}(t))\ddot{x}(t) + g(x(t-h), \dot{x}(t-h)) + i(x(t-h)) = 0 \tag{28}$$

is uniformly asymptotic stable. It is note that this result was not mentioned in his work [3].

Remark 3. If $a(t) = b(t) = c(t) = 1$, then equation (1) is a special case of equation (28). In this case our hypotheses are not the same as Chukwu's. Because Lyapunov functions that we used are different from each other.

Example 1. Consider the equation

$$\dddot{x}(t) + (e+1)\ddot{x}(t) + \dot{x}(t-1) + e^{-3}x^3(t-1) = e^{-3t} \tag{29}$$

with the initial conditions

$$\begin{aligned} x(t) &= e^{-t}, \quad -1 \le t \le 0, \\ \dot{x}(t) &= -e^{-t}, \quad -1 \le t \le 0, \\ \ddot{x}(t) &= e^{-t}, \quad -1 \le t \le 0. \end{aligned} \tag{30}$$

Equation (29) is a kind of (2) providing that $a(t) = e+1$, $b(t) = 1$, $c(t) = e^{-3}$, $p(t) = e^{-3t}$, $f(x(t), \dot{x}(t)) = 1$, $g(x(t-1), \dot{x}(t-1)) = \dot{x}(t-1)$, $h(x(t-1)) = x^3(t-1)$, $\tau = 1$, $t_0 = 0$. Therefore, all hypotheses of Theorem 3 are satisfied for the initial value problem (29)-(30). Then by Theorem 3, the solution $x(t)$ of (29)-(30) together with its derivatives of the first and second order are bounded for $t \ge 0$. Indeed, the problem (29)-(30) has the unique solution $x(t) = e^{-t}$ which is bounded in $[0, \infty)$.

Example 2. In this example, we show the numerical verification of Theorem 2. Consider the equation

$$\begin{aligned} &\dddot{x}(t) + \left(\frac{3}{2} + \cos t\right)\left(2 - x(t)\dot{x}(t)\right)\ddot{x}(t) \\ &+ (2 + \sin t)\left(2 - x(t-\frac{1}{2})\dot{x}(t-\frac{1}{2})\right)\dot{x}(t-\frac{1}{2}) + \frac{1}{3}x(t-\frac{1}{2}) = 0 \end{aligned} \tag{31}$$

which is a special case of (1) providing that $a(t) = \frac{3}{2} + \cos t$, $b(t) = 2 + \sin t$, $c(t) = 1$, $f(x(t), \dot{x}(t)) = (2 - x(t)\dot{x}(t))$, $g(x(t-\frac{1}{2}), \dot{x}(t-\frac{1}{2}))$ $= (2 - x(t-\frac{1}{2})\dot{x}(t-\frac{1}{2}))\dot{x}(t-\frac{1}{2})$, $h(x(t-\frac{1}{2})) = \frac{1}{3}x(t-\frac{1}{2})$, $\tau = \frac{1}{2}$, $t_0 = 0$.

We take into account the equation (31) together with the following three initial functions, respectively,

$$\phi(t) = t^3, \qquad -0.5 \le t \le 0, \tag{32}$$

$$\phi(t) = 0.5t^3, \qquad -0.5 \le t \le 0, \tag{33}$$

$$\phi(t) = 0.1t^3, \qquad -0.5 \le t \le 0. \tag{34}$$

In the following figure, the numerical solutions of the initial value problems (31)-(32), (31)-(33) and (31)-(34) are given respectively by the bolded lines, segmented lines and dotted lines, by using Runge-Kutta fourth order method: This figure illustrates the uniform asymptotic stability of the zero

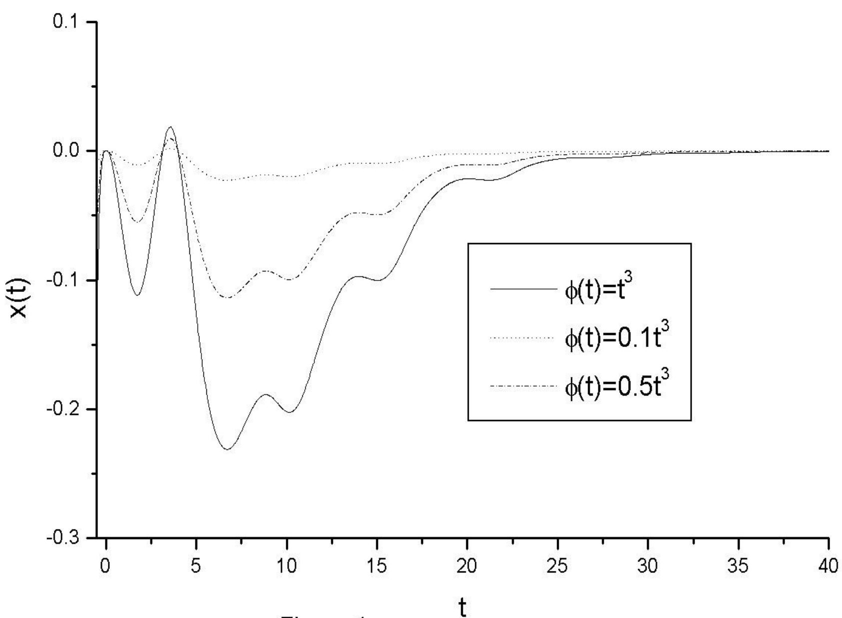

Figure 1.

solution of equation (1). From the figure, we observe as expected that, the closer the initial function to zero function in $[-0.5, 0]$ means closer the solution of the corresponding initial value problem to the zero solution. Indeed, the solution of initial value problem (31)-(34) almost coincides with the zero solution.

Acknowledgement. Authors would like to thank to Dr. Afet Golayoglu (Baskent University, Ankara, Turkey) for his kind support of taking the numerical outputs.

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Received: 15.IX.2004

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