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WEAK TANGENCY AND WEAK DERIVATIVES IN BANACH SPACES*

BY

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Abstract. We get characterizations of decreasing functions and Lyapunov pairs along solutions to semilinear differential inclusions in Banach spaces in terms of a suitable weak derivative. The weak derivative introduced here corresponds to a weak tangency concept used in [5] in order to get viability with respect to differential inclusions for a locally weakly closed set.

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1. Introduction. Let X be a Banach space equipped with norm $\|\cdot\|$. If the sequence $\{v_n\}$ converges weakly to v we write $v_n \xrightarrow{w} v$. Finally, we use $\mathbb{R}_+$ for $[0,\infty)$.

Let $f : X \to \mathbb{R} \cup \{\infty\}$ be a function and F a multifunction from X to X. In [6] we studied the (weak) decreasing property of the function f for the differential inclusion

$$x'(t) \in F(x(t)). \tag{1}$$

Recall that f is (weakly) decreasing for (1) if for any $\xi \in X$ there exists a solution $x(\cdot)$ to the differential inclusion (1) on $\mathbb{R}_+$ with $x(0) = \xi$ which satisfies

$$f(x(t)) \leq f(\xi)\ \forall t \geq 0. \tag{2}$$

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See [9, p.211]. We also recall that f is strongly decreasing for (1) if for any $\xi \in X$ and for every solution to the differential inclusion (1) with $x(0) = \xi$ we have (2). See [9, p.217] To avoid confusions, in this paper we shall use the term "decreasing" instead of "weakly decreasing".

This is a basic tool in the study of asymptotic stability properties [9, p.208] as well as in the study of contingent or viscosity solutions to Bellman equations (see, for example [10], [11], [4]). It is closely related to the classical method of characteristics [3]. In [6] we presented characterizations of the decreasing property in Hilbert spaces, in terms of contingent derivatives, proximal subdifferential and Fréchet subdifferential. It is clear that the decreasing property (2) is equivalent to the viability (or weak invariance in other terminology) of the subset (epi(f) in $X \times \mathbb{R}$ with respect to the differential inclusion $z' \in F \times \{0\}$) where epi(f) means the epigraph of f,

$$\text{epi}\,(f) = \{(x, \alpha) \in X \times \mathbb{R} : f(x) \leq \alpha\}.$$

Our approach in [6] was based on recent results on viability in infinite dimensions [7], [5], [2]. Let us recall some facts from [6]. The set K is said to be viable with respect to the differential inclusion (1) if for any $\xi \in K$ there exists a solution $x(\cdot)$ of (1) on $\mathbb{R}_+$ with $x(0) = \xi$ such that $x(t) \in K$, $\forall t \geq 0$. The viability involves tangency concepts and tangency conditions.

The weak (sequential) tangent cone to the set K at $x \in K$ is

$$T_S^*(x) = \{v \in X \ : \ \exists\, t_n \downarrow 0,\, v_n \overset{w}{\to} v;\ x + t_n v_n \in K\}.$$

Recall that the Bouligand tangent cone is

$$T_K(x) = \{v \in X \ : \ \exists\, t_n \downarrow 0, v_n \to v;\ x + t_n v_n \in K\}.$$

For $x \in D(f) = \{x \in X : f(x) < \infty\}$ and $u \in X$, consider the extended real number

$$D\,f(x)(u) = \liminf_{\substack{t \downarrow 0 \\ p \to 0}} \frac{f(x + t(u + p)) - f(x)}{t},$$

called the Dini subderivative of f at x in the direction u, and recall the equality [1, p. 226]

$$\text{epi}\,(D\,f(x)) = T_{\text{epi}(f)}(x, f(x)),$$

for any $x \in D(f)$. It is clear that

$$D\,f(x)(u) = \inf\left\{\liminf_{n\to\infty} \frac{f(x+t_nu_n)-f(x)}{t_n} \,:\, t_n \downarrow 0,\, u_n \to u\right\}.$$

Moreover, the inf is in fact min in the relation above and, moreover, we may take lim instead of lim inf.

