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STRONG INVARIANCE FOR DISCONTINUOUS HILBERT SPACE DIFFERENTIAL INCLUSIONS*

BY

T. DONCHEV, V. RIOS and P. WOLENSKI

Abstract. This paper presents necessary and sufficient conditions, for strong invariance with Hamilton-Jacobi inequalities for a differential inclusion in a Hilbert space. We study the cases when the right-hand side is almost upper semicontinuous with convex compact values o when it is almost upper hemicontinuous with convex weakly compact values. In both cases we essentially use the so called one sided Lipschitz condition.

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1. Introduction. This paper is devoted to characterizations of strongly invariant systems modelled as a differential inclusion

$$\text{(DI)} \qquad \dot{x}(t) \in F(t, x(t)) \quad \text{a.e. } t \in I := [0, 1],$$

with constant state constraints. For our setting we shall consider a Hilbert space E, and the multifunction $F : I \times E \rightrightarrows E$, is assumed with nonempty convex (weakly) compact values. As usual, a solution (or trajectory) to (DI) is an absolutely continuous (AC) $x : I \to E$ satisfying (DI) for almost all $t \in I$. In the sequel $\mathbb{B}$ will denote the unit (open) ball in E. For ease of exposition, and since it is more convenient for our purposes, we first consider the case of a compact valued right-hand side. State constraints are given by $x(t) \in S$, where S is a closed subset of E. We will provide this and others important definitions and hypotheses later.

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The issue about invariant systems has its origins in the flow-invariant sets theory, also called by some authors "viability theory" (see [1, 2]). We briefly recall its main concepts. The pair (S, F) is called a *weakly invariant* system if for each $x_0 \in S$ there exists a solution $x(\cdot)$ of (DI) with $x(0) = x_0$ satisfying $x(t) \in S$, $\forall t \geq 0$. Analogously, the pair (S, F) is called a *strongly invariant* system when for each $x_0 \in S$ there exists a solution $x(\cdot)$ on (DI) with $x(0) = x_0$ and every such a solution satisfies $x(t) \in S$, $\forall t \geq 0$.

Given a closed set S and $x \in S$ we denote $d_S(x) := \inf\limits_{a \in S} |x - s|$.

Denote $T_S^B(x) := \left\{ v : \liminf\limits_{h \downarrow 0} \dfrac{d_S(x + hv)}{h} = 0 \right\}$ the *contingent cone* of S at x, which was introduced by Bouligand.

Define

$$T_S^w(x) := \{v : \ S^w d_D(x, v) = 0\}$$

Here $D^w f(x, v) := \inf\limits_{v_i} \liminf\limits_{i \to \infty,\ t \to 0^+} \dfrac{f(x + tv_i) - f(x)}{t}$, where the infimum is taken over all sequences $\{v_i\}$ converging weakly to v (see [8]). Obviously $T_S^B(x) \subset T_S^w(x)$. A large number of works show how such a generalization provides criteria that are consistent with the single valued case. The most popular is the following tangential condition:

There exists a null set $A \subset I$ such that:

$$F(t, x) \bigcap T_S^B(x) \neq \emptyset, \ \forall x \in S, \ \forall t \in I \setminus A \tag{1}$$

was obtained as a characterization for weakly invariant autonomous control systems.

Another criterion for weakly invariance was given in 1993 by Veliov [25]. This used the notion of perpendicular directions to a set, or "proximal normals". The most pedestrian version of this criteria establishes that weakly invariance of (S, F) is equivalent to

$$h_F(t, x, \zeta) \leqslant 0 \quad \forall \zeta \in N_S^P(x), \ \forall x \in S, \ \forall t \in I \setminus A, \tag{2}$$

where $h_F : I \times E \times E \to \mathbb{R}$ is the *lower Hamiltonian* defined as

$$h_F(t, x, p) = \inf\big\{ \langle p, v \rangle : v \in F(t, x) \big\}, \tag{3}$$

and

$$N_S^P(x) := \{\zeta \in E :\ \exists\ \sigma > 0, \eta > 0|\ \langle \zeta, y - x\rangle \leqslant \sigma |y - x|^2,\ \forall y \in S \cap x + \eta \mathbb{B}\}$$

is the proximal normal cone of S at $x \in S$. An alternative proximal aiming approach of this equivalence is also presented in the works by Clarke et. al. [8], [10] and [11] (1998). Among others we refer to the recent book [20] (chapter I.4 and remarks after chapter I with a number of references on the subject) where infinite dimensional systems are considered. In 1995 a strong counterpart of the criterion (2) was obtained independently by Krastanov [21] and Clarke et. al. [10] when $E \equiv \mathbb{R}^n$. The extension of this result to Hilbert spaces was also obtained by Clarke et. al. in [8]. These results establish that strong invariance is equivalent to

$$H_F(x, \zeta) \leqslant 0 \quad \forall x \in S,\ \zeta \in N_S^P(x), \tag{4}$$

where the *upper Hamiltonian* $H_F :\ I \times E \times E \to \mathbb{R}$ is defined as

$$H_F(t, x, p) = \sup\big\{\langle p, v\rangle : v \in F(t, x)\big\}. \tag{5}$$

Notice also the new paper [18], where weak and flow invariance in $\mathbb{R}^n$ is studied with the help of generalized proximal normals.

