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ON SYMPLECTIC MANIFOLDS ENDOWED WITH AN EXTERIOR RECURRENT STRUCTURE

BY

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Abstract. Geometrical and structural properties are proved for symplectic manifolds which are structured by the presence of a torse forming and exterior concurrent field.

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1. Introduction. Let (M, g) be a $2m$-dimensional Riemannian manifold and let ∇ be the covariant differential operator defined with respect to the metric tensor g. We assume in the sequel that M is oriented and that ∇ is the Levi-Civita connection. We assume that M is a locally symplectic manifold, carrying structural elements $(\Omega, \phi, \mathcal{P}, d^\omega)$ such that:

(i) Ω is a 2-form of rank $2m$, for which we have

$$\Omega(Z, Z') = g(\phi Z, Z');$$

(ii) ϕ is a tensor field of type (1.1) which satisfies

$$\phi^2 = -Id;$$

(iii) $\mathcal{P}(\mathcal{P}^A)$ is a torse forming vector field [15], which we will call the principal vector field on M;

(iv) $d^\alpha = d + c(\alpha)$ denotes the cohomological operator [4];

(v) $f \in \Lambda^0 M$.

Let $\Gamma TM = \chi(M)$ be the set of sections of the tangent bundle TM, and

$$\flat : TM \xrightarrow{\flat} T^*M \text{ and } \sharp : TM \xleftarrow{\sharp} T^*M$$

the classical isomorphisms defined by the metric tensor g (i.e. $^\flat$ is the index lowering operator, and $^\sharp$ is the index raising operator).

Following [9], we denote by

$$\Lambda^q(M, TM) = \Gamma \operatorname{Hom}(\Lambda^q TM, TM),$$

the set of vector valued q-forms ($q < \dim M$), and we write for the covariant derivative operator with respect to ∇

$$d^\nabla : A^q(M, TM) \to A^{q+1}(M, TM).$$

It should be noticed that in general $d^{\nabla^2} = d^\nabla \circ d^\nabla \neq 0$, unlike $d^2 = d \circ d = 0$. We denote by $dp \in A^1(M, TM)$ the canonical vector valued 1-form of M, which is also called the soldering form of M [3]. Let $\mathcal{O} = vect\{e_A \mid \Lambda = 1, \dots 2m\}$ be a local field of adapted vectorial frames over M and let $\mathcal{O}^* = covect\{\omega^A \mid A = 1, \dots 2m\}$ be its associated coframe. Then the soldering form dp is expressed by

$$dp = \sum_{A=1}^{2m} \omega^A \otimes e_A, \tag{1}$$

and we recall that E. Cartan's structure equations can be written as

$$\nabla e_A = \sum_{B=1}^{2m} \theta_A^B \otimes e_B, \tag{2}$$

$$d\omega^A = -\sum_{B=1}^{2m} \theta_B^A \wedge \omega^B, \tag{3}$$

$$d\theta_B^A = -\sum_{C=1}^{2m} \theta_B^C \wedge \theta_C^A + \Theta_B^A. \tag{4}$$

In the above equations θ (respectively Θ) are the local connection forms in the tangent bundle TM (respectively the curvature 2-forms on M).

A vector field $\mathcal{T}$ whose covariant differential satisfies

$$\nabla \mathcal{T} = r dp + v \otimes \mathcal{T}, \quad r \in \Lambda^0 M, v \in \Lambda^1 M, \tag{5}$$

is called a torse forming vector field (see also [13], [10], [6]).

In the present paper we assume that the vector basis $\{e_A\}$ satisfies

$$\nabla e_A = \mathcal{P}^A dp - \omega^A \otimes \mathcal{P}, \text{ where } \mathcal{P} = \sum_{A=1}^{2m} \mathcal{P}^A e_A, \tag{6}$$

for some vector field $\mathcal{P}$ on the manifold M; we will also the notations

$$\beta = \mathcal{P}^\flat = \sum_{A=1}^{2m} \mathcal{P}^A \omega^A, \text{ and } 2\rho = \sum_{A=1}^{2m} (\mathcal{P}^A)^2,$$

for the principal form β and the principal scalar ρ respectively.

