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$\mathcal{K}$–TIGHT MULTISUBMEASURES, $\mathcal{K}-\mathcal{D}$-REGULAR MULTISUBMEASURES

BY

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Abstract. The concept of a $\mathcal{K}$-tight set function has been introduced for measures by KISYNSKI in [7]. Since then, mathematicians like LIPECKI, ADAMSKI and others were interested in tight set functions, pointing out in [1], [9] and [10] important properties of them.

In this paper, taking as starting point the works [8], [9] and [10] of LIPECKI concerning group-valued additive set functions which are $\mathcal{K}$-tight, we define and study the notion of $\mathcal{K}$-tightness for a special subadditive type of multivalued set functions that we have introduced in [5] and called them there to be multisubmeasures.

We also establish for multisubmeasures an extension theorem by preserving the properties, more exactly, by preserving the $\mathcal{K}$-tightness, respectively, the $\mathcal{K}$-$\mathcal{D}$-regularity.

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0. Basic notions, terminology and notations. Let T be a nonvoid, abstract set, $\mathcal{C}$ a ring of subsets of T, $\mathcal{K} \subset \mathcal{C}$ an arbitrary, nonvoid subfamily of $\mathcal{C}$ and X a real normed space with the origin 0. We also consider the families $\mathcal{P}_0(X)$ of all nonvoid sets of X, $\mathcal{P}_f(X)$ of all nonempty, closed subsets of X and $\mathcal{P}_{bf}(X)$ of all nonvoid, closed, bounded sets of X.

Let also $\overset{\bullet}{+}$ be the Minkowski addition on $\mathcal{P}_0(X)$, h, the Hausdorff pseudometric on $\mathcal{P}_f(X)$, which becomes a metric on $\mathcal{P}_{bf}(X)$ and $|A| = h(A, \{0\})$, for every $A \in \mathcal{P}_f(X)$.

We recall the following classical notions (see, for details, [5]):

Definition 1. *If $\mu : \mathcal{C} \mapsto \mathcal{P}_f(X)$ is a multivalued set function, then μ is said to be:*

i) exhaustive if $\lim|\mu(A_n)| = 0$*, for every disjoint sequence* $(A_n)_n \subset \mathcal{C}$.

ii) o-continuous if $\lim|\mu(A_n)|=0$*, for every decreasing sequence* $(A_n)_n \subset \mathcal{C}$*, with* $\bigcap\limits_{n=1}^{\infty} A_n = \emptyset$.

iii) increasing convergent if $\lim h(\mu(A_n), \mu(A)) = 0$*, for every increasing sequence* $(A_n)_n \subset \mathcal{C}$*, with* $\bigcup\limits_{n=1}^{\infty} A_n = A \in \mathcal{C}$.

iv) $h-\sigma$*-subadditive if* $|\mu(A)| \leq \sum\limits_{n=1}^{\infty} |\mu(A_n)|$*, for every (disjoint) sequence* $(A_n)_n \subset \mathcal{C}$*, with* $\bigcup\limits_{n=1}^{\infty} A_n = A \in \mathcal{C}$.

All over this paper we consider the following special type of multivalued set functions that we have introduced and studied in [5]:

Definition 2. *If* $\mu : \mathcal{C} \mapsto \mathcal{P}_f(X)$ *is a multivalued set function, then* μ *is said to be a multisubmeasure if*

i) $\mu(\emptyset) = \{0\}$,

ii) $\mu(A \cup B) \subseteq \mu(A) \overset{\bullet}{+} \mu(B)$*, for every* $A, B \in \mathcal{C}$*, with* $A \cap B = \emptyset$ *(or, equivalently, for every* $A, B \in \mathcal{C}$*) and*

iii) $\mu(A) \subseteq \mu(B)$*, for every* $A, B \in \mathcal{C}$*, with* $A \subseteq B$.

Let then $\mu : \mathcal{C} \mapsto \mathcal{P}_f(X)$ be an arbitrary multisubmeasure.

We denote by $\sigma(\mathcal{C})$, the σ-ring generated by $\mathcal{C}$, $\mathcal{C}_\sigma = \{A \subset T$; there exists an increasing sequence of sets $(A_n)_n \subset \mathcal{C}$ such that $A = \bigcup\limits_{n=1}^{\infty} A_n\}$ and $\mathcal{C}_\delta = \{A \subset T$; there exists a decreasing sequence of sets $(A_n)_n \subset \mathcal{C}$ so that $A = \bigcap\limits_{n=1}^{\infty} A_n\}$.