This suggests to define the weak Dini subderivative as

$$D^*\,f(x)(u) := \inf\left\{\liminf_{n\to\infty} \frac{f(x+t_nu_n)-f(x)}{t_n} \,:\, t_n \downarrow 0,\, u_n \overset{w}{\to} u\right\}, \tag{3}$$

for $x \in D(f)$. Here the inf is not necessarily a min but we also may take lim instead of lim inf. It is easy to see that

$$D^*\,f(x)(u) = \inf\left\{\liminf_{\substack{t\downarrow 0\\ n\to\infty}} \frac{f(x+tu_n)-f(x)}{t} \,:\, u_n \overset{w}{\to} u\right\},$$

in which form was introduced in [7]. Further, it follows directly from the definition of weak tangency that for any $x \in D(f)$ we have

$$T^*_{\mathrm{epi}(f)}(x, f(x)) \subseteq \mathrm{epi}\,(D^*\,f(x)) \subseteq \mathrm{cl}\; T^*_{\mathrm{epi}(f)}(x, f(x)).$$

As a matter of fact, the rightmost inclusion holds with the closure of vertical sections, i.e.,

$$(u,\lambda) \in \mathrm{epi}\,(D^*\,f(x)) \Rightarrow \exists \lambda_n \downarrow \lambda, (u,\lambda_n) \in T^*_{\mathrm{epi}(f)}(x, f(x)).$$

Our aim here is to extend some notions and results in [6] to differential inclusions of the form

$$y'(t) \in Ay(t) + F(y(t)), \tag{4}$$

where A generates a C_0-semigroup on the Banach space X.

We recall that the function $y : [\,0,T\,] \to X$ is a *mild solution* of (4) with $y(0) = x$ if there exists $f \in L^1([\,0,T\,];X)$, with $f(t) \in F(y(t))$ a.e. for $t \in (0,T)$ and such that

$$y(t) = S(t)x + \int_0^t S(t-s)f(s)\,ds$$

for $t \in [\,0, T\,]$.

2. Decreasing functions along solutions of (4)**.** Given $x \in K$ and $M(x) > 0$, let us introduce the $(A, M(x))$-weak sequential tangent set to K at $x \in K$ as

$$(*)T_S^{A,M(x)}(x) = \{v \in X \ : \ \exists\, t_n \downarrow 0,\, v_n \overset{w}{\to} 0;\ \|v_n\| \le M(x);\ S(t_n)x + t_n(v + v_n) \in K\}, \tag{5}$$

and the $(A, M(x))$-weak tangent set to K at $x \in K$, $(w)T_S^{A,M(x)}(x)$, as the set of all $v \in X$ such that for each $\delta > 0$ and each weak neighborhood V of 0 there exist $t \in (0, \delta)$ and $p \in V$ with $\|p\| \le M(x)$ and

$$S(t)x + t(v + p) \in K.$$

We have to mention that if the sequence (v_n) converges weakly to 0 then it is a fortiori bounded. The presence of $M(x)$ in (5) is motivated by the weak sequential tangency condition below, where the function $M(x)$ should be locally bounded.

Suggested by (3), define now the $(A, M(x))$-weak sequential Dini subderivative as

$$\begin{aligned}&(*)D^{A,M(x)}\, f(x)(u)\\ &= \inf\left\{\liminf_{n\to\infty} \frac{f(S(t_n)x + t_n(u + u_n)) - f(x)}{t_n} \ : \ t_n \downarrow 0,\, u_n \overset{w}{\to} 0,\right.\\ &\left.\|u_n\| \le M(x)\right\},\end{aligned} \tag{6}$$

for $x \in D(f)$.

Lemma 2.1. *For any $x \in D(f)$ we have*

$$\begin{aligned}(*)T_{\mathrm{epi}(f)}^{\mathcal{A},\mathcal{M}(x,f(x))}(x, f(x)) &\subseteq \mathrm{epi}\,((*)D^{A,M(x)}\, f(x))\\ &\subseteq (w)T_{\mathrm{epi}(f)}^{\mathcal{A},\overline{\mathcal{M}}(x,f(x))}(x, f(x))\end{aligned} \tag{7}$$

where

$$\mathcal{A} = \begin{pmatrix} A & 0\\ 0 & 0\end{pmatrix}, \tag{8}$$

$M(x) = \mathcal{M}(x, f(x))$ *and* $\overline{\mathcal{M}}(x, \mu) = M(x) + 1$, *for each* $(x, \mu) \in \mathrm{epi}(f)$.

