ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" IAŞI
Tomul LI, s.I, Matematică, 2005, f.2

ON THE CURVATURE TENSOR OF NEARLY COSYMPLECTIC MANIFOLDS OF CONSTANT Φ-SECTIONAL CURVATURE

BY

HIROSHI ENDO

Abstract. We show that the curvature tensor on nearly cosymplectic manifolds is determined by the pointwise constant ϕ-sectional curvature perfectly and give some result induced from it.

Mathematics Subject Classification 2000: 53D15.

Key words: Nearly cosymplectic manifold, constant ϕ-sectional curvature, harmonic vector field.

1. Introduction. An almost contact metric structure (ϕ, ξ, η, g) satisfying $(\nabla_X \phi)X = 0$ is called a nearly cosymplectic structure (see [1]). If we consider S^5 as a totally geodesic hypersurface of S^6, then it is known that S^5 has a non cosymplectic nearly cosymplectic structure. In this paper, by the way of Olszak [5], we shall show that the curvature tensor on nearly cosymplectic manifolds is determined by the pointwise constant ϕ-sectional curvature perfectly and give some result induced from it. These results are a generalization of ones of Blair and the present author ([2], [3], [4]).

2. Nearly cosymplectic manifolds. Let M be a (2n+1)-dimensional almost contact metric manifold with structure tensors (ϕ, ξ, η, g). Then we have ([8] p.252)

$$\eta(\xi) = 1,\ \phi^2 = -I + \xi \otimes \eta,\ \phi\xi = 0,\ \eta(\phi X) = 0,$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),\ \eta(X) = g(X, \xi),\ rank\ \phi = 2n$$

for any vector fields X and Y on M. Let ∇ denote the Levi-Civita connection with respect to the metric g on M. If ϕ satisfies $(\nabla_X\phi)X = 0$ for any vector field X on M i.e., $(\nabla_X\phi)Y + (\nabla_Y\phi)X = 0$ for any vector fields X and Y on M, then M is said to have a nearly cosymplectic structure (ϕ, ξ, η, g) and is called a nearly cosymplectic manifold [1]. It is known that a nearly cosymplectic structure with the normality ($[\phi, \phi] + 2d\eta \otimes \xi = 0$) is a cosymplectic structure ([1]).

It is also known that an almost contact metric structure (ϕ, ξ, η, g) is cosymplectic if and only if ϕ is parallel ([1]). From now on, we assume that M is a nearly cosymplectic manifold. Then the vector field ξ is Killing ([1]), that is,

$$g(\nabla_Y\xi, Z) + g(Y, \nabla_Z\xi) = 0. \tag{2.1}$$

We define a tensor field H by putting

$$\nabla_X\xi = HX. \tag{2.2}$$

Then H is skew-symmetric by means of (2.1).

If at any point $p \in M$ the sectional curvature $K(X, \phi X)$ (denote it by K_p) is independent of the choice of the tangent vector $X \in T_p(M)$, $0 \neq X \perp \xi$. It is said that M has the pointwise constant ϕ-sectional curvature K_p.

We use the following symbol from now on:

$$\nabla^2_{X,Y}\phi : \nabla_X(\nabla_Y\phi) - (\nabla_{\nabla_X Y}\phi).$$

Then the Ricci identity can be written by

$$g((\nabla^2_{X,Y}\phi)W, Z) - g((\nabla^2_{Y,X}\phi)W, Z) = g(R(X, Y)\phi W, Z) + g(R(X, Y)W, \phi Z),$$

where R is the curvature tensor on M. Moreover, by calculating

$$W(g((\nabla_Z\phi)X, Y)) = -W(g((\nabla_Z\phi)Y, X)),$$

we notice that $\nabla^2_{X,Y}\phi$ is anti-symmetric.

3. Some lemma

Lemma 3.1. *For a nearly cosymplectic manifold we have*

$$(\nabla_X\phi)\xi = -\phi HX \tag{3.1}$$

$$\phi H + H\phi = 0 \tag{3.2}$$

$$H\xi = 0. \tag{3.3}$$

Proof. (3.1): By $\phi\xi = 0$, we see that $(\nabla_X\phi)\xi + \phi(\nabla_X\xi) = 0$. From (2.2), we get (3.1).

(3.2): From (3.1) we have

$$g((\nabla_X\phi)\xi, Y) = -g(\xi, (\nabla_X\phi)Y) = -g(\phi HX, Y).$$

Interchanging X and Y in the above equation, we have

$$g((\nabla_Y\phi)\xi, X) = -g(\xi, (\nabla_Y\phi)X) = -g(\phi HY, X).$$

Adding these equations, we get (3.2).

(3.3): From (2.2), $H\xi = \nabla_\xi\xi$. On the other hand, by (2.1), we have

$$g(\nabla_\xi\xi, Z) + g(\nabla_Z\xi, \xi) = 0.$$

Here, differentiating $g(\xi, \xi) = 1$ covariantly, we get $g(\nabla_Z\xi, \xi) = 0$, from which $H\xi = 0$.