It is easy to observe that $\mathcal{C} \subset \mathcal{C}_\sigma, \mathcal{C} \subset \mathcal{C}_\delta, \mathcal{C}_\sigma \subset \sigma(\mathcal{C}), \mathcal{C}_\delta \subset \sigma(\mathcal{C})$ and $(\mathcal{C}_\sigma)_\sigma = \mathcal{C}_\sigma$ (for the last assertion, see the proof of theorem 1.13). We also note that if $\mathcal{C}$ is a δ-ring, then $\mathcal{C}_\sigma = \sigma(\mathcal{C})$.

If, particularly, T is a locally compact space, let us consider the Baire δ-ring (σ-ring, respectively) $\mathcal{B}_0$ ($\mathcal{B}_0'$, respectively) generated by the G_δ-compact sets (that is, compact sets which are countable intersections of open sets) and the borelian δ-ring (σ-ring, respectively) $\mathcal{B}$ ($\mathcal{B}'$, respectively) generated by all compact sets of T.

Taking as starting point the work [4] of DINCULEANU, we have introduced in [5] different types of regularity for multisubmeasures μ defined on a ring $\mathcal{C}$ of a locally compact space T:

Definition 3. i) *A set $A \in \mathcal{C}$ is said to be R'_l-regular with respect to μ if for every $\varepsilon > 0$, there exists a compact set $K \in \mathcal{C}, K \subset A$ so that $|\mu(B)| < \varepsilon$, for every $B \in \mathcal{C}$, with $B \subset A \backslash K$.*

ii) *A set $A \in \mathcal{C}$ is said to be R'_r-regular with respect to μ if for every $\varepsilon > 0$, there exists an open set $D \in \mathcal{C}, A \subset D$ so that $|\mu(B)| < \varepsilon$, for every $B \in \mathcal{C}$, with $B \subset D \backslash A$.*

iii) *A set $A \in \mathcal{C}$ is said to be R'-regular with respect to μ if for every $\varepsilon > 0$, there exist a compact set $K \in \mathcal{C}$ and an open set $D \in \mathcal{C}, K \subset A \subset D$ so that $|\mu(B)| < \varepsilon$, for every $B \in \mathcal{C}$, with $B \subset D \backslash K$.*

iv) *μ is said to be R'_l-regular (R'_r-regular, R'-regular, respectively) on $\mathcal{C}$ if every set $A \in \mathcal{C}$ is R'_l-regular (R'_r-regular, R'-regular, respectively) with respect to μ.*

1. $\mathcal{K}-$**tight multisubmeasures.** Let T be a nonvoid, abstract set, $\mathcal{C}$ a ring of subsets of T, $\mathcal{K} \subset \mathcal{C}$ an arbitrary, nonvoid subfamily of $\mathcal{C}$ and $\mu : \mathcal{C} \mapsto \mathcal{P}_f(X)$ a multisubmeasure.

Definition 1.1. *μ is said to be $\mathcal{K}$-tight if for every $A \in \mathcal{C}$ and every $\varepsilon > 0$, there exists a set $K \in \mathcal{K}, K \subset A$ so that $|\mu(B)| < \varepsilon$, for every $B \in \mathcal{C}$, with $B \subset A \backslash K$.*

Remark 1.2. I) Obviously, any multisubmeasure $\mu : \mathcal{C} \mapsto \mathcal{P}_f(X)$ is $\mathcal{C}$-tight. Also, if $\mathcal{K}_1 \subset \mathcal{K}_2$ are two arbitrary subfamilies of $\mathcal{C}$ and if μ is $\mathcal{K}_1$-tight, then μ is also $\mathcal{K}_2$-tight.

II) If T is a locally compact space, $\mathcal{B}$ is the borelian δ-ring and $\mu : \mathcal{B} \mapsto \mathcal{P}_f(X)$ is a R'_l-regular multisubmeasure, then μ is $\mathcal{K}$-tight, $\mathcal{K}$ being the family of all compact sets of $\mathcal{B}$. Thus, when T is an arbitrary space, the property of $\mathcal{K}$-tightness is the natural generalization of the property of R'_l-regularity for multisubmeasures defined on rings of a locally compact space T.