The usual characterizations of strongly invariant systems (Theorem 3.1 in [10], and Theorem 4.3.8 in [11]), require the data to be *Locally Lipschitz* (LL) with respect to the Hausdorff metric. In the language of Hamiltonians this means that for each bounded subset C of $I \times E$ there is a $k_C \geq 0$, such that, for all $(t_i, x_i) \in C$ $(i = 1, 2)$, we have

$$\big|H_F(t_1, x_1, p) - H_F(t_2, x_2, p)\big| \leqslant k_C |p| \Big(|t_1 - t_2| + |x_1 - x_2|\Big) \quad \forall p \in E. \tag{6}$$

A recent paper [14] characterizes strong invariance of moving constrains system by means of (4) and others tangential conditions and using the weaker assumptions of continuity plus a *Locally One Sided Lipschitz* (LOSL) condition. This last means that for each $z \in E$ there exists an open set $U \ni z$ and a constant $k_U \in \mathbb{R}$, such that, for all $x, y \in U$, we have

$$H_F(t, x, x - y) - H_F(t, y, x - y) \leqslant k_U\, |x - y|^2 \quad a.e.,\ t \in I. \tag{7}$$

The multifunction $F(t, \cdot)$ is called OSL when there exists a constant $K = k_E$, i.e. (7) hold on the whole space E. It is easy to show (6) implies

(7), but the converse is not true in general, as can be seen by letting $F(x) = \{-\text{sign}(x)\sqrt{|x|}\}$. This example, with $S = \{0\}$, is covered by the strong invariance result in [14] but not by any of the ones in [10], [11]. The sufficiency of (4) for strong invariance is also proved in [15] when the right-hand side of (DI) is LOSL and satisfies other very mild conditions.

Now we briefly state give some definitions and notations. We refer to [11, 12] for all the concepts used here but not described in details.

The multifunction $R : E \to K(E)$ is said to be upper semicontinuous (USC) at $x \in E$ when for every $\varepsilon > 0$ there exists $\delta > 0$ such that $R(x) + \varepsilon\mathbb{B} \supset R(x + \delta\mathbb{B})$. The multifunction $F : I \times E \to K(E)$ is called almost USC, when for every $\varepsilon > 0$ there exists a closed $I_\varepsilon \subset I$, with Lebesgue measure $meas(I \setminus I_\varepsilon) < \varepsilon$, such that F is USC at every point $(t, x) \in I_\varepsilon \times E$. The multifunction $S : E \rightrightarrows E$ with convex weakly compact values will be called upper hemi continuous (UHC) when for every $l \in E^*$ the upper hamiltonian $H_S(l, \cdot)$ is upper semicontinuous as a real valued function. The almost UHC property of $F : I \times E \rightrightarrows E$ is defined analogously. We will say that F satisfies the linear growth condition when there exists an integrable $\lambda(\cdot)$ such that $|F(t, x)| \leq \lambda(t)(1 + |x|)$. Thu multifunction $G(t, x) \subset F(t, x)$ is said to be a sub-multifunction (of F) if it is with nonempty convex weakly compact values (obviously if F is compact valued than so is and G).

Definition 1. *The absolutely continuous (AC) function $x(\cdot)$ is said to be ε-solution of (DI) when $\dot{x}(t) \in F(t, x(t) + \varepsilon\mathbb{B})$ for a.a. $t \in I$.*

We denote $proj_S(x) = \{s \in S : |x - s| = d_S(x)\}$ the (possible empty) set of all closest points to x in S.

The infinitesimal characterization (4) for strong invariance in the LOSL case is no longer necessary for discontinuous F.

We mention our finite dimensional paper [16], where the strong invariance of (F, S) is studied. The aim of this paper is to extend the results of [16] to the case of more general spaces and more general systems.

2. The results. First we impose two general properties on the differential inclusion (DI). We will assume that every (local) solution is extendable on the whole interval I and that $F(\cdot, \cdot)$ is bounded on the bounded sets.

Consider the following perturbed system:

$$\dot{y}(t) \in F(t, y(t) + \varepsilon_N\mathbb{B}),\ y(0) = x_0. \tag{8}$$

Definition 2. *The system (DI) is said to satisfy a Perturbed Compact Condition (PCC) if every sequence* $\{x^N(\cdot)\}_{N=1}^{\infty}$ *of solution of* (8) *is precompact when* $\varepsilon_N \to 0^+$.