Next we get some relations for the curvature tensor on M. □

Lemma 3.2. *On a nearly cosymplectic manifold, the curvature tensor satisfies the following:*

$$\begin{aligned} g(R(\phi X, Y)Z, W) &+ g(R(X, \phi Y)Z, W) + g(R(X, Y)\phi Z, W) \\ &+ g(R(X, Y)Z, \phi W) = 0 \end{aligned} \tag{3.4}$$

$$\begin{aligned} g(R(\phi X, \phi Y)\phi Z, \phi W) &= g(R(X, Y)Z, W) - \eta(X)g(R(\xi, Y)Z, W) \\ &- \eta(Y)g(R(X, \xi)Z, W) \end{aligned} \tag{3.5}$$

$$g(R(\phi X, \phi Y)Z, W) = g(R(X, Y)\phi Z, \phi W). \tag{3.6}$$

Proof. (3.4): From the assumption, we have

$$(\nabla_Y\phi)Z + (\nabla_Z\phi)Y = 0.$$

Differentiating this equation covariantly, we find

$$(\nabla^2_{X,Y}\phi)Z + (\nabla^2_{X,Z}\phi)Y = 0.$$

Applying the Ricci identity, from the above equation and the anti-symmetry of $(\nabla^2_{X,Y}\phi)$,

$$\begin{aligned} g(R(X,Y)Z,\phi W) \; & -g(R(X,Y)W,\phi Z) + g((\nabla^2_{X,Z}\phi)Y,W) \\ & -g((\nabla^2_{Y,Z}\phi)X,W) = 0. \end{aligned} \tag{3.7}$$

Here, by the Bianchi and Ricci identities, we find

$$\begin{aligned} g(R(X,Y)Z,\phi W) &= -g(R(Y,Z)X,\phi W) - g(R(Z,X)Y,\phi W) \\ &= g((\nabla^2_{Y,Z}\phi)W,X) - g((\nabla^2_{Z,Y}\phi)W,X) \\ &\quad -g(R(Y,Z)W,\phi X) - g(R(Z,X)Y,\phi W). \end{aligned} \tag{3.8}$$

Substituting (3.8) into (3.7), it follows that

$$\begin{aligned} & g(R(X,Z)Y,\phi W) - g(R(X,Y)W,\phi Z) - g(R(Y,Z)W,\phi X) \\ & -g((\nabla^2_{Z,Y}\phi)W,X) - g((\nabla^2_{X,Z}\phi)W,Y) = 2g((\nabla^2_{Y,Z}\phi)X,W). \end{aligned} \tag{3.9}$$

On the other hand, using the Ricci identity, we see that

$$\begin{aligned} g(R(X,Z)Y,\phi W) \; & -g(R(X,Z)W,\phi Y) - g((\nabla^2_{X,Z}\phi)Y,W) \\ & -g((\nabla^2_{Z,X}\phi)Y,W) = 0. \end{aligned} \tag{3.10}$$

Adding (3.10) to (3.9), we get

$$\begin{aligned} 2g(R(X,Z)Y,\phi W) \; & -g(R(X,Y)W,\phi Z) - g(R(Y,Z)W,\phi X) \\ & -g(R(X,Z)W,\phi Y) = 2g((\nabla^2_{Y,W}\phi)Z,X). \end{aligned} \tag{3.11}$$

Interchanging Y and W, we find

$$\begin{aligned} & 2g(R(X,Z)W,\phi Y) - g(R(X,W)Y,\phi Z) - g(R(W,Z)Y,\phi X) \\ & -g(R(X,Z)Y,\phi W) = 2g((\nabla^2_{W,Y}\phi)Z,X). \end{aligned} \tag{3.12}$$

Subtracting (3.12) from (3.11), and using the Bianchi and Ricci identities, we get

$$\begin{aligned} & g(R(\phi X,Z)Y,W) + g(R(X,\phi Z)Y,W) + g(R(X,Z)\phi Y,W) \\ & +g(R(X,Z)Y,\phi W) = 0, \end{aligned}$$

which implies (3.4).

(3.5): Replacing X by ϕX in (3.4), we have

$$\begin{aligned}(3.13)\qquad &-g(R(X,Y)Z,W)+\eta(X)g(R(\xi,Y)Z,W)+g(R(\phi X,\phi Y)Z,W)\\ &+g(R(\phi X,Y)\phi Z,W)+g(R(\phi X,Y)Z,\phi W)=0.\end{aligned}$$

Exchanging X and Y, we see that

$$\begin{aligned}(3.14)\qquad &g(R(X,Y)Z,W)+\eta(Y)g(R(\xi,X)Z,W)-g(R(\phi X,\phi Y)Z,W)\\ &+g(R(\phi Y,X)\phi Z,W)+g(R(\phi Y,X)Z,\phi W)=0.\end{aligned}$$

Subtracting (3.14) from (3.13), we find

$$\begin{aligned}(3.15)\qquad &2g(R(\phi X,\phi Y)Z,W)-2g(R(X,Y)Z,W)+\eta(X)g(R(\xi,Y)Z,W)\\ &-\eta(Y)g(R(\xi,X)Z,W)+g(R(\phi X,Y)\phi Z,W)-g(R(\phi Y,X)\phi Z,W)\\ &+g(R(\phi X,Y)Z,\phi W)-g(R(\phi Y,X)Z,\phi W)=0.\end{aligned}$$

On the other hand, replacing Z by ϕZ, and W by ϕW in (3.4) respectively, it follows that

$$\begin{aligned}(3.16)\qquad &-g(R(X,Y)Z,W)=-\eta(Z)g(R(X,Y)\xi,W)-g(R(X,Y)\phi Z,\phi W)\\ &-g(R(X,\phi Y)\phi Z,W)-g(R(\phi X,Y)\phi Z,W).\end{aligned}$$

$$\begin{aligned}(3.17)\qquad &-g(R(X,Y)Z,W)=-\eta(W)g(R(X,Y)Z,\xi)-g(R(X,Y)\phi Z,\phi W)\\ &-g(R(\phi X,Y)Z,\phi W)-g(R(X,\phi Y)Z,\phi W).\end{aligned}$$

Adding (3.16) to (3.17), and substituting the gotten equation into (3.15), we have

(3.18)

$$\begin{aligned}&2g(R(\phi X,\phi Y)Z,W)-2g(R(X,Y)\phi Z,\phi W)-\eta(Z)g(R(\xi,W)X,Y)\\ &-\eta(W)g(R(\xi,Z)Y,X)+\eta(X)g(R(\xi,Y)Z,W)-\eta(Y)g(R(\xi,X)Z,W)=0.\end{aligned}$$