As we shall prove in the sequel, the property of $\mathcal{K}$-tightness always implies the exhaustivity:

Theorem 1.3. *If $\mu : \mathcal{C} \mapsto \mathcal{P}_f(X)$ is $\mathcal{K}$-tight, then μ is exhaustive on $\mathcal{C}$.*

Proof. Let $\varepsilon > 0$ and $(A_n)_n \subset \mathcal{C}$, with $A_n \cap A_m = \emptyset, m \neq n$. Since μ is $\mathcal{K}$-tight, for every $n \in N$ there exists $K_n \in \mathcal{K}$ so that $K_n \subset A_n$ and $|\mu(B)| < \frac{\varepsilon}{2^{n+1}}$, for every $B \in \mathcal{C}$, with $B \subset A_n \backslash K_n$. Then $|\mu(A_n \backslash K_n)| < \frac{\varepsilon}{2^n}$, for every $n \in \mathbb{N}$. Because $K_n \cap K_m = \emptyset, m \neq n$ we get that $A_n \subset \bigcup_{i=1}^{n} (A_i \backslash K_i)$ for every $n \in \mathbb{N}$, which yields $|\mu(A_n)| \leq \sum_{i=1}^{n} |\mu(A_i \backslash K_i)| < \sum_{i=1}^{n} \frac{\varepsilon}{2^i} < \varepsilon$. This implies that μ is exhaustive, as claimed. □

Since in [5] we have established that any exhaustive multisubmeasure $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is also bounded (that is, there exists $M > 0$ such that $|\mu(A)| \leq M$, for every $A \in \mathcal{C}$), we immediately get from theorem 1.3:

Corollary 1.4. *Let $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ be a $\mathcal{K}$-tight multisubmeasure. Then μ is bounded.*

Remark 1.5. Generally, if $\mathcal{K}_1 \subset \mathcal{K}_2$ are two arbitrary subfamilies of $\mathcal{C}$, and if $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is $\mathcal{K}_2$-tight, we do not get that μ is $\mathcal{K}_1$-tight. Although, as we shall see in the following theorem, if, moreover, $\mathcal{C}$ is a σ-ring and μ is o-continuous on $\mathcal{C}$, then for two particular subfamilies, we obtain such a result.

Theorem 1.6. *Let $\mathcal{C}$ be a σ-ring and $\mathcal{K} \subset \mathcal{C}$ be a ring of sets. If $\mu : \mathcal{C} \mapsto \mathcal{P}_f(X)$ is o-continuous on $\mathcal{C}$ and $\mathcal{K}_\sigma$-tight, then μ is $\mathcal{K}$-tight.*

Proof. Let $A \in \mathcal{C}$ and $\varepsilon > 0$. Because μ is $\mathcal{K}_\sigma$-tight, there exists $K_1' \in \mathcal{K}_\sigma$, $K_1' \subset A$ so that $|\mu(B)| < \frac{\varepsilon}{2}$, for every $B \in \mathcal{C}$, with $B \subset A \backslash K_1'$. Since $K_1' \in \mathcal{K}_\sigma$, there exists an increasing sequence of sets $(K_n)_n \subset \mathcal{K}$ such that $K_1' = \bigcup_{n=1}^{\infty} K_n$. This implies $K_1' \backslash K_n \searrow \emptyset$ and, taking into account the o-continuity of μ on $\mathcal{C}$ hence also on $\mathcal{K}_\sigma$, we get that there exists $n_0(\varepsilon) \in N$ so that $|\mu(K_1' \backslash K_n)| < \frac{\varepsilon}{2}$, for every $n \geq n_0$. Particularly, $|\mu(K_1' \backslash K_{n_0})| < \frac{\varepsilon}{2}$.

Consequently, there exists $K_{n_0} \in \mathcal{K}$, with $K_{n_0} \subset K_1' \subset A$. Let $B \in \mathcal{C}, B \subset A \backslash K_{n_0} = (A \backslash K_1') \cup (K_1' \backslash K_{n_0})$. Obviously, there are two sets $B_1, B_2 \in \mathcal{C}$, so that $B_1 \subset A \backslash K_1', B_2 \subset K_1' \backslash K_{n_0}$ and $B = B_1 \cup B_2$. It follows that $|\mu(B)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, which means that μ is $\mathcal{K}$-tight. □

Corollary 1.7. *Let $\mathcal{C}$ be a σ-ring, $\mathcal{K} \subset \mathcal{C}$ a ring and $\mu : \mathcal{C} \mapsto \mathcal{P}_f(X)$ an o-continuous multisubmeasure (on $\mathcal{C}$). Then μ is $\mathcal{K}$-tight if and only if it is $\mathcal{K}_\sigma$-tight.*

Proof. The if part is immediately from remark 1.2. For the only if part, we use theorem 1.6. □

Theorem 1.8. *Let $\mu : \mathcal{C} \mapsto \mathcal{P}_f(X)$ be a $\mathcal{K}$-tight multisubmeasure. Then for every $A \in \mathcal{C}$ and every $\varepsilon > 0$, there exists $K \in \mathcal{K}$, $K \subset A$ so that $h(\mu(B), \mu(A)) < \varepsilon$, for every $B \in \mathcal{C}$, with $K \subset B \subset A$.*

Proof. Since μ is $\mathcal{K}$-tight, for every $A \in \mathcal{C}$ and every $\varepsilon > 0$, there exists $K \in \mathcal{K}, K \subset A$ such that $|\mu(B)| < \varepsilon$, for every $B \in \mathcal{C}$, with $B \subset A \backslash K$.