We will show that the elements of the covector basis $\{\omega^A\}$ are exterior recurrent 1-forms [2] and have the closed Plaffian β as recurrence form:

$$d\omega^A = \beta \wedge \omega^A \Longrightarrow d\beta = 0. \tag{7}$$

With Θ^A_B denoting the curvature 2-forms on M, we will prove that

$$d\Theta^A_B = \frac{2f}{\rho} \beta \wedge \Theta^A_B, \tag{8}$$

and

$$\mathcal{L}_{\mathcal{P}} \Theta^A_B = 2f\rho\Theta^A_B. \tag{9}$$

In consequence of the above equations, one can state the following facts. In view of (8) we agree to call the manifold under consideration a recurrent Riemannian manifold and by (9) it follows that the principal vector field $\mathcal{P}$ defines an infinitesimal conformal transformation of the curvature forms.

In addition we will show that the principal vector field $\mathcal{P}$ is both

(i) a torse forming vector field, i.e.

$$\nabla \mathcal{P} = (f + 2\rho) dp - 2\beta \otimes \mathcal{P};$$

(ii) and an exterior concurrent vector field, i.e.

$$\nabla^2 \mathcal{P} = -\frac{1}{2m-1} Ric(\mathcal{P}) \mathcal{P}^\flat \otimes dp, \text{ where } \mathcal{P}^\flat = \beta.$$

Finally, we will also find the following properties regarding the local symplectic form Ω:

(iii) $\mathcal{L}_{\mathcal{P}}\Omega = 2(f + 2\rho)\Omega$,
(iv) $\mathcal{L}_{\phi\mathcal{P}}\Omega = 0$.

This proves that $\mathcal{P}$ defines an infinitesimal conformal transformation of Ω and that Ω is an absolute invariant of $\phi\mathcal{P}$.

2. Recurrent manifolds. In the present paper we assume that the vector basis $\{e_A|A = 1, \dots 2m\}$ satisfies

$$\nabla e_A = \mathcal{P}^A dp - \omega^A \otimes \mathcal{P}, \tag{10}$$

for some vector field

$$\mathcal{P} = \sum_{A=1}^{2m} \mathcal{P}^A e_A,$$

which is called the principal vector field on M.

For later use in the sequel we also introduce the notation

$$\beta = \mathcal{P}^\flat = \sum_{A=1}^{2m} \mathcal{P}^A \omega^A; \tag{11}$$

the Pfaffian β is called the principal form on M.

From (2) one first gets that the connection forms θ_B^A can be expressed as

$$\theta_B^A = \mathcal{P}^B \omega^A - \mathcal{P}^A \omega^B. \tag{12}$$

Next, taking the exterior differential of the cobasis $\{\omega^A\}$, one finds by (3) and (12) that

$$d\omega^A = \beta \wedge \omega^A; \tag{13}$$

from which it follows by exterior differentiation that

$$d\beta = 0.$$

This shows that the elements of the covector basis $\{\omega^A\}$ are exterior recurrent 1-forms [2] and have the closed Pfaffian β as recurrence form.

If we take the exterior derivative of $\beta = \sum_{A=1}^{2m} \mathcal{P}^A \omega^A$, and since $d\beta = 0$, we find that

$$d\mathcal{P}^A = f\omega^A - \mathcal{P}^A \beta, \;\; f \in \Lambda^0 M, \tag{14}$$

for some function f on the manifold M.

From (14) it now follows by exterior differentiation that

$$df + 2f\beta = 0. \tag{15}$$

We define the scalar ρ by

$$2\rho := \|\mathcal{P}\|^2 = \sum_{A=1}^{2m} (\mathcal{P}^A)^2$$

and agree to call ρ the principal scalar of M.