The system (S,F) *is said to be approximately weakly invariant when for every* $\varepsilon > 0$ *there exists an* ε*–solution* $y(\cdot)$ *such that* $d_S(y(t)) < \varepsilon$ *for every* $t \in I$.

Notice that (DI) satisfies a PCC condition, when F satisfies compactness type assumptions (see [12, 24] for instance), which are not studied in the paper (see Remark 1).

We will use the following proposition and definitions taken from [8].

Definition 3. *Given* $\delta > 0$, *the* δ*-metric projection of* x *onto* S *is defined as* $proj_S^{\delta}(x) = \{s \in S : |x-s|^2 < d_S^2(x) + \delta^2\}$. *Obviously this set is always nonempty.*

Proposition 1. (Proposition 2.2 of [8]) *Let* $x \in E \setminus S$, $\delta > 0$ *and* $s_\delta \in proj_S^{\delta}(x)$. *Then there exists* $y_\delta \in E \setminus S$ *and* $\bar{s}_\delta \in S$ *such that* $y_\delta - \bar{s}_\delta \in N_S^P(\bar{s}_\delta)$, $|(y_\delta - \bar{s}_\delta) - (x - s_\delta)| \leqslant 2\delta$ *and* $|s_\delta - \bar{s}_\delta| \leqslant \delta$.

First we we investigate the weak invariance. The following proposition is a partial answer of problem 8 p.126 of [12].

Here we assume that $F(t,\cdot)$ is UHC with nonempty convex weakly compact values.

Proposition 2. *Suppose that at least one of the following two conditions holds:*

1) $F(t,\cdot)$ *is OSL and bounded on the bounded sets.*

2) *There exists an integrable* $\lambda(\cdot)$ *such that* $|F(t,x)| \leq \lambda(t)\,(1+|x|)$ *(linear growth condition)*

If for every $x \in S$ *there exists a strongly measurable selection* $f_x(t) \in F(t,x)$ *such that:*

$$\langle p, f_x(t)\rangle \leq 0,\ \forall p \in N_S^P(x),\ \textit{for a.a. } t \in I, \tag{9}$$

then the system (S,F) *is approximately weakly invariant (on* I*).*

Proof. Let $\tau > 0$ and $x_\tau \in S$. From Theorem 1 of [13] (in case of 1)) we know that for every interval $[0,T]$ there exist constants K and M such that, for every absolutely continuous (AC) function $x(\cdot)$ with $x(\tau) = x_\tau$

and $\dot{x}(t) \in F(t, x(t) + \mathbb{B})$, we have $|x(t)| \leq K(x_\tau, T)$ and $|F(t, x(t) + \mathbb{B})| \leq M(x_\tau, T)$. In case of 2) one can deal as in [3].

We will show that if $F(\cdot, \cdot)$ satisfies assumptions of the proposition, then the system

$$\dot{x}(t) \in F(t, x(t)),\ x(\tau) = x_\tau \in S \tag{10}$$

is approximately weakly invariant. It is sufficient to prove the claim for a fixed (but arbitrary) x_τ and $T > 0$ on the interval $[\tau, \tau + T)$. Moreover, one can replace τ by 0 and $\tau + T$ by 1 considering the function $z(t) = x((t - \tau)/T)$. In the sequel we will denote $[0, 1]$ by I.

Let $\delta \in \left(0, \frac{1}{4}\right)$ be given. Consider the set $D = \{x \in K\mathbb{B} : d_S(x) < \frac{1}{2}\}$. For $x \in S$ we choose $s_\delta(x) \in proj_S^\delta(x)$ (if $x \in S$ we can take $s_\delta(x) = x$). By Proposition 1 there are $y_\delta(x)$ and $\bar{s}_\delta(x)$ with $y_\delta(x) - \bar{s}_\delta(x) \in N_S^P(\bar{s}_\delta(x))$. Let $g_\delta(\cdot, x)$ be a strongly measurable selection satisfying (9). In case that $x \notin D$, we set $g_\delta(\cdot, x) \equiv 0$.

It is easy to check that

1) $g_\delta(t, x) \in F(t, s_\delta(x) + \delta\mathbb{B})$ - indeed $|s_\delta(x) - \bar{s}_\delta(x)| < \delta$,

2) $|g_\delta(t, x)| \leqslant M$, it is because $|F(t, x)| \leq M$,

3) $\langle g_\delta(t, x), x - s_\delta(x)\rangle \leq 2M\delta$, because $\langle g_\delta(t, x), y_\delta(x) - \bar{s}_\delta(x)\rangle \leq 0$ $(y_\delta(x) - \bar{s}_\delta(x) \in N_S^P(\bar{s}_\delta(x)))$. Moreover, due to Proposition 3 $|(y_\delta - \bar{s}_\delta) - (x - s_\delta)| \leq 2\delta$.