Let $B \in \mathcal{C}, K \subset B \subset A$. Then $A \backslash B \subset A \backslash K$, which implies that $h(\mu(A), \mu(B)) \leq |\mu(A \backslash B)| < \varepsilon$. □

Theorem 1.9. *Let $\mathcal{C}$ be a ring of subsets of T, $\mathcal{K} \subset \mathcal{C}$ a subfamily which is closed with respect to finite unions and $\mu_1, \mu_2 : \mathcal{C} \to \mathcal{P}_f(X)$ two $\mathcal{K}$-tight multisubmeasures so that $\mu_1 = \mu_2$ on $\mathcal{K}$. Then $\mu_1 = \mu_2$ on $\mathcal{C}$.*

Proof. Let $A \in \mathcal{C}$ and $\varepsilon > 0$.

Because μ_1 is $\mathcal{K}$-tight, there exists $K_1 \in \mathcal{K}, K_1 \subset A$ so that $|\mu_1(B)| < \frac{\varepsilon}{2}$, for every $B \in \mathcal{C}$, with $B \subset A \backslash K_1$.

Analogously, since μ_2 is $\mathcal{K}$-tight, there is $K_2 \in \mathcal{K}, K_2 \subset A$ such that $|\mu_2(B)| < \frac{\varepsilon}{2}$, for every $B \in \mathcal{C}$, with $B \subset A \backslash K_2$.

Let $K_1 \cup K_2 = K \in \mathcal{K}$. Then $K \subset A$ and, since $A \backslash K \subset A \backslash K_1$ and $A \backslash K \subset A \backslash K_2$, it follows that $|\mu_1(A \backslash K)| < \frac{\varepsilon}{2}$ and $|\mu_2(A \backslash K)| < \frac{\varepsilon}{2}$. Therefore,

$$e(\mu_1(A), \mu_2(A)) \leq e(\mu_1(A), \mu_1(K)) + e(\mu_1(K), \mu_2(K))$$
$$+e(\mu_2(K), \mu_2(A)) \leq |\mu_1(A \backslash K)| + |\mu_2(A \backslash K)| < \varepsilon.$$

Similarly, $e(\mu_2(A), \mu_1(A)) < \varepsilon$, hence $\mu_1(A) = \mu_2(A)$ because μ_1, $\mu_2 : \mathcal{C} \mapsto \mathcal{P}_f(X)$ are exhaustive and h is a metric on $\mathcal{P}_{bf}(X)$. □

In the sequel, we establish that the o-continuity on $\mathcal{K}$ always implies the o-continuity on $\mathcal{C}$ if μ is, additionally, $\mathcal{K}$-tight.

Theorem 1.10. *If $\mathcal{C}$ is a ring of subsets of T, $\mathcal{K} \subset \mathcal{C}$ is a subring of $\mathcal{C}$ and $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is a $\mathcal{K}$-tight multisubmeasure which is o-continuous on $\mathcal{K}$, then μ is o-continuous on $\mathcal{C}$.*

Proof. Let $\varepsilon > 0$ and $(A_n)_{n \in \mathbb{N}} \subset \mathcal{C}$ be a sequence so that $A_n \searrow \emptyset$.

Since μ is $\mathcal{K}$-tight, for every $n \in N$ there exists $K_n \in \mathcal{C}$ such that $K_n \subset A_n$ and $|\mu(B)| < \frac{\varepsilon}{2^{n+1}}$, for every $B \in \mathcal{C}$, with $B \subset A_n \backslash K_n$.

Obviously, $\bigcap_{n=1}^{\infty} K_n = \emptyset$ because $K_n \subset A_n$, for every $n \in N$.

We shall prove that, for every $n \in N$ there exists $K'_n \in \mathcal{K}, K'_n \subset A_n$ so that $K'_n \supset K'_{n+1}$ and $|\mu(B)| < \frac{\varepsilon}{2}$, for every $B \in \mathcal{C}$, with $B \subset A_n \backslash K'_n$.