Proof. Let us mention first that the semigroup generated by $\mathcal{A}$ is $\mathcal{S}(t)(x,\mu) = (S(t)x, \mu)$, for all $t \geq 0$, $x \in X$ and $\mu \in \mathbb{R}$. Let $(u,\theta) \in (*)T^{\mathcal{A},\mathcal{M}(x,f(x))}_{\mathrm{epi}(f)}(x, f(x))$. There exist $t_n \downarrow 0$, $u_n \xrightarrow{w} 0$, $\theta_n \to 0$ such that $\|(u_n,\theta_n)\| \leq \mathcal{M}(x,f(x))$ and

$$(S(t_n)x + t_n(u+u_n), f(x) + t_n(\theta+\theta_n)) \in \mathrm{epi}(f).$$

Therefore, there exists (u_n) with $\|u_n\| \leq M(x) = \mathcal{M}(x,f(x))$ such that

$$\liminf_{n\to\infty} \frac{f(S(t_n)x + t_n(u+u_n)) - f(x)}{t_n} \leq \theta.$$

This implies that $(*)D^{A,M(x)} f(x)(u) \leq \theta$, hence $(u,\theta) \in \mathrm{epi}\,((*)D^{A,M(x)} f(x))$.

Take now $(u,\theta) \in \mathrm{epi}\,((*)D^{A,M(x)} f(x))$. Consider $\varepsilon > 0$, $0 < \delta < \varepsilon$, $\delta < 1$, and a weak neighborhood V of 0 in X. We get

$$\inf \left\{ \liminf_{n\to\infty} \frac{f(S(t_n)x + t_n(u+u_n)) - f(x)}{t_n} \,:\, t_n \downarrow 0,\ u_n \xrightarrow{w} 0,\ \|u_n\| \leq M(x) \right\} < \theta + \delta,$$

hence there exist $t_n \downarrow 0$, $u_n \xrightarrow{w} 0$ with $\|u_n\| \leq M(x)$ such that

$$\liminf_{n\to\infty} \frac{f(S(t_n)x + t_n(u+u_n)) - f(x)}{t_n} < \theta + \delta.$$

Therefore, there exists $(p,s) \in X \times \mathbb{R}$ with $\|(p,s)\| \leq M(x) + 1$, $p \in V$, $s \in (0,\varepsilon)$ and there exists $t \in (0,\varepsilon)$ such that

$$(S(t_n)x + t(u+p), f(x) + t(\theta+s)) \in \mathrm{epi}(f).$$

This implies that

$$(u,\theta) \in (w)T^{\mathcal{A},\overline{\mathcal{M}}(x,f(x))}_{\mathrm{epi}(f)}(x, f(x))$$

and the proof is complete. □

Let us recall a viability result from [5]. We mention that a multifunction $F : K \to 2^X$ is weakly-weakly upper semicontinuous (u.s.c.) if for each $u \in K$ and each neighborhood V of $F(u)$ in the weak topology there exists a neighborhood W of u also in the weak topology, such that $F(v) \subset V$ for

each $v \in W$. Further, K is locally weakly closed if for each $x \in K$ there exists $r > 0$ such that $K \cap B(x, r)$ is weakly closed in X, where, as usual, $B(x, r)$ denotes the closed ball with center x and radius r. Finally, K is a viable domain for (4) if for each $x \in K$ there exists at least one mild solution $y : [\,0, T\,] \to K$ of (4) satisfying the initial condition $y(0) = x$.

Theorem 2.1. *Let X be a reflexive and separable Banach space, K a nonempty, weakly locally closed subset in X and $F : K \to 2^X$ a nonempty, closed, convex and bounded valued mapping which is weakly-weakly u.s.c. Let $A : D(A) \subset X \to X$ be the generator of a C_0-semigroup $S(t) : X \to X$, $t \geq 0$. Then a necessary and sufficient condition in order that K be a viable domain with respect to* (4) *is the bounded weak tangency condition* $(BwTC)$ *bellow.*
$(BwTC)$ *There exists a locally bounded function $M : K \to \mathbf{R}_+^*$ such that for each $x \in K$ there exists $y \in F(x)$ and such that for each $\delta > 0$ and each weak neighborhood V of 0 there exist $t \in (0, \delta\,]$ and $p \in V$ with $\|p\| \leq M(x)$ satisfying*

$$S(t)x + t(y + p) \in K.$$

The $(BwTC)$ can be stated as follows: there exists a locally bounded function $M : K \to \mathbb{R}_+^*$ such that for each $x \in K$,

$$F(x) \cap (w)T_K^{A,M(x)}(x) \neq \emptyset.$$

Remark 2.1. As proved in [5] (See Remark 3.1), the following stronger condition, $(B * TC)$, i.e. the sequential counterpart of the $(BwTC)$, is necessary (as well as sufficient) for viability.