Given a subdivision $\pi := \{0 = t_0 < t_1 < \cdots < t_N = 1\}$ we define $y(t) = y(t_i) + \int_{t_i}^{t} g_\delta(s, y_i)\, ds$, (the Euler polygonal corresponding to the π partition). Denote $y_i = y(t_i)$ and $s_i = s_\delta(y_i)$. We let $d(\Delta) = \max_i |t_{i+1} - t_i|$. Let $\delta < \min_{0 \leq i \leq N-1}(t_{i+1} - t_i)$ and let $d(\Delta) \leq \min\left\{\frac{\tilde{\varepsilon}}{2M}, \frac{\tilde{\varepsilon}^2}{4(M^2 + 8M + 1), \tilde{\varepsilon}}\right\}$. Obviously

$$\begin{aligned} d_S^2(y_{i+1}) &\leqslant |y_{i+1} - s_i|^2 \\ &= |y_{i+1} - y_i|^2 + |y_i - s_i|^2 + 2\big\langle y_{i+1} - y_i, y_i - s_i\big\rangle \\ &\leqslant M^2|t_{i+1} - t_i|^2 + d_S^2(y_i) + \delta^2 + 2\int_{t_i}^{t_{i+1}} \big\langle g_\delta(t, y_i), y_i - s_i\big\rangle\, dt. \end{aligned}$$

Due to $\delta < d(\Delta)$ and 3)

$$d_S^2(y_{i+1}) - d_S^2(y_i)\big(M^2 d(\Delta) + d(\Delta) + 4M d(\Delta)\big)(t_{i+1} - t_i)\tilde{\varepsilon}(t_{i+1} - t_i),$$

Consequently $d_S(y_{i+1}) \leqslant \frac{\tilde{\varepsilon}}{2}$ for $i = 0, 1, 2, \ldots, N-1$, since $d_S(x_0) = 0$. As it was shown $y(\cdot)$ is M–Lipschitz and hence $|y_i - y(t)| \leq Md(\Delta)$. Furthermore, $|y - s_\delta(y)| \leq \frac{\tilde{\varepsilon}}{2} + \delta$. Since $\dot{y}(t) = g_\delta(t, y_i) \in F(t, \bar{s}_\delta)$ with $|s_\delta - \bar{s}_\delta| < \delta$, one has that $\dot{y}(t) \in F(t, y(t) + \mu B)$, where $\mu = \tilde{\varepsilon} + 2\delta + Md(\Delta)$. Therefore the system (10) is approximately weakly invariant. □

Corollary 1. *Let F satisfy PCC. Under the assumptions of Proposition 2 the system* (10) *is weakly invariant.*

Proof. Due to Proposition 2 the system (10) is approximately weakly invariant. It follows from PCC that passing to subsequences, if necessary, $y_i(\cdot) \to y(\cdot)$ uniformly on I, which is a solution of (10) and $y(t) \in S$ for all $t \in I$. Consequently the system (10) is weakly invariant. □

Throughout this section we will assume that $F(\cdot, \cdot)$ is convex compact valued and the following hypotheses:

A1. F is OSL and maps bounded sets into bounded.

A1'. $F(t, \cdot)$ is LOSL and satisfies the linear growth condition.

A2. $F : I \times E \to CK(E)$ is almost USC.

A2'. $F : I \times E \to CK(E)$, $F(t, \cdot)$ is USC and $F(\cdot, x)$ is strongly measurable.

We will use the following theorem which is proved in [13] (see the proof of Theorem 1 of [13] for instance):

Theorem 2.1. *If* **A1** *and* **A2** *hold then the following condition is valid:*

C. Assume that $G(\cdot, x)$ is strongly measurable and $H_G(t, \cdot, l)$ is USC as a real valued function for every $l \in E^*$. If $G(t, x) \subset F(t, x)$ is nonempty convex and compact valued, then the solution set R_1 of

$$\dot{x}(t) \in G(t, x), \quad x(0) = x_0 \tag{DG}$$

is nonempty $C(I, H)$ compact and (DG) is PCC. Moreover, for every $\varepsilon > 0$ there exists a constant $K(\varepsilon)$ such that $|G(t, x + \varepsilon\mathbb{B}) + \varepsilon\mathbb{B}| \leq K$ for every ε solution $x(\cdot)$.

Remark 1. Notice that Theorem 2.1 is proved in [13] in case of a Banach space E with uniformly convex dual E^*.