Replacing X by ϕX and Y by ϕY in (3.18), we find

$$\begin{aligned}(3.19)\qquad &2g(R(X,Y)Z,W)-2\eta(X)g(R(\xi,Y)Z,W)\\ &-2\eta(Y)g(R(X,\xi)Z,W)-2g(R(\phi X,\phi Y)\phi Z,\phi W)\\ &-\eta(Z)g(R(\xi,W)\phi X,\phi Y)+\eta(W)g(R(\xi,Z)\phi X,\phi Y)=0.\end{aligned}$$

Replacing W by ξ, we have

$$2g(R(X,Y)Z,\xi) - 2\eta(X)g(R(\xi,Y)Z,\xi) - 2\eta(Y)g(R(X,\xi)Z,\xi) + g(R(\xi,Z)\phi X,\phi Y) = 0. \tag{3.20}$$

Replacing X and Y by ϕX and ϕY respectively, it follows that

$$4g(R(\phi X,\phi Y)Z,\xi) + 2g(R(\xi,Z)X,Y) - 2\eta(X)g(R(\xi,Z)\xi,Y) - 2\eta(Y)g(R(\xi,Z)X,\xi) = 0. \tag{3.21}$$

Adding (3.20) and (3.21), we get

$$g(R(\xi,Z)\phi X,\phi Y) = 0. \tag{3.22}$$

From (3.22) we obtain (3.5).

(3.6): Replacing Z by ϕZ and W by ϕW, and using (3.22), we get (3.6).□

Lemma 3.3. *For a nearly cosymplectic manifold we have*

$$(\nabla_X H)\xi = -H^2 X \tag{3.23}$$

$$g(R(\xi,X)\,Y,Z) = -g((\nabla_X H)\,Y,Z) = \eta(\,Y)g(H^2X,Z) - \eta(Z)g(H^2X,\,Y) \tag{3.24}$$

$$g(Q\xi,Z) = Ric(\xi,Z) = -\eta(Z)\,TrH^2 \tag{3.25}$$

In particular $TrH^2 = constant.$

$$Ric(\phi\,Y,Z) = Ric(\,Y,\phi Z) \text{ i.e., } \phi Q = Q\phi. \tag{3.26}$$

$$Ric(\phi\,Y,\phi Z) = Ric(\,Y,Z) + \eta(\,Y)\eta(Z)\,TrH^2. \tag{3.27}$$

where Ric denotes the Ricci tensor and Q is the Ricci operator.

Proof. (3.23): Differentiating (3.3) covariantly, we get (3.23).

(2.24): The equality $g(R(\xi,X)Y,Z) = -g((\nabla_X H)Y,Z)$ can be obtained as a consequence of (2.1) and (2.2), by well-known argument (see for example [7]).

Replacing Y by ϕY and Z by ϕZ in $g(R(\xi,X)Y,Z) = -g((\nabla_X H)Y,Z)$ respectively, using (3.22), we see that

$$g((\nabla_X H)\phi Y, \phi Z) = 0. \tag{3.28}$$

Moreover, replacing Y by ϕY and Z by ϕZ in (3.28) respectively, using (3.23), we get

$$g(R(\xi,X)Y,Z) = -g((\nabla_X H)Y,Z) = -\eta(Y)g((\nabla_X H)\xi,Z)$$
$$-\eta(Z)g((\nabla_X H)Y,\xi) = \eta(Y)g(H^2X,Z) - \eta(Z)g(H^2X,Y).$$

(3.25): Putting $X{=}Y{=}E_i$ in (3.24) and summing over $i{=}1,2,.....,2n{+}1$, we get (3.25). Replacing Y by HY in (3.24), putting $Y = Z = E_i$ in the gotten equation and summing over $i = 1,2,.....,2n+1$, we get

$$Tr\ (\nabla_X H)H = 0,$$

which implies TrH^2 =constant.

(3.26): Taking a ϕ-basis (i.e., an orthonormal frame $E_0, E_1, ...E_{2n}$ such that $E_0 = \xi$ and $E_{i+n} = \phi E_i$ for $i = 1,2,.....,n$) $X = W = E_i$ in (3.4) and summing over $i = 0,2,.....,2n$, we get (3.26).

(3.27): Replacing Z by ϕZ in (3.26) and using (3.25), we get (3.27). □

Lemma 3.4. *For a nearly cosymplectic manifold, we have*

$$g((\nabla_X\phi)\phi Y, Z) = g((\nabla_X\phi)Y, \phi Z) + \eta(Z)g(HX, Y) + \eta(Y)g(HX, Z), \tag{3.29}$$

$$g((\nabla_{\phi X}\phi)Y, Z) = g((\nabla_X\phi)Y, \phi Z) + \eta(Z)g(HX, Y) + \eta(X)g(HZ, Y), \tag{3.30}$$

$$g((\nabla_{\phi X}\phi)\phi Y, Z) = -g((\nabla_X\phi)Y, Z) + \eta(X)g(HZ, \phi Y) + \eta(Y)g(HX, \phi Z). \tag{3.31}$$

Proof. (3.29): Differentiating $g(\phi Y, \phi Z) = g(Y,Z) - \eta(Y)\eta(Z)$ covariantly and using (3.2), we have (3.29).

(3.30): (3.30) follows from (3.29) by the assumption.