Invoking (14) there follows that

$$d\rho = (f - 2\rho)\beta. \tag{16}$$

Regarding the vector basis $\{e_A\}$ one also finds by applying the Lie bracket that

$$\nabla[e_A, e_B] = -2\beta \otimes [e_A, e_B] - [e_A, e_B]^\flat \otimes \mathcal{P}. \tag{17}$$

Further, by (14) one also gets that

$$\nabla\mathcal{P} = (f + 2\rho)dp + 2\beta \otimes \mathcal{P}, \tag{18}$$

which proves that $\mathcal{P}$ is a closed torse forming vector field. Operating on (18) by d^∇ yields

$$d^\nabla(\nabla\mathcal{P}) = (\nabla)^2\mathcal{P} = 2f\beta \wedge dp, \tag{19}$$

which shows that $\mathcal{P}$ is an exterior concurrent vector field [11], [18]. Consequently, one may now write

$$(\nabla)^2\mathcal{P} = -\frac{1}{2m-1} Ric(\mathcal{P})\mathcal{P}^\flat \otimes dp. \tag{20}$$

It is also interesting to draw the attention on the fact that $\mathcal{P}$ possesses the property

$$\nabla_{\mathcal{P}}\mathcal{P} = (f - 2\rho)\mathcal{P}, \tag{21}$$

which means that $\mathcal{P}$ is an affine geodesic vector field. On the other hand, exterior differentiation of (12) gives on behalf of (14) that

$$d\theta_B^A = -2f\omega^A \wedge \omega^B, \tag{22}$$

which is indeed consistent and in agreement with (15). Now making use of E. Cartan's structure equations involving the curvature forms Θ^A_B, one finds by (4)

$$\Theta^A_B = 2\rho\omega^A \wedge \omega^B + \beta \wedge \theta^A_B, \tag{23}$$

and by (22) and (16) exterior differentiation of (12) yields

$$d\Theta^A_B = \frac{2f}{\rho}\beta \wedge \theta^A_B. \tag{24}$$

This shows the relevant fact that the curvature forms θ^A_B are exterior recurrent. In view of this observation, we agree to call the manifold M under consideration a recurrent Riemannian manifold.

Taking moreover the Lie derivative of Θ^A_B with respect to the principal vector field $\mathcal{P}$, and since $i_{\mathcal{P}}\Theta^A_B = 0$, one derives that

$$\mathcal{L}_{\mathcal{P}}\Theta^A_B = 2f\rho\Theta^A_B. \tag{25}$$

Hence, following a known definition [5], the above equation means that $\mathcal{P}$ defines an infinitesimal conformal transformation of the curvature forms Θ^A_B.

On the other hand, regarding the curvature forms, we recall that the Bianchi forms in the sense of Tachibana [12] are defined by

$$\Theta^{(p)}_{u^1,\ldots,u^{2p}} = \Theta^{u_2}_{u_1} \wedge \Theta^{u_3}_{u_2} \wedge \ldots \wedge \Theta^{2p}_{2p-1}. \tag{26}$$

Noticing that on basis of (23) Bianchi's identities

$$\sum_{B=1}^{2m} \omega^B \wedge \Theta^A_B = 0, \quad A \in \{1, \ldots 2m\}, \tag{27}$$

are indeed satisfied, one can conclude from this that the Bianchi forms (26) do identically vanish, i.e.

$$\Theta^{(p)}_{u^1,\ldots,u^{2p}} = 0. \tag{28}$$

Introducing the operator ϕ defined as

$$\phi e_A = e_{A^*} \;\; A \in \{1, \ldots m\}, \qquad \phi e_{A^*} = -e_A, \;\; A^* = A + m,$$

one can check that indeed $\phi^2 = -Id$. Acting with ϕ on the principal vector field $\mathcal{P}$, one obtains in a first step that

$$\phi\mathcal{P} = \sum_{A=1}^{m} \mathcal{P}^A e_{A^*} - \mathcal{P}^{A^*} e_A \qquad A^* = A + m. \tag{29}$$

Next, operating on $\phi\mathcal{P}$ by ∇ and taking (10) into account, one gets

$$\nabla(\phi\mathcal{P}) = f\phi dp - \beta \otimes \phi\mathcal{P} - (\phi\mathcal{P})^\top \otimes \mathcal{P}. \tag{30}$$