It is easy to see that **C** holds under the growth condition and when F satisfies some compactness type assumption, i.e. there exists a Kamke function $w(\cdot, \cdot)$ such that $\beta(F(t, A) \leq w(t, \beta(A))$ for every bounded set A.

Here $\beta(A) := \inf\{\varepsilon > 0 : \ A \text{ can be covered by finite number of balls with radius} \leq \varepsilon\}$ is the Hausdorff measure of non-compactness.

Recall that the Caratheodory function $w(\cdot,\cdot)$ is called a Kamke function when it is integrally bounded on the bounded sets, $w(t,0) \equiv 0$ and the unique solution of $\dot{r}(t) = w(t, r(t))$, $r(0) = 0$ is $r(t) \equiv 0$. We refer to [24] section 2.5 for more details (see also [8]).

It is easy to see that the condition **C** holds also under **A1', A2** for sufficiently small $\varepsilon > 0$.

Proposition 3. *Let $G(\cdot,\cdot)$ be an almost USC sub-multifunction of F and let $F(\cdot,\cdot)$ satisfy the linear growth condition. Under* **C** *the system (S,G) is weakly invariant (on $[0,T]$) iff there exists a null set $A_G \subset I$ such that $h_G(t,x,p) \leq 0 \ \forall \ p \in N_S^P(x) \ \forall \ t \in I \setminus A_G$.*

Proof. I. We will prove $\Leftarrow$.

Let $G(\cdot,\cdot)$ be sub-multifunction. Let the system (10) be weakly invariant. Therefore due to Theorem 1 of [14] the lower Hamiltonian condition holds, i.e. there exits a null set $A \subset A_G \subset I$ such that $h_G(t,x,p) \leqslant 0$ $\forall p \in N_S^P(x)$ $\forall x \in S$, $\forall t \in I \setminus A_G$.

Now consider the case when **A1', A2** hold. First we will show that the solution set of (DI) exists and is $C([0,T],E)$ compact for every $T > 0$.

Fix $T > 0$. As it is shown in [3] one can replace $\lambda(t)$ by 1 preserving the other hypotheses. Therefore the solution set (if nonempty) is uniformly bounded on $[0,T]$. Let the open set $U \ni x_0$ be such that $R(t,\cdot)$ is OSL with a constant L_U on U. Due to Theorem 1 of [13] there exists $\tau > 0$ such that the solution set is nonempty and $C([0,\tau],E)$ compact on $[0,\tau]$. Furthermore the reachable set of (DI) $Reach_\tau$ at $t = \tau$ is compact. Hence there exists an open set $U_\tau \supset Reach_\tau$ such that $F(t,\cdot)$ is OSL on U_τ. Due to Theorem 1 of [13] there exists $\tau_1 > \tau$ such that the conclusion of the lemma holds also on $[0,\tau_1]$. We continue in the same fashion. Thus there exists a maximal $S \leq T$ such that the conclusions of the lemma holds on $[0,S)$. Due to **A1'** there exists a constant K_S such that $|F(t,x(t))| \leq K_S$ for every solution $x(\cdot)$ on $[0,S]$. Thus the set valued map $t \to Reach_t$ is Lipschitz. Hence $Reach_S$ exists and it is a compact set. If $S < T$ there exists an open $U_S \supset Reach_S$ such that $R(t,\cdot)$ is OSL on U_S. Dealing as above one can conclude that there exists $S_1 > S$ such that the conclusion of the lemma holds also on $[0,S_1]$. Due to Zorn' lemma the proof is complete.

The rest of the proof is similar is in case of **A1, A2** presented above.

II. $\Rightarrow$. We have to see that $F(\cdot,\cdot)$ satisfies PCC and that there exists a strongly measurable $f_x(\cdot)$ satisfying (9). Fix $x \in S$ and consider $G(\cdot,x)$. The conjugate cone to $\left(N_S^P(x)\right)^*$ is closed convex valued. Since $H(t) = \left(N_S^P(x)\right)^* \bigcap G(\cdot,x)$ is compact valued and (almost) USC, one has that there exists a strongly measurable selection $f_x(t) \in H(t)$. Hence Proposition 2 applies. The fact that $F(\cdot,\cdot)$ is PCC follows from Theorem 1 of [13]. The proof is therefore complete thanks to Corollary 1. □

Now we state a characterization for strong invariance in terms of lower hamiltonians (3). We use the same "proximal aiming" argument presented in the proof of Proposition 1.

Theorem 2.2. *Assume that* **A2** *holds. Let also either* **A1** *or* **A1'** *be true. The system (S,F) is strongly invariant iff for every almost USC sub-multifunction $G(t,x) \subset F(t,x)$ there exists a null set $A_G \subset I$ such that $h_G(t,x,p) \leq 0$, $\forall p \in N_S^P(x)$, $\forall x \in S$, $\forall t \in I \setminus A_G$.*

Proof. Due to Theorem 2.1 $F(\cdot,\cdot)$ satisfy **C**. It follows from Proposition 3 that the system (10) is weakly invariant for every sub-multifunction G.