Indeed, let $K'_1 = K_1$ and $K'_2 = K_1 \cap K_2$. Then $K'_2 \subset A_2, K'_2 \in \mathcal{K}, K'_2 \subset K'_1$ and

$$|\mu(B)| \leq |\mu(A_2 \backslash K_1)| + |\mu(A_2 \backslash K_2)| \leq |\mu(A_1 \backslash K_1)| + |\mu(A_2 \backslash K_2)| < \frac{\varepsilon}{4} + \frac{\varepsilon}{8} < \frac{\varepsilon}{2},$$

for every $B \in \mathcal{C}$ with $B \subset A_2 \backslash K'_2$.

Continuing by induction, we suppose that we have allready constructed $K'_1, K'_2 \ldots K'_{p-1}$. Let $K'_p = K_1 \cap \ldots \cap K_p \in \mathcal{K}$. Obviously, $K'_p \subset A_p, K'_p \subset K'_{p-1}$ and

$$|\mu(B)| \leq |\mu(A_1 \backslash K_1) + \ldots + |\mu(A_p \backslash K_p)| < \frac{\varepsilon}{4} + \frac{\varepsilon}{8} + \ldots + \frac{\varepsilon}{2^{p+1}} < \frac{\varepsilon}{2},$$

for every $B \in \mathcal{C}$ with $B \subset A_p \backslash K'_p$.

Consequently, we get a sequence of sets $(K'_n)_n \subset \mathcal{K}$ such that $K'_n \searrow \emptyset$ and $K'_n \subset A_n$, for every $n \in N$. It also holds the inequality $|\mu(B)| < \frac{\varepsilon}{2}$, for every $B \in \mathcal{C}$,with $B \subset A_n \backslash K'_n$.

On the other hand, since μ is o-continuous on $\mathcal{K}$, there exists $n_0(\varepsilon) \in N$ such that $|\mu(K'_n)| < \frac{\varepsilon}{2}$, for every $n \geq n_0$. Then

$$|\mu(A_n)| \leq |\mu(A_n \backslash K'_n)| + |\mu(K'_n)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

for every $n \geq n_0$, which means that μ is o-continuous on $\mathcal{C}$. □

Theorem 1.11. *Let $\mathcal{K}_1$, $\mathcal{K}_2 \subset \mathcal{C}$ be two subfamilies of $\mathcal{C}$ so that $\mathcal{K}_1 \subset \mathcal{K}_2$ and $\mathcal{K}_2$ is a ring. Let $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ be a multisubmeasure. Then μ is $\mathcal{K}_1$-tight if and only if μ is $\mathcal{K}_2$-tight and $\mu|_{\mathcal{K}_2}$ is $\mathcal{K}_1$-tight.*

Proof. For the if part, let $A \in \mathcal{C}$ and $\varepsilon > 0$.

Because μ is $\mathcal{K}_1$-tight and $\mathcal{K}_1 \subset \mathcal{K}_2$, we immediately get from remark 1.2 that μ is also $\mathcal{K}_2$-tight.

We prove now that $\mu|_{\mathcal{K}_2}$ is $\mathcal{K}_1$-tight. Indeed, let $A \in \mathcal{K}_2$. Since μ is $\mathcal{K}_1$-tight, there exists $K_1 \in \mathcal{K}_1, K_1 \subset A$ such that $|\mu(B)| < \varepsilon$, for every $B \in \mathcal{C}, B \subset A \backslash K_1$. Particularly, $|\mu(B)| < \varepsilon$, for every $B \in \mathcal{K}_2$, with $B \subset A \backslash K_1$. This means $\mu|_{\mathcal{K}_2}$ is $\mathcal{K}_1$-tight.

For the only if part, let $A \in \mathcal{C}$ and $\varepsilon > 0$. Because μ is $\mathcal{K}_2$-tight, there exists $K_2 \in \mathcal{K}_2, K_2 \subset A$ so that $|\mu(B)| < \frac{\varepsilon}{2}$, for every $B \in \mathcal{C}, B \subset A \backslash K_2$. Particularly, $|\mu(A \backslash K_2)| < \frac{\varepsilon}{2}$.