$(B * TC)$ *There exists a locally bounded function $M : K \to \mathbf{R}_+^*$ such that for each $x \in K$,*

$$F(x) \cap (*)T_K^{A,M(x)}(x) \neq \emptyset.$$

We are now ready to state and to prove the main result of this paper. Recall that the function f is said to be decreasing for problem (4) if for any $x \in X$ there exist $T > 0$ and a solution $y : [0, T] \to X$ for the differential inclusion (4) with $y(0) = x$ and $f(y(t)) \leq f(x)$ for all $t \in [0, T]$.

Theorem 2.2. *Let X be a reflexive and separable Banach space, $F : X \to 2^X$ a nonempty, closed, convex and bounded valued mapping which is*

weakly-weakly u.s.c., $A : D(A) \subset X \to X$ *the generator of a* C_0*-semigroup* $S(t) : X \to X$, $t \geq 0$, *and let the weakly lower semicontinuous function* $f : X \to (-\infty, +\infty]$. *Then the function* f *is decreasing for problem* (4) *if and only if the following contingent inequality,* (CI) *is satisfied.*

(CI) *There exists a locally bounded function* $M : D(f) \to \mathbb{R}_+^*$ *such that, for each* $x \in D(f)$, *there exists* $u \in F(x)$ *such that*

$$(*)D^{A,M(x)} f(x)(u) \leq 0. \tag{9}$$

Proof. Notice first that $K = \text{epi}(f)$ is weakly closed in $X \times \mathbb{R}$ because $f : X \to (-\infty, +\infty]$ is weakly lower semicontinuous. Condition (CI) is equivalent to the following one:

$(CI1)$ *There exists a locally bounded function* $M : D(f) \to \mathbb{R}_+^*$ *such that, for each* $x \in D(f)$,

$$(F(x) \times \{0\}) \cap epi(*)D^{A,M(x)} f(x) \neq \emptyset.$$

In case condition (CI) is satisfied, then, by (7),

$$(F(x) \times \{0\}) \cap (w)T^{\mathcal{A},\overline{\mathcal{M}}(x,f(x))}_{\text{epi}(f)}(x, f(x)) \neq \emptyset.$$

Since

$$(w)T^{\mathcal{A},\overline{\mathcal{M}}(x,f(x))}_{\text{epi}(f),}(x, f(x)) = \bigcap_{\mu \geq f(x)} (w)T^{\mathcal{A},\overline{\mathcal{M}}(x,\mu)}_{\text{epi}(f)}(x, \mu)$$

and since $\overline{\mathcal{M}}(x, \mu) = M(x) + 1$, we deduce that the $(BwTC)$ condition is satisfied for the subset $\text{epi}(f)$ of $X \times \mathbb{R}$ and the evolution equation

$$\begin{pmatrix} y' \\ z' \end{pmatrix} \in \mathcal{A} \begin{pmatrix} y \\ z \end{pmatrix} + \begin{pmatrix} F(y) \\ 0 \end{pmatrix} \tag{10}$$

on the space $X \times \mathbb{R}$, with the operator $\mathcal{A}$ in (8). We obtain that $\text{epi}(f)$ is viable with respect to (10), which clearly implies that the function f is decreasing for problem (4).

Conversely, suppose that the function f is decreasing for problem (4). This implies that $\text{epi}(f)$ is viable with respect to (10), hence, by Remark 2.1, the bounded sequential weak tangency condition $(B * TC)$ is satisfied for the subset $\text{epi}(f)$ of $X \times \mathbb{R}$ and the evolution equation (10). Hence,

there exists a locally bounded function $\mathcal{M} : \text{epi}(f) \to \mathbf{R}_+^*$ such that for each $(x, \mu) \in \text{epi}(f)$,

$$(F(x) \times \{0\}) \cap (*)T_{\text{epi}(f)}^{\mathcal{A},\mathcal{M}(x,\mu))}(x, \mu) \neq \emptyset.$$

In particular,

$$(F(x) \times \{0\}) \cap (*)T_{\text{epi}(f)}^{\mathcal{A},\mathcal{M}(x,f(x))}(x, f(x)) \neq \emptyset.$$