Now we will show the only if case, i.e. the system (S,F) is strongly invariant.

Suppose $y(\cdot)$ is a solution of (DI). Consider the following multifunction:

$$F_0(t,x) = \{v \in F(t,x) : \ \langle y(t) - x, \dot{v}(t) - v\rangle \leqslant k|y(t) - x|^2\}.$$

It is easy to see that $F_0(\cdot,x)$ is almost USC. Since there exists a null set $A \subset [0,1]$ such that $h_{F_0}(t,x,p) \leqslant 0$ $\forall p \in N_S^P(x)$ $\forall x \in S$, $\forall t \in I \setminus A$, one has that the system:

$$\dot{x}(t) \in F_0(t,x), \ x(\tau) = y_\tau$$

is weakly invariant. Let $x(\cdot)$ be an invariant solution to the last system, i.e. $x(t) \in S$. Thus $\langle y(t) - x(t), \dot{y}(t) - \dot{x}(t)\rangle \leq k|y(t) - x(t)|^2$, i.e. $\dfrac{d}{dt}|y(t) - x(t)|^2 \leqslant 2k|y(t) - k(t)|^2$. Using Gronwall' inequality one obtains $x(t) \equiv y(t)$, i.e. $y(t) \in S$. That is the system (DI) is strongly invariant.

The if case. Since (F,S) is strongly invariant one has that for every sub-multifunction the system (10) is also strongly and hence weakly invariant. Due to proposition 3 and Theorem 2.1 the lower Hamiltonian condition holds on $I \setminus A_G$. Now we will extend G. □

The following Corollary is a consequence of Theorem 2.2. It provides an alternative necessary and sufficient condition for strongly invariant systems

in terms of its original data. For this purpose, we shall utilize the following notation: For a given nonzero vector $\zeta \in N_S^P(x)$, $y \to_\zeta x$ denotes the limit of y approaching x along the vector ζ, which signifies that for $\zeta = z - x$ with $x \in proj_S(z)$, we have $y = \delta(z - x) + x$, where $\delta \to 0^+$.

Corollary 2. *Under assumptions* **A1** *and* **A2**, *the following assertions hold:*
i) *If there is a null set $A \subset I$ such that*

$$\limsup_{y \to_\zeta x,\ \tau \to t} H_F(\tau, y, \zeta) \leq 0, \tag{11}$$

for all $t \in I \setminus A$, all $x \in S$, and all $\zeta \in N_S^P(x)$, then the system (S, F) is strongly invariant.
ii) *If the system (S, F) is strongly invariant, then there is a null set $A \subset I$ such that*

$$\liminf_{y \to_\zeta x,\ \tau \to t} H_F(\tau, y, \zeta) \leq 0, \tag{12}$$

for all $t \in I \setminus A$, all $x \in S$, and all $\zeta \in N_S^P(x)$.

The proof follows very closely (with obvious modifications) the proof of Theorem 4 of [17] and it is omitted.

Remark 2. The sufficient condition in Theorem 2.2 can be proved when **A2** is replaced by:

$F(\cdot, x)$ is strongly measurable or $F(\cdot, x)$ is measurable and E is separable and $F(t, \cdot)$ is USC. However to prove the necessity we have to assume that F is almost USC.

Theorem 2.2 can be proved also when in **A1'** the LOSL condition is replaced by

A $F(t, \cdot)$ is Kamke OSL, i.e. there exists a Kamke function $\omega(\cdot, \cdot)$ such that

$$H_F(t, x, x - y) - H_F(t, y, x - y) \leq \omega(t, |x - y|^2),\ a.e.\, t \in I.$$

3. Almost UHC case. In this section we assume that $F(\cdot, \cdot)$ is almost UHC with nonempty convex weakly compact values. It is supposed to be OSL (with a constant L) and bounded on the bounded sets.

First we will prove the following variant of the lemma of Pliss.

Lemma 1. *Let $F(t,\cdot)$ be OSL (with a constant L) and let $F(\cdot,\cdot)$ be almost UHC with nonempty convex weakly compact values. If $F(\cdot,\cdot)$ is bounded on the bounded sets, then there exists a constant k_F such that for every AC $y(\cdot)$ such that $y(0)=x_0$ and $\dot{y}(t)\in \overline{co}\ F(t,x+\varepsilon\mathbb{B})$ there exists a solution $x(\cdot)$ of (DI) such that $|x(t)-y(t)|\le k_F\sqrt{\varepsilon}$.*

Proof. As it was pointed out above there exit constants M and N such that $|x(t)|\le M$ and $|F(t,x(t)+\mathbb{B})|\le N$ for every AC $x(0)=x_0$ and $\dot{x}(t)\in \overline{co}\ F(t,x(t)+\mathbb{B})$.