(3.31): Replacing Y by ϕY in (3.30) and using (3.29), we get (3.31). □

Lemma 3.5. *In a nearly cosymplectic manifold, we have the following identity:*

$$g(R(\phi W, \phi X)Y, Z) = g(R(W,X)Y,Z)$$
$$+g((\nabla_W\phi)X, (\nabla_Y\phi)Z) + g(HW,X)g(HY,Z).$$

Proof. Differentiating (3.29) covariantly, we see that

$$
\begin{aligned}
&g(\nabla_W(\nabla_X\phi)\phi Y,Z)+g((\nabla_X\phi)(\nabla_W\phi)Y,Z)=g(\nabla_W(\nabla_X\phi)Y,\phi Z)\\
&+g((\nabla_X\phi)Y,(\nabla_W\phi)Z)+g(HW,Z)g(HX,Y)+\eta(Z)g((\nabla_W H)X,Y)\\
&+g(HW,Y)g(HX,Z)+\eta(Y)g((\nabla_W H)X,Z).
\end{aligned}
\tag{3.32}
$$

On the other hand, applying (3.12), (3.24) and (3.6), the sum of the first and second terms in the right hand members of (3.32) imply

$$
\begin{aligned}
&g(\nabla_W(\nabla_X\phi)Y,\phi Z)+g((\nabla_X\phi)Y,(\nabla_W\phi)Z)=g((\nabla_X\phi)Y,(\nabla_W\phi)Z)\\
&+g(R(X,Y)W,Z)-\eta(X)\eta(Z)g(HW,HY)+\eta(Y)\eta(Z)g(HW,HX)\\
&-\frac{1}{2}g(R(W,Z)\phi X,\phi Y)-\frac{1}{2}g(R(X,Z)\phi W,\phi Y)-\frac{1}{2}g(R(W,X)\phi Z,\phi Y).
\end{aligned}
\tag{3.33}
$$

Exchanging Y and Z in (3.33), taking the minus of it and considering the left hand members of (3.32), we get

$$
\begin{aligned}
&g((\nabla_X\phi)Z,(\nabla_W\phi)Y)+g((\nabla_X\phi)Y,(\nabla_W\phi)Z)+g(R(X,Z)W,Y)\\
&+g(R(X,Y)W,Z)-g(R(W,Z)\phi X,\phi Y)-g(R(X,Z)\phi W,\phi Y)\\
&+g(HX,Z)g(HW,Y)+g(HW,Z)g(HX,Y)=0.
\end{aligned}
\tag{3.34}
$$

Replacing in (3.34) Z and W by ϕZ and ϕW respectively, we find

$$
\begin{aligned}
&g((\nabla_X\phi)\phi Z,(\nabla_{\phi W}\phi)Y)+g((\nabla_X\phi)Y,(\nabla_{\phi W}\phi)\phi Z)\\
&+g(R(X,\phi Z)\phi W,Y)+g(R(X,Y)\phi W,\phi Z)\\
&-g(R(\phi W,\phi Z)\phi X,\phi Y)-g(R(X,\phi Z)\phi^2 W,\phi Y)\\
&+g(HX,Y)g(H\phi W,\phi Z)+g(HX,\phi Z)g(H\phi W,Y)=0.
\end{aligned}
\tag{3.35}
$$

Using (3.29), (3.30) and Lemma 3.1, we find

$$
\begin{aligned}
g((\nabla_X\phi)\phi Z,(\nabla_{\phi W}\phi)Y)&=g((\nabla_X\phi)Z,(\nabla_W\phi)Y)-g(X,H\phi Z)g(W,H\phi Y)\\
&+\eta(W)g(\phi(\nabla_X\phi)Z,HY)+g(HX,Z)g(HW,Y)\\
&-\eta(Z)g(HX,\phi(\nabla_W\phi)Y)-\eta(Z)\eta(W)g(HX,HY),
\end{aligned}
\tag{3.36}
$$

$$
\begin{aligned}
&g((\nabla_X\phi)Y,(\nabla_{\phi W}\phi)\phi Z)=-g((\nabla_X\phi)Y,(\nabla_W\phi)Z)\\
&+g(\phi HX,Y)g(HW,\phi Z)-\eta(W)g(H\phi Z,(\nabla_X\phi)Y)\\
&+g(\phi HX,Y)g(Z,\phi HW)+\eta(Z)g(HW,\phi(\nabla_X\phi)Y).
\end{aligned}
\tag{3.37}
$$

From (3.24) and Lemma 3.1, we have

$$\begin{aligned} g(R(X,\phi Z)\phi W,Y)-g(R(X,\phi Z)\phi^2 W,\phi Y)=g(R(X,\phi Z)\phi W,Y) \\ +g(R(X,\phi Z)W,\phi Y)-\eta(X)\eta(W)g(H^2Z,Y) \end{aligned} \tag{3.38}$$

On the other hand, by (3.4) and (3.6) it follows that

$$\begin{aligned} g(R(X,Z)\phi W,\phi Y)+g(R(X,Z)\phi^2 W,\phi Y) \\ +g(R(X,\phi Z)\phi W,Y)+g(R(\phi X,Z)\phi W,Y)=0. \end{aligned} \tag{3.39}$$

$$g(R(\phi X,Z)\phi W,\phi^2 Y)=g(R(\phi^2 X,\phi Z)W,\phi Y), \tag{3.40}$$

from which

$$\begin{aligned} -g(R(\phi X,Z)\phi W,Y)+\eta(Y)g(R(\phi X,Z)\phi W,\xi) \\ =-g(R(X,\phi Z)W,\phi Y)+\eta(X)g(R(\xi,\phi Z)W,\phi Y). \end{aligned}$$

using (3.24), (3.39), (3.40) and Lemma 3.1, we see that

$$\begin{aligned} g(R(X,\phi Z)\phi W,Y)+g(R(X,\phi Z)W,\phi Y)=-\eta(W)\eta(X)g(HY,HZ) \\ +\eta(Z)\eta(Y)g(HX,HW)+g(R(X,Z)W,Y) \\ -g(R(X,Z)\phi W,\phi Y)+\eta(W)g(R(\xi,Y)Z,X). \end{aligned} \tag{3.41}$$