Taking into account (7), we have that there exists a locally bounded function $M : D(f) \to \mathbb{R}_+^*$, $M(x) = \mathcal{M}(x, f(x))$ for all $x \in D(f)$, such that, for each $x \in D(f)$,

$$(F(x) \times \{0\}) \cap \text{epi}(*)D^{A,M(x)} f(x) \neq \emptyset.$$

Hence, (CI) is satisfied and this completes our proof. □

Remark 2.2. As in [6], we can prove that, in case $f : X \to \mathbb{R}$ is weakly continuous, Condition (CI) is equivalent to the fact that for any $x \in X$ there exist $T > 0$ and a solution $y : [0, T] \to X$ for the differential inclusion (4) with $y(0) = x$ such that the function $t \mapsto f(y(t))$ is decreasing on $[0, T]$. Moreover, by Zorn's Lemma one can prove that the decreasing property holds for the existence interval of $y(\cdot)$.

3. Lyapunov pairs. The results of the previous section can be extended to characterizations of Lyapunov pairs. Recall that the pairs of functions $f, g : H \to (-\infty, +\infty]$ form a Lyapunov pair for problem (4) if for every $x \in X$ there exists a solution $y(\cdot)$ of (4) with $y(0) = x$ such that

$$f(y(t)) + \int_0^t g(y(s))ds \leq f(x), \tag{11}$$

for all t on its existence interval. In [12] the authors obtain a characterization for a Lyapunov pair associated with a similar problem but with F single valued. That characterization refers to an inequality for which the solution is understood in the viscosity sense. We give here a characterization by an inequality involving the weak sequential Dini subderivative defined in (6).

Theorem 3.1. *Let X be a reflexive and separable Banach space, $F : X \to 2^X$ a nonempty, closed, convex and bounded valued mapping which is weakly-weakly u.s.c., $A : D(A) \subset X \to X$ the generator of a C_0-semigroup $S(t) : X \to X$, $t \geq 0$, the weakly lower semicontinuous function $f : X \to$*

$(-\infty, +\infty]$ *and the continuous function* $g : X \to (-\infty, +\infty)$. *Then* (f, g) *forms a Lyapunov pairs for problem* (4) *if and only if there exists a locally bounded function* $M : D(f) \to \mathbb{R}^*_+$ *such that, for each* $x \in D(f)$, *there exists* $u \in F(x)$ *such that*

$$(*)D^{A,M(x)} f(x)(u) + g(x) \leq 0. \tag{12}$$

Proof. Suppose that (f, g) forms a Lyapunov pair for problem (4). Hence for every $x \in D(f)$ there exists a solution $y(\cdot)$ of (4) with $y(0) = x$ such that (11) is satisfied. This means that the function $\tilde{f} : X \times \mathbb{R} \to (\infty, +\infty]$ defined by $\tilde{f}(x, y) = f(x) - y$ is decreasing for the problem

$$\begin{pmatrix} y' \\ z' \end{pmatrix} \in \mathcal{A} \begin{pmatrix} y \\ z \end{pmatrix} + \begin{pmatrix} F(y) \\ -g(y) \end{pmatrix} \tag{13}$$

We now apply Theorem 2.2 on $X \times \mathbb{R}$ to $\mathcal{A}$ defined in (8), $\tilde{F}(x, t) = F(x) \times \{-g(x)\}$, and $\tilde{f}(x, t) = f(x) - t$, and deduce that there exists a locally bounded function $M : D(f) \times \mathbb{R} \to \mathbb{R}^*_+$ such that, for each $(x, y) \in D(f) \times \mathbb{R}$, there exists $u \in F(x)$ such that

$$(*)D^{\mathcal{A},M(x,y)} \tilde{f}(x, y)(u, -g(x)) \leq 0.$$

It is easy to see that

$$g(x) + (*)D^{A,M(x,y)} f(x)(u) \leq (*)D^{\mathcal{A},M(x,y)} \tilde{f}(x, y)(u, -g(x))$$

thus we get

$$g(x) + (*)D^{A,M(x,0)} f(x)(u) \leq 0.$$

Conversely, suppose there exists a locally bounded function $M : D(f) \to \mathbb{R}^*_+$ such that, for each $x \in D(f)$, there exists $u \in F(x)$ such that (12) is satisfied. This implies

$$(*)D^{\mathcal{A},M(x)+1} \tilde{f}(x, y)(u, -g(x)) \leq 0$$

for every $y \in \mathbb{R}$. We apply again Theorem 2.2 to deduce that the function $\tilde{f}$ is decreasing for the problem (13). This implies that (f, g) forms a Lyapunov pair for (4), and this completes the proof. □

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