Consider the partition $\Delta=\{t_i\}_{i=0}^m$ with $t_i=ih=\dfrac{i}{m}$. Let $\dot{y}(t)\in F(t,y(t)+l(t))$, where $|l(t)|\le\varepsilon$.

We define an approximate solution $z(\cdot)$ such that $z(t)=z_i+\displaystyle\int_{t_i}^{t} f_i(\tau)\,d\tau$, where $z_i=z(t_i)$ and $f_i(t)\in F(t,z_i)$ is strongly measurable selection on $[t_i,t_{i+1}]$ such that:

$$\begin{aligned}&\langle y(t)-z_i,\dot{y}(t)-f_i(t)\rangle\\ &\le\langle y(t)+l(t)-x_i,\dot{y}(t)-f_i(t)\rangle\le L|x(t)+l(t)-x_i|^2.\end{aligned}$$

It is standard to show that such a strongly measurable selection exists, because the set of strongly measurable selection of $F(\cdot,x)$ is weakly $L_1(I,E)$ compact and $\langle x,\cdot\rangle$ is continuous on E-weak. Further $x_{i+1}=x(t_{i+1})$ and we deal on $[t_{i+1},t_{i+2}]$ in the same way.

Consequently:

$$\begin{aligned}&\langle x(t)-y(t),\dot{x}(t)-\dot{y}(t)\rangle\\ &\le L|x(t)-y(t)|^2+(|L|+1)\left||x(t)-y(t)|^2-|y(t)+l(t)-x_i|^2\right|\\ &\le L|x(t)-y(t)|^2+(|L|+1)\{|x(t)|+|y(t)|+\varepsilon\}\{\varepsilon+|x(t)-x_i|\}\\ &\le L|x(t)-y(t)|^2+(4M+\varepsilon)(|L|+1)(\varepsilon+Nh).\end{aligned}$$

Thus $|x(t)-y(t)|^2\le r(t)$, where $r(0)=0$ and $\dot{r}(t)=2Lr(t)+2(4M+\varepsilon)(|L|+1)(\varepsilon+Nh)$. As it is shown in [15] there exists a constant C such that for every approximate solution with the length of partition h one has $\operatorname{dist}(z(\cdot),Sol(DI))\le C\sqrt{h}$, where $Sol(DI)$ is the solution set of (DI). Choosing $h<\dfrac{\varepsilon}{N+1}$ one completes the proof. □

First we will prove the following:

Proposition 4. *(***Invariance Principle***) The system (DI) is strongly invariant iff for every almost UHC sub-multifunction $G(t,x) \subset F(t,x)$ with nonempty convex weakly compact values the system:*

$$\dot{y}(t) \in G(t, y(t)),\ y(0) = x_0 \in S, \tag{13}$$

is approximately weakly invariant.

Proof. It is shown in [15] that the solution set of (DI) is nonempty and $C(I,E)$ closed.

1) Let (DI) be strongly invariant. Due to Lemma 1 if $y(\cdot)$ is ε-solution of (13) then $d_S(y(t)) \leq k_F\sqrt{\varepsilon}$ for every $t \in I$. Thus the system (13) is approximately weakly invariant, because every $\frac{\varepsilon^2}{(k_F+1)^2}$-solution is also ε-solution.

2) Let (S,G) be approximately weakly invariant for every submultifunction $G(t,x)$. Suppose (DI) is not strongly invariant, i.e. there exists a solution $x(\cdot)$ of (DI) and $s \in I$ such that $x(s) \notin S$, i.e. $d_S(x(s)) = a > 0$. Define the multifunction:

$$G(t,u) := \{v \in F(t,u) :\ \langle x(t) - u, \dot{x}(t) - v\rangle \leq L|x(t) - u|^2\}.$$

It is standard to see that $G(\cdot,\cdot)$ is almost UHC with nonempty convex weakly compact values. Thus for every $\varepsilon > 0$ there exists δ such that every δ – solution of (13) belongs to $S + \varepsilon\mathbb{B}$. It follows from Lemma 1 that there exists a solution $y(\cdot)$ of (DI) such that $|x(t) - y(t)| \leq C_F\sqrt{\delta}$. Obviously one can choose ε and δ so small that $\varepsilon + C\sqrt{\delta} < a$ which is contradiction. □

The following theorem is our strong invariance result in the UHC case:

Theorem 3.1. *The system (S,F) is strongly invariant iff for every almost UHC sub-multifunction $G(t,x) \subset F(t,x)$ with nonempty convex weakly compact values there exists a null set $A_G \subset I$ such that*

$$h_G(t,x,\zeta) \leq 0, \tag{14}$$

for all $t \in I \setminus A_G$, $x \in S$, and $\zeta \in N_S^P(x)$.