Substituting (3.41) into (3.38), we get

$$\begin{aligned} g(R(X,\phi Z)\phi W,Y)-g(R(X,\phi Z)\phi^2 W,\phi Y)=g(R(X,Z)W,Y) \\ -g(R(X,Z)\phi W,\phi Y)+\eta(W)\eta(Z)g(H^2Y,X) \\ -\eta(Z)\eta(Y)g(H^2X,W)-\eta(X)\eta(W)g(H^2Z,Y). \end{aligned} \tag{3.42}$$

By means of (3.24) and (3.5), we have

$$\begin{aligned} g(R(X,Y)\phi W,\phi Z)-g(R(\phi W,\phi Z)\phi X,\phi Y)=g(R(W,Z)\phi X,\phi Y) \\ -g(R(W,Z)X,Y)-\eta(Y)\eta(W)g(\phi HZ,X)+\eta(X)\eta(W)g(\phi HZ,Y) \\ -\eta(Y)\eta(W)g(X,H^2Z)+\eta(X)\eta(W)g(H^2Z,Y) \\ -\eta(Z)\eta(X)g(\phi HW,Y)+\eta(Z)\eta(Y)g(\phi HW,X) \\ -\eta(Z)\eta(X)g(Y,H^2W)+\eta(Z)\eta(Y)g(H^2W,X). \end{aligned} \tag{3.43}$$

Substituting (3.36), (3.37), (3.42) and (3.43) into (3.35), and using Lemma 3.1, we obtain

$$\begin{aligned}
&g((\nabla_X\phi)Z,(\nabla_W\phi)Y)-g((\nabla_X\phi)Y,(\nabla_W\phi)Z)\\
&-\eta(W)g((\nabla_X\phi)Z,\phi HY)+\eta(W)g((\nabla_X\phi)Y,\phi HZ)\\
&+\eta(Z)g((\nabla_W\phi)Y,\phi HX)-\eta(Z)g((\nabla_X\phi)Y,\phi HW)\\
&+g(R(X,Z)W,Y)-g(R(X,Z)\phi W,\phi Y)\\
&+g(R(W,Z)\phi X,\phi Y)-g(R(W,Z)X,Y)\\
&-\eta(Y)\eta(W)g(X,H^2Z)-\eta(Z)\eta(X)g(Y,H^2W)\\
&+\eta(W)\eta(Z)g(H^2Y,X)+\eta(Z)\eta(W)g(H^2Y,X)\\
&-g(HX,Y)g(HW,Z)+g(HX,Z)g(HW,Y)=0.
\end{aligned}\tag{3.44}$$

Adding (3.44) to (3.34), we obtain

$$\begin{aligned}
&2g((\nabla_X\phi)Z,(\nabla_W\phi)Y)-\eta(W)g((\nabla_X\phi)Z,\phi HY)\\
&+\eta(W)g((\nabla_X\phi)Y,\phi HZ)+\\
&+\eta(Z)g((\nabla_W\phi)Y,\phi HX)-\eta(Z)g((\nabla_X\phi)Y,\phi HW)\\
&+2g(R(X,Z)W,Y)-2g(R(X,Z)\phi W,\phi Y)\\
&+2g(HX,Z)g(HW,Y)+\eta(Y)\eta(W)g(HX,HZ)\\
&-2\eta(W)\eta(Z)g(HX,HY)+\eta(X)\eta(Z)g(HY,HW)=0.
\end{aligned}\tag{3.45}$$

Interchanging X and Z, and W and Y in (3.45), moreover subtracting the gotten equation from (3.45), we have

$$\begin{aligned}
&\eta(Y)g((\nabla_W\phi)Z,\phi HX)-\eta(X)g((\nabla_W\phi)Z,\phi HY)\\
&+\eta(Z)g((\nabla_W\phi)Y,\phi HX)+\eta(X)g((\nabla_W\phi)Y,\phi HZ)\\
&-\eta(Y)g((\nabla_X\phi)Z,\phi HW)-\eta(W)g((\nabla_X\phi)Z,\phi HY)\\
&-\eta(Z)g((\nabla_X\phi)Y,\phi HW)+\eta(W)g((\nabla_X\phi)Y,\phi HZ)\\
&+2\eta(Y)\eta(X)g(HZ,HW)-2\eta(W)\eta(Z)g(HX,HY)=0.
\end{aligned}\tag{3.46}$$

Putting ξ on W of (3.46), and using Lemma 3.1, we get

$$\begin{aligned}
g((\nabla_X\phi)Y,\phi HZ)&=g((\nabla_X\phi)Z,\phi HY)+\eta(Z)g(HY,HX)\\
&\quad-\eta(Y)g(HZ,HX),
\end{aligned}\tag{3.47}$$

from which

$$g((\nabla_Y\phi)X,\phi HZ) = -g((\nabla_X\phi)Z,\phi HY) - \eta(Z)g(HY,HX) + \eta(Y)g(HZ,HX). \tag{3.48}$$

Interchanging X and Y in (3.48) and substituting (3.48) into the gotten equation, we have

$$g((\nabla_Z\phi)Y,\phi HX) + g((\nabla_Z\phi)X,\phi HY) = 2\eta(Z)g(HY,HX) - \eta(X)g(HZ,HY) - \eta(Y)g(HX,HZ). \tag{3.49}$$

Interchanging Z and X in (3.49), and which together with (3.47) leads to

$$g((\nabla_X\phi)Y,\phi HZ) = \eta(X)g(HY,HZ) - \eta(Y)g(HX,HZ). \tag{3.50}$$

Substituting (3.50) into (3.45), we obtain Lemma 3.5. □

Lemma 3.6. *For a nearly cosymplectic manifold, we have*

$$g((\nabla_X\phi)\,Y, HZ) = \eta(\,Y)g(H^2X,\phi Z) - \eta(X)g(H^2\,Y,\phi Z). \tag{3.51}$$

$$g((\nabla_X H)\,Y, HZ) = -\eta(\,Y)g(H^2X, HZ). \tag{3.52}$$

Proof. (3.51): Replacing Z in (3.50) by ϕZ , and using Lemma 3.1, we get (3.51).