Since $\mu|_{\mathcal{K}_2}$ is $\mathcal{K}_1$-tight, for $K_2 \in \mathcal{K}_2$, there exists $K_1 \in \mathcal{K}_1, K_1 \subset K_2$ such that $|\mu(B)| < \frac{\varepsilon}{2}$, for every $B \in \mathcal{K}_2, B \subset K_2 \backslash K_1$. Hence, $K_1 \subset A$, and if $B \in \mathcal{C}, B \subset A \backslash K_1 = (A \backslash K_2) \cup (K_2 \backslash K_1)$, then $|\mu(B)| \leq |\mu(A \backslash K_2)| + |\mu(K_2 \backslash K_1)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Consequently, μ is $\mathcal{K}_1$-tight. □

In the following, let $\mathcal{C}$ be an arbitrary ring, $\mathcal{K}$ a subfamily of $\mathcal{C}$ and $\mu_1, \mu_2 : \mathcal{C} \mapsto \mathcal{P}_f(X)$ be two multisubmeasures. We introduce the multivalued set function $\mu : \mathcal{C} \mapsto \mathcal{P}_f(X)$, defined by:

$$\mu(A) = \mu_1(A) \overset{\bullet}{+} \mu_2(A), \text{for every } A \in \mathcal{C},$$

which will be called the *Minkowski addition of* μ_1 and μ_2.

It is easy to prove that μ is a multisubmeasure and it has the following properties:

Theorem 1.12. i) *If μ_1 and μ_2 are o-continuous (exhaustive or increasing convergent, respectively), then so is μ.*

ii) *If $\mathcal{K}$ is a family which is closed with respect to finite unions and* if *μ_1 and μ_2 are $\mathcal{K}$-tight, then μ is $\mathcal{K}$-tight.*

More generally, if μ_1 is $\mathcal{K}_1$-tight and μ_2 is $\mathcal{K}_2$-tight, $\mathcal{K}_1$ and $\mathcal{K}_2$ being two arbitrary families, then μ is $\mathcal{K}_1 \cup \mathcal{K}_2$-tight, where $\mathcal{K}_1 \cup \mathcal{K}_2 = \{K_1 \cup K_2; K_1 \in \mathcal{K}_1, K_2 \in \mathcal{K}_2\}$.

In the sequel, using some ideas of Dennenberg [3] (chapter 2), we shall establish an extension result from $\mathcal{C}$ to $\mathcal{C}_\sigma$ by preserving the properties of the multisubmeasure.

Theorem 1.13. *Let $\mathcal{C}$ be a ring of sets of T and X be a Banach space.*

i) *If $\mu : \mathcal{C} \mapsto \mathcal{P}_f(X)$ is an increasing convergent and exhaustive multisubmeasure, then μ uniquely extends to an increasing convergent, exhaustive multisubmeasure $\mu^* : \mathcal{C}_\sigma \mapsto \mathcal{P}_{bf}(X)$.*

ii) *If, additionally, $\mathcal{K} \subset \mathcal{C}$ is a family which is closed with respect to countable intersections and μ is increasing convergent and $\mathcal{K}$-tight, then μ^* is also increasing convergent and $\mathcal{K}$-tight.*

Proof. i) Let $\varepsilon > 0$ and $A \in \mathcal{C}_\sigma$. There is an increasing sequence of sets $(A_n)_n \subset \mathcal{C}$ so that $A = \bigcup\limits_{n=1}^{\infty} A_n$. Since μ is exhaustive, we get from [5] (theorem 2.8) that $\lim |\mu(A_n \Delta A_m)| = 0$, hence $(\mu(A_n))_n$ is a Cauchy sequence in $\mathcal{P}_f(X)$. On the other hand, the exhaustivity of μ also implies that μ is bounded; therefore, $(\mu(A_n))_n \subset \mathcal{P}_{bf}(X)$. Because $\mathcal{P}_{bf}(X)$ is a complete metric space, it follows that $(\mu(A_n))_n$ is convergent. Let us denote then

$$\mu^*(A) = \lim_{n\to\infty} \mu(A_n) \text{ (with respect to } h), \text{ for every } A \in \mathcal{C}_\sigma.$$

Obviously, $\mu^* : \mathcal{C}_\sigma \mapsto \mathcal{P}_{bf}(X)$.

Now, we prove that the limit does not depend on the sequence $(A_n)_n$. More precisely, if $(A_n)_n$ and $(B_n)_n$ are two increasing sequences of sets of $\mathcal{C}$ so that $A = \bigcup\limits_{n=1}^{\infty} A_n = \bigcup\limits_{n=1}^{\infty} B_n$ and if we denote by $\mu_1^*(A) = \lim\limits_{n\to\infty} \mu(B_n)$, then $\mu^*(A) = \mu_1^*(A)$, for every $A \in \mathcal{C}_\sigma$.

Indeed, since $A_m \nearrow A$, there exists $m_0 \in N$ such that $h(\mu(A_m), \mu^*(A)) < \frac{\varepsilon}{3}$, for every $m \geq m_0$. Particularly, $h(\mu(A_{m_0}), \mu^*(A)) < \frac{\varepsilon}{3}$.