Proof. In view of the **Invariance Principle** it suffices to show that (14) is necessary and sufficient condition for the approximate weak invariance of $G(\cdot,\cdot)$.

First we will prove the necessity. Let $I = \bigcup\limits_{n\geq 0} I_n$ with $\mathrm{meas}(I_0) = 0$, I_n are pairwise disjoint compacts, such that $G(\cdot,\cdot)$ is UHC on $I_n \times H$ for every

$n > 0$. Let $\tau \in I_n$ be its point of density. For $x \in S$ consider a sequence $u_k(\cdot)$ of ε_k solution of (13) such that $d_S(u_k(t)) \leq \varepsilon_k$ and $u_n(\tau) = x$. We will follow with modifications the proof of Theorem 3.2 of [8]. Due to Lemma 1 of [15] $|G(t, u_n(t) + \mathbb{B})| \leq N$. Hence $u_k(t) \in G(t, x + (\varepsilon_k + (t - \tau))\mathbb{B})$. Let $t_k \to 0^+$ be such that $\tau + t_k \in I_n$. Fix $\varepsilon > 0$. We let $v_k := \dfrac{u_k(\tau + t_k) - x}{t_k}$. Obviously $v_k \in \overline{co}\ F([\tau, \tau + t_k] \bigcap I_n, x + \varepsilon\mathbb{B})$. The sequence v_k is bounded and hence passing to subsequences $v_k \rightharpoonup v$. Since $F(\cdot, \cdot)$ is UHC at (τ, x) one has that $v \in F(\tau, x)$. Let $t_k \leq \sqrt{\varepsilon_k}$. Further $\dfrac{1}{t_k} d_S(x + t_k v_k) = \dfrac{1}{t_k} d_S(x_k(t_k)) \leq \sqrt{\varepsilon_k}$. Hence $v \in T_D^w(x)$.

When the state space E is Hilbert one has that if $z \in T_D^w(x)$ then $\langle z, \varsigma \rangle \leq 0$ for every $\varsigma \in N_D^P(x)$ (see [8] for instance).

To prove the sufficiency we will use Proposition 1.

The cone $(N_D^P(x))^*$ is closed and convex, i.e. $S(t) := G(t, x) \bigcap (N_D^P(x))^*$ has a strongly measurable selection $f_x(t) \in S(t)$ such that (9) holds. Now Proposition 2 applies. The proof is therefore complete thanks to Proposition 4. □

4. Concluding remarks. It will be interesting to prove similar result to Corollary 2 in case of almost UHC $F(\cdot, \cdot)$.

Some of the results can be extended to the case of Banach space E with uniformly convex dual E^*, however, in this case one can not exploit the proximal normal cone to prove weak invariance.

When $F(\cdot, \cdot)$ is almost USC one can easily prove the following:

Proposition 5. *Let $G(\cdot, \cdot)$ be almost USC sub-multifunction of F and let $F(\cdot, \cdot)$ satisfy the linear growth condition. Under* **C** *the system (S, G) is weakly invariant (on $[0, T]$) iff there exists a null set $A_G \subset I$ such that $G(t, x) \bigcap T_S(x) \neq \emptyset$ for every $x \in S$ and $\forall\ t \in I \setminus A_G$.*

In case of almost UHC $F(\cdot, \cdot)$, however, the commonly used condition for weak invariance is $F(t, x) \bigcap T_D(x) \neq \emptyset$ for every $x \in D$ and a.a. $t \in I$ does not necessarily hold. We can not prove that this condition is necessary or that it is sufficient for weak invariance. Also the commonly used (for strong invariance) condition $F(t, x) \subset T_D(x)$ is quite demanding.

Using the **Invariance Principle** one can prove:

Proposition 6. *If for every almost UHC sub-multifunction G with convex weakly compact values there exists a null set I_G such that $G(t,x) \cap T_S(x) \neq \emptyset$, $\forall x \in S$, $t \in I \setminus I_G$, then the system (DI) is strongly invariant. If (DI) is strongly invariant then $G(t,x) \bigcap T_S^w(x) \neq \emptyset$, $\forall x \in S$, $t \in I \setminus I_G$ for every sub-multifunction G.*

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Department of Mathematics,
University of Architecture and Civil Engineering,
1 "Hr. Smirnenski" str., 1046 Sofia,
BULGARIA,
tdd51us@yahoo.com

Departamento de Matematicas,
Universidad del Zulia,
Maracaibo, Edo Zulia, 4001,
VENEZUELA,
vrios@demat.org.ve

Mathematics Department,
Louisiana State University,
Baton Rouge, Louisiana 70803,
USA
wolenski@math.lsu.edu