(3.52): Replacing Z in (3.24) by HZ , and, by virtue of Lemma 3.1, we get (3.52). □

4. Some results

Theorem 4.1. *Let M be a nearly cosymplectic manifold of pointwise constant ϕ-sectional curvature K. Then the curvature tensor has the following form:*

$$\begin{aligned}
4\,g(R(W,X)Y,Z) &= g((\nabla_W\phi)Z,(\nabla_X\phi)Y) - g((\nabla_W\phi)Y,(\nabla_X\phi)Z)\\
&-2g((\nabla_W\phi)X,(\nabla_Y\phi)Z)+g(\nabla_W\xi,Z)g(\nabla_X\xi,Y)-g(\nabla_W\xi,Y)g(\nabla_X\xi,Z)\\
&-2g(\nabla_W\xi,X)g(\nabla_Y\xi,Z) - \eta(W)\eta(Y)g(\nabla_X\xi,\nabla_Z\xi)\\
&+\eta(W)\eta(Z)g(\nabla_X\xi,\nabla_Y\xi) + \eta(X)\eta(Y)g(\nabla_W\xi,\nabla_Z\xi)\\
&-\eta(X)\eta(Z)g(\nabla_W\xi,\nabla_Y\xi) + K\,\{g(X,Y)g(Z,W) - g(Z,X)g(Y,W)\\
&+\eta(Z)\eta(X)g(Y,W) - \eta(Y)\eta(X)g(Z,W) + \eta(Y)\eta(W)g(Z,X)\\
&-\eta(Z)\eta(W)g(Y,X) + g(\phi Y,X)g(\phi Z,W)\\
&-g(\phi Z,X)g(\phi Y,W) - 2g(\phi Z,Y)g(\phi X,W)\}\,.
\end{aligned}$$

Proof. By the assumption we have

$$g(R(\phi X, X)\phi X, X) + K_p \|X\|^4 = 0$$

at any point $p \in M$ and for any $X \in T_p(M)$, $X \perp \xi$. It is clear that this condition implies

$$g(R(\phi^2 X, \phi X)\phi^2 X, \phi X) + K_p g(\phi X, \phi X) g(\phi X, \phi X) = 0 \tag{4.1}$$

at any point $p \in M$ and for any $X \in T_p(M)$. Set

$$A(W, X, Y, Z) = g(R(\phi^2 W, \phi X)\phi^2 Y, \phi Z) + K_p g(\phi W, \phi Y) g(\phi X, \phi Z). \tag{4.2}$$

In view of (3.3) and (3.5), we see that

$$g(R(\phi^2 W, -X + \eta(X)\xi)\phi^2 Y, -Z + \eta(Z)\xi) = g(R(\phi W, \phi X)\phi Y, \phi Z).$$

Putting $X = \phi S$ and $Z = \phi T$ in the above equation, we find

$$g(R(\phi^2 W, \phi S)\phi^2 Y, \phi T) = g(R(\phi W, \phi^2 S)\phi Y, \phi^2 T). \tag{4.3}$$

Using (4.3), we have

$$A(X, W, Z, Y) = A(W, X, Y, Z). \tag{4.4}$$

Moreover, from (4.2), we find

$$A(W, X, Y, Z) = A(Y, Z, W, X). \tag{4.5}$$

By means of (4.4) and (4.5), (4.1) is equivalent to

$$\begin{aligned} &A(U, X, V, Z) + A(U, V, Z, X) + A(U, Z, X, V) \\ &+A(U, X, Z, V) + A(U, Z, V, X) + A(U, V, X, Z) = 0. \end{aligned} \tag{4.6}$$

Replacing U and V by ϕW and ϕY respectively, and using (4.2) together with (3.2), (3.3), (3.5), (3.6), (3.24) and (3.28), it follows that

$$\begin{aligned} &g(R(W, X)Y, Z) + g(R(W, Z)Y, X) - g(R(W, \phi Y)\phi Z, X) \\ &-g(R(W, \phi Y)\phi X, Z) - g(R(W, Z)\phi X, \phi Y) - g(R(W, X)\phi Z, \phi Y) \\ &-\eta(Y)\eta(Z)g(HW, HX) - \eta(X)\eta(Y)g(HW, HZ) \\ &+2\eta(Y)\eta(W)g(HX, HZ) + 2\eta(X)\eta(Z)g(HW, HY) \\ &+2K_p(g(W, Y) - \eta(W)\eta(Y))(g(X, Z) - \eta(X)\eta(Z)) \\ &+2K_p(g(\phi W, X)g(\phi Y, Z) - g(\phi W, Z)g(\phi X, Y)) = 0. \end{aligned} \tag{4.7}$$

Interchanging W and X, we see that

$$
\begin{aligned}
&g(R(X,W)Y,Z)+g(R(X,Z)Y,W)-g(R(X,\phi Y)\phi Z,W)\\
&-g(R(X,\phi Y)\phi W,Z)-g(R(X,Z)\phi W,\phi Y)-g(R(X,W)\phi Z,\phi Y)\\
&-\eta(Y)\eta(Z)g(HX,HW)-\eta(Y)\eta(W)g(HX,HZ)\\
&+2\eta(X)\eta(Y)g(HW,HZ)+2\eta(Z)\eta(W)g(HX,HY)\\
&+2K_p(g(X,Y)-\eta(X)\eta(Y))(g(W,Z)-\eta(W)\eta(Z))\\
&+2K_p(g(\phi X,W)g(\phi Y,Z)-g(\phi X,Z)g(\phi W,Y))=0.
\end{aligned}
\tag{4.8}
$$