Because $A_{m_0} \cap B_n \nearrow A_{m_0} \cap A = A_{m_0}$ and μ is increasing convergent, there exists $n_1 \in N$ so that $h(\mu(A_{m_0}), \mu(A_{m_0} \cap B_n)) < \frac{\varepsilon}{3}$, for every $n \geq n_1$.

Also, since $B_n \nearrow A$, there exists $n_2 \in N$ such that $h(\mu_1^*(A), \mu(B_n)) < \frac{\varepsilon}{3}$, for every $n \geq n_2$.

Consequently, if $n_3 = \max\{n_1, n_2\}$, we get that

$$\begin{aligned} e(\mu^*(A), \mu_1^*(A)) &\leq e(\mu^*(A), \mu(A_{m_0})) + e(\mu(A_{m_0}), \mu(A_{m_0} \cap B_{n_3})) \\ &+ e(\mu(A_{m_0} \cap B_{n_3}), \mu(B_{n_3})) + e(\mu(B_{n_3}), \mu_1^*(A)) \\ &\leq h(\mu^*(A), \mu(A_{m_0})) + h(\mu(A_{m_0}), \mu(A_{m_0} \cap B_{n_3})) \\ &+ h(\mu(B_{n_3}), \mu_1^*(A)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Analogously, $e(\mu_1^*(A), \mu^*(A)) < \varepsilon$, for every $\varepsilon > 0$, which finally yields $\mu^*(A) = \mu_1^*(A)$, for every $A \in \mathcal{C}_\sigma$.

In the following we demonstrate that μ^* is a multisubmeasure. Obviously, $\mu^*(\emptyset) = \{0\}$.

If $A, B \in \mathcal{C}_\sigma, A \subset B$, there exist two sequences $(A_n)_n$ and $(B_n)_n$ in $\mathcal{C}$ with $A_n \nearrow A$ and $B_n \nearrow B$; consequently,

$$\begin{aligned} e(\mu^*(A), \mu^*(B)) &\leq e(\mu^*(A), \mu(A_n \cap B_n)) + e(\mu(A_n \cap B_n), \mu(B_n)) \\ &+ e(\mu(B_n), \mu^*(B)) \leq h(\mu^*(A), \mu(A_n \cap B_n)) + h(\mu(B_n), \mu^*(B)). \end{aligned}$$

Since $A_n \cap B_n \nearrow A$ and $B_n \nearrow B$, from the definition of μ^* we immediately get that $\mu^*(A) \subseteq \mu^*(B)$.

We consider now two sets $A, B \in \mathcal{C}_\sigma$ such that $A \cap B = \emptyset$. Let $(A_n)_n$ and $(B_n)_n$ be two increasing sequences in $\mathcal{C}$ so that $A = \bigcup\limits_{n=1}^{\infty} A_n$ and $B = \bigcup\limits_{n=1}^{\infty} B_n$. Obviously, $A_n \cap B_n = \emptyset$, for every $n \in N$. Then, using again the definition of μ^*, we get that $\mu^*(A \cup B) \subseteq \mu^*(A) \overset{\bullet}{+} \mu^*(B)$, hence, indeed, μ^* is a multisubmeasure.

Let us prove now that μ^* is increasing convergent. For this, let $(A_n)_n \subset \mathcal{C}_\sigma$, with $A_n \nearrow A = \bigcup\limits_{n=1}^{\infty} A_n$. Because $e(\mu^*(A_n), \mu^*(A)) = 0$, for every $n \in N$, we have to demonstrate only that for every $\varepsilon > 0$ there exists $n_0 \in N$, so that $e(\mu^*(A), \mu^*(A_n)) < \varepsilon$, for every $n \geq n_0$.

Since $A_n \in \mathcal{C}_\sigma$, there exists an increasing sequence of sets $(A^n_m)_m \subset \mathcal{C}$, with $A_n = \bigcup\limits_{m=1}^{\infty} A^n_m$, for every $n \in N$.

Let $C_m = A^1_m \cup A^2_m \cup \ldots \cup A^m_m$. Then $(C_m)_m \subset \mathcal{C}$ and $A^n_m \subset C_m \subset A_m$, for every $m \geq n$. It follows that

$$A_n = \bigcup_{m=n}^{\infty} A^n_m \subset \bigcup_{m=n}^{\infty} C_m \subset \bigcup_{m=n}^{\infty} A_m = \bigcup_{n=1}^{\infty} A_n, \text{ for every } n \in N,$$

hence $\bigcup\limits_{n=1}^{\infty} A_n = \bigcup\limits_{n=1}^{\infty} C_n = A$.