In this way we are led to the following relations:

$$\nabla_{\mathcal{P}}(\phi\mathcal{P}) = (f - 2\rho)\phi\mathcal{P}, \quad \nabla_{\phi\mathcal{P}}(\mathcal{P}) = (f + 2\rho)\phi\mathcal{P}. \tag{31}$$

Now, with the Lie bracket $[,]$, we find from (31) that

$$[\mathcal{P}, \phi\mathcal{P}] = -4\rho\phi\mathcal{P} \Longleftrightarrow \mathcal{L}_{\mathcal{P}}(\phi\mathcal{P}) = -4\rho\phi\mathcal{P}. \tag{32}$$

Hence, as is well known, the above relation proves that $\phi\mathcal{P}$ admits an infinitesimal transformation with generator $\mathcal{P}$ (see also [1]).

In the same perspective, and making use of the operator $d^{2\beta}$, one derives

$$d^{2\beta}(\phi\mathcal{P})^\flat = 2f\Omega. \tag{33}$$

Taking now the Lie derivatives of

$$\Omega = \sum_{A=1}^{m} \omega^A \wedge \omega^{A^*}, \quad A^* = A + m,$$

with respect to $\mathcal{P}$, one finds that

$$\mathcal{L}_{\mathcal{P}}\Omega = 2(f + 2\rho)\Omega. \tag{34}$$

Accordingly one may say, as it is also case for the curvature forms Θ^A_B, that $\mathcal{P}$ defines an infinitesimal conformal transformation of the symplectic form Ω.

In view of (29), we further see that

$$i_{\phi\mathcal{P}}\Omega = -\beta, \tag{35}$$

from which one consecutively gets

$$\mathcal{L}_{\phi\mathcal{P}}\Omega = 0. \tag{36}$$

Therefore, following a known definition [11] (see also [8]), one may say that Ω is an absolute integral invariant of $\phi\mathcal{P}$.

Summarizing, we can formulate the following

Theorem 2.1. *Let $M(\Omega, \phi, \mathcal{P}, \rho)$ be a 2m-dimensional locally symplectic manifold carrying a principal vector $\mathcal{P}(\mathcal{P}^A)(A \in \{1, \dots 2m\})$ with principal scalar ρ $(2\rho = \sum_{\Lambda=1}^{2m} (\mathcal{P}^A)^2)$, such that the vector basis $\{e_A\}$ satisfies*

$$\nabla e_A = \mathcal{P}^A dp - \omega^A \otimes \mathcal{P}$$

(dp is the soldering form of M, and $\{\omega^A\}$ is the cobasis of $\{e_A\}$). Then, by the classical isomorphisms and with the notation $\beta = \mathcal{P}^\flat$, the following properties hold:

(i) *$\mathcal{P}$ is both a torse forming and an exterior recurrent vector field, i.e.*

(a) $\nabla\mathcal{P} = (f + 2\rho)dp - 2\beta \otimes \mathcal{P}$,

(b) $\nabla^2\mathcal{P} = 2f\beta \wedge dp, \quad f \in \Lambda^0 M$;

(ii) *if Θ_B^A denote the curvature forms, one has*

$$d\Theta_B^A = \frac{2f}{\rho}\beta \wedge \Theta_B^A,$$

which means that all curvature forms Θ_B^A are exterior recurrent;

(iii) *$\mathcal{P}$ defines an infinitesimal conformal transformation of the curvature forma Θ_B^A and of the symplectic form Ω:*

$$\mathcal{L}_\mathcal{P}\Theta_B^A = 2f\rho\Theta_B^A, \quad \mathcal{L}_\mathcal{P}\Omega = 2(f + 2\rho)\Omega;$$

(iv) *Ω is an absolute integral invariant of $\phi\mathcal{P}$:*

$$\mathcal{L}_{\phi\mathcal{P}}\Omega = 0, \quad \phi^2 = -Id;$$

(v) *$\phi\mathcal{P}$ admits an infinitesimal transformation with generator $\mathcal{P}$, i.e.*

$$\mathcal{L}_\mathcal{P}(\phi\mathcal{P}) = -4\rho\phi\mathcal{P}.$$

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