Subtracting (4.8) from (4.7), we find

$$
\begin{aligned}
&2g(R(W,X)Y,Z)+2g(R(W,X)\phi Y,\phi Z)-g(R(W,\phi Y)\phi Z,X)\\
&+g(R(W,Z)Y,X)-g(R(X,Z)Y,W)-g(R(W,\phi Y)\phi X,Z)\\
&-g(R(W,Z)\phi X,\phi Y)+g(R(X,\phi Y)\phi Z,W)+g(R(X,\phi Y)\phi W,Z)\\
&+g(R(X,Z)\phi W,\phi Y)-\eta(X)\eta(Y)g(HW,HZ)+\eta(Y)\eta(W)g(HX,HZ)\\
&+2\eta(Y)\eta(W)g(HX,HZ)+2\eta(X)\eta(Z)g(HW,HY)\\
&-2\eta(X)\eta(Y)g(HW,HZ)-2\eta(Z)\eta(W)g(HX,HY)\\
&+2K_p(g(W,Y)g(X,Z)-g(X,Y)g(W,Z)-\eta(W)\eta(Y)g(X,Z)\\
&-\eta(X)\eta(Z)g(W,Y)+\eta(X)\eta(Y)g(W,Z)+\eta(W)\eta(Z)g(X,Y))\\
&+2K_p(g(\phi W,X)g(\phi Y,Z)-g(\phi W,Z)g(\phi X,Y)\\
&-g(\phi X,W)g(\phi Y,Z)+g(\phi X,Z)g(\phi W,Y))=0.
\end{aligned}
\tag{4.9}
$$

On the other hand, using (3.4) and (3.24), we have

$$
\begin{aligned}
&g(R(W,Z)Y,X)=g(R(\phi W,Z)Y,\phi X)+g(R(W,\phi Z)Y,\phi X)\\
&+g(R(W,Z)\phi Y,\phi X)-\eta(X)\eta(W)g(Z,H^2Y)+\eta(X)\eta(Z)g(H^2Y,W).
\end{aligned}
\tag{4.10}
$$

$$
\begin{aligned}
&-g(R(X,Z)Y,W)=-g(R(\phi X,Z)Y,\phi W)-g(R(X,\phi Z)Y,\phi W)\\
&-g(R(X,Z)\phi Y,\phi W)+\eta(W)\eta(X)g(Z,H^2Y)-\eta(W)\eta(Z)g(H^2Y,X).
\end{aligned}
\tag{4.11}
$$

Here, from (3.6) and (3.24), we have generally

$$
\begin{aligned}
&g(R(\phi X,Y)\phi Z,W)=g(R(X,\phi Y)Z,\phi W)\\
&+\eta(Z)\eta(X)g(HW,HY)-\eta(Y)\eta(W)g(HX,HZ).
\end{aligned}
\tag{4.12}
$$

By means of the first Bianchi identity for the first term of the right hand members of (4.10),(4.12) and (3.6) we note

$$\begin{aligned} g(R(\phi W,Z)Y,\phi X) &= g(R(X,W)\phi Z,\phi Y)-g(R(\phi Y,W)\phi Z,X)\\ &\quad +\eta(Z)\eta(Y)g(HW,HX)-\eta(X)\eta(W)g(HY,HZ). \end{aligned} \tag{4.13}$$

Substituting (4.13) into (4.10), and substituting the gotten result and (4.11) into (4.9), and using (4.12) and the first Bianchi identity for the third term of (4.9), we get

$$\begin{aligned} &g(R(W,X)Y,Z)+2g(R(W,X)\phi Y,\phi Z)-g(R(W,Z)\phi X,\phi Y)\\ &+g(R(X,Z)\phi W,\phi Y)+\eta(W)\eta(Y)g(HX,HZ)+\eta(X)\eta(Z)g(HW,HY)\\ &-\eta(X)\eta(Y)g(HW,HZ)-\eta(Z)\eta(W)g(HX,HY)+K_p(g(W,Y)g(X,Z)\\ &-g(X,Y)g(W,Z)-\eta(W)\eta(Y)g(X,Z)-\eta(X)\eta(Z)g(W,Y)\\ &+\eta(X)\eta(Y)g(W,Z)+\eta(W)\eta(Z)g(W,Z))+2K_p(g(\phi W,X)g(\phi Y,Z)\\ &-g(\phi W,Z)g(\phi X,Y)-g(\phi X,W)g(\phi Y,Z)+g(\phi X,Z)g(\phi W,Y))=0. \end{aligned} \tag{4.14}$$

By means of Lemma 3.5, (4.14) yields

$$\begin{aligned} &g(R(W,X)Y,Z)+2g(R(W,X)Y,Z)+2g((\nabla_W\phi)X,(\nabla_Y\phi)Z)\\ &+2g(HW,X)g(HY,Z)-g(R(W,Z)X,Y)-g((\nabla_W\phi)Z,(\nabla_X\phi)Y)\\ &-g(HW,Z)g(HX,Y)+g(R(X,Z)W,Y)+g((\nabla_X\phi)Z,(\nabla_W\phi)Y)\\ &+g(HX,Z)g(HW,Y)+\eta(W)\eta(Y)g(HX,HZ)\\ &+\eta(X)\eta(Z)g(HW,HY)-\eta(Y)\eta(X)g(HW,HZ)\\ &-\eta(W)\eta(Z)g(HX,HY)+K_p(g(W,Y)g(X,Z)\\ &-g(X,Y)g(W,Z)-\eta(W)\eta(Y)g(X,Z)\\ &-\eta(X)\eta(Z)g(W,Y)+\eta(X)\eta(Y)g(W,Z)+\eta(W)\eta(Z)g(X,Y))\\ &+2K_p(g(\phi W,X)g(\phi Y,Z)-g(\phi W,Z)g(\phi X,Y)\\ &-g(\phi X,W)g(\phi Y,Z)+g(\phi X,Z)g(\phi W,Y))=0, \end{aligned} \tag{4.15}$$

from which, by the first Bianchi identity,

$$\begin{aligned} 4\ g(R(W,X)Y,Z) &= g((\nabla_W\phi)Z,(\nabla_X\phi)Y)-g((\nabla_W\phi)Y,(\nabla_X\phi)Z)\\ &-2g((\nabla_W\phi)X,(\nabla_Y\phi)Z)+g(HW,Z)g(HX,Y)-g(HW,Y)g(HX,Z) \end{aligned} \tag{4.16}$$

$$-2g(HW,X)g(HY,Z)-\eta(W)\eta(Y)g(HX,HZ)+\eta(W)\eta(Z)g(HX,HY)$$
$$+\eta(X)\eta(Y)g(HW,HZ)-\eta(X)\eta(Z)g(HW,HY)+K_p\{g(X,Y)g(Z,W)$$
$$-g(Z,X)g(Y,W)+\eta(Z)\eta(X)g(Y,W)-\eta(Y)\eta(X)g(Z,W)$$
$$+\eta(Y)\eta(W)g(Z,X)-\eta(Z)\eta(W)g(Y,X)+g(\phi Y,X)g(\phi Z,W)$$
$$-g(\phi Z,X)g(\phi Y,W)-2g(\phi Z,Y)g(\phi X,W)\},$$

which implies our result. □

From (4.16), the Ricci curvature Ric and the scalar curvature S on M are as follows:

$$\begin{aligned} Ric(X,Y) &= -\tfrac{3}{4}Tr\ (\nabla_X\phi)(\nabla_Y\phi)+\tfrac{3}{4}g((\nabla_X\xi),(\nabla_Y\xi)) \\ &+\eta(X)\eta(Y)\left|\nabla\xi\right|^2+\tfrac{(n+1)}{2}K_p(g(X,Y)-\eta(X)\eta(Y)) \end{aligned} \tag{4.17}$$

$$S=\frac{3}{4}\left|\nabla\phi\right|^2+\left|\nabla\xi\right|^2+n(n+1)K. \tag{4.18}$$

If $K\geq 0$, by (4.17), $g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y)$ and the skew-symmetry of $\nabla\phi$, it follows that $Ric(X,X)\geq 0$ for any vector field X, so that, from Bochner's theorem [6], we have the following theorem.

Theorem 4.2. *On a compact nearly cosymplectic manifold M of pointwise constant ϕ-sectional curvature $K\geq 0$, a harmonic vector field X in M has a vanishing covariant derivative.*

From (4.18) we can see that

Theorem 4.3. *Let M be a nearly cosymplectic manifold of pointwise constant ϕ-sectional curvature K, then the scalar curvature on M is*

$$S\geq n(n+1)K.$$

Equality fold if and only if M is cosymplectic.

Remark 4.1. Theorem 4.1 is a generalization of Theorem 1 in [2], and (4.18) and Theorem 4.3 give a generalization of Theorem 2.1 and Corollary 2.1 in [3].

Remark 4.2. If M *is a cosymplectic manifold of pointwise constant* ϕ*-sectional curvature* K*, it follows that* $\nabla\phi = 0$ *and* $\nabla\xi = 0$*, so that, the curvature tensor has the following form (see [4]):*

$$\begin{aligned} 4\, g(R(W,X)Y,Z) = K\,\{&g(X,Y)g(Z,W) - g(Z,X)g(Y,W) \\ &+\eta(Z)\eta(X)g(Y,W) - \eta(Y)\eta(X)g(Z,W) \\ &+\eta(Y)\eta(W)g(Z,X) - \eta(Z)\eta(W)g(Y,X) \\ &+g(\phi Y,X)g(\phi Z,W) - g(\phi Z,X)g(\phi Y,W) - 2g(\phi Z,Y)g(\phi X,W)\}\,. \end{aligned}$$

REFERENCES

1. Blair, D.E. – *Riemannian Geometry of Contact and Symplectic Manifolds,* Progress in Math. 203, Birkhauser Boston 2002.
2. Endo, H. – *Nearly cosymplectic manifolds of constant* ϕ*-sectional curvature with a submersion of geodesic fibres,* Lagrange and Hamilton Geometries and Their Applications, Radu Miron (Ed.), Handbooks Treatises, Monographs, 49 (2004), 103-114, Fair Partners Publishers (Romania).
3. Endo, H. – *Remarks on nearly cosymplectic manifolds of constant* ϕ*-sectional curvature with a submersion of geodesic fibres,* to appear in Tensor, N.S.
4. Ludden, G.D. – *Submanifolds of cosymplectic manifolds,* J. Diff. Geom. 4(1970), 237-244.
5. Olszak, Z. – *Nearly Sasakian manifolds,* Tensor, N.S., 33(1979), 277-286.
6. Yano, K. – *Integral Formulas in Riemannian Geometry,* Marcel Dekker Inc., New York, 1970.
7. Yano, K.; Bochner, S. – *Curvature and Betti Numbers,* Princeton (1953).
8. Yano, K.; Kon, M. – *Structures on Manifolds,* World Sci., 1984.

Received: 12.IV.2005

Department of Mechanical Eng.,
Utsunomiya University,
yoto 7-1-2, Utsunomiya-shi 321-8585,
JAPAN
hsk-endo@cc.utsunomiya-u.ac.jp