On the other hand,
$e(\mu^*(A), \mu^*(A_n)) \leq e(\mu^*(A), \mu(C_n)) + e(\mu(C_n), \mu^*(A_n))$ and $\mu^*(C_n) = \mu(C_n)$, for every $n \in N$, because $(C_n)_n \subset \mathcal{C}$.

Therefore, since $C_n \nearrow A$, there exists $n_0 \in N$ so that
$e(\mu^*(A), \mu^*(A_n)) \leq h(\mu^*(A), \mu(C_n)) + e(\mu^*(C_n), \mu^*(A_n)) = h(\mu^*(A), \mu(C_n)) < \varepsilon$, for every $n \geq n_0$.

We demonstrate now that μ^* is exhaustive on $\mathcal{C}_\sigma$. Indeed, let $(A_n)_n \subset \mathcal{C}_\sigma$ be an arbitrary disjoint sequence. For every $n \in N$ there exists $(A^k_n)_k \subset \mathcal{C}$ so that $A^k_n \nearrow A_n$. Since for every $n \in N$, $\mu^*(A_n) = \lim\limits_{k\to\infty} \mu(A^k_n)$, we get that there exists $k^n_0 \in N$ so that

$|\mu^*(A_n)| \leq h(\mu^*(A_n), \mu(A^{k^n_0}_n)) + |\mu(A^{k^n_0}_n)| < \frac{\varepsilon}{2} + |\mu(A^{k^n_0}_n)|$.

But $A^{k^n_0}_n \cap A^{k^m_0}_m = \emptyset$, $m \neq n$; therefore, from the exhaustivity of μ on $\mathcal{C}$, it follows that $\lim |\mu(A^{k^n_0}_n)| = 0$.

Thus, there exists $n_0 \in N$ so that $|\mu^*(A_n)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, for every $n \geq n_0$. It means that μ^* is indeed exhaustive on $\mathcal{C}_\sigma$.

It only remains to prove the uniqueness. Let us suppose that, on the contrary, there also exists another increasing convergent, exhaustive multisubmeasure $\mu_1^* : \mathcal{C}_\sigma \mapsto \mathcal{P}_f(X)$ which extends μ. Let $A \in \mathcal{C}_\sigma$ and $\varepsilon > 0$ be arbitrarily.

There is an increasing sequence of sets $(A_n)_n \subset \mathcal{C}$, with $A_n \nearrow A$.

Since μ^* and μ_1^* are both increasing convergent, there exists $n_0(\varepsilon) \in N$ so that $h(\mu^*(A_n), \mu^*(A)) < \frac{\varepsilon}{2}$ and $h(\mu_1^*(A_n), \mu_1^*(A)) < \frac{\varepsilon}{2}$, for every $n \geq n_0$. Then, also, $h(\mu^*(A_{n_0}), \mu^*(A)) < \frac{\varepsilon}{2}$ and $h(\mu_1^*(A_{n_0}), \mu_1^*(A)) < \frac{\varepsilon}{2}$.

Because $\mu(A_{n_0}) = \mu^*(A_{n_0}) = \mu_1^*(A_{n_0})$, we get that

$h(\mu^*(A), \mu_1^*(A)) \leq h(\mu^*(A), \mu(A_{n_0})) + h(\mu(A_{n_0}), \mu_1^*(A)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$,

which implies that $\mu^*(A) = \mu_1^*(A)$.

ii) Let us note first that, since μ is $\mathcal{K}$-tight, it is also exhaustive, hence bounded.

Let $\varepsilon > 0$ and $A \in \mathcal{C}_\sigma$. There exists an increasing sequence $(A_n)_n \subset \mathcal{C}$, with $A_n \nearrow A$. We have already seen that, in this case, $\mu^*(A) = \lim\limits_{n\to\infty} \mu(A_n)$.

Because $(A_n)_n \subset \mathcal{C}$ and μ is $\mathcal{K}$-tight, for every $n \in N$ there exists $K_n \in \mathcal{K}$ so that $K_n \subset A_n$ and $|\mu(B)| < \frac{\varepsilon}{2^{n+2}}$, for every $B \in \mathcal{C}$ with $B \subset A_n \backslash K_n$.

Then, $|\mu(A_n \backslash K_n)| < \frac{\varepsilon}{2^{n+1}}$, for every $n \in N$.

Let $K = \bigcap\limits_{n=1}^{\infty} K_n$. Then $K \in \mathcal{K}$ and $K \subset A$.

If $B \in \mathcal{C}_\sigma, B \subset A \backslash K = (\bigcup\limits_{n=1}^{\infty} A_n) \backslash (\bigcap\limits_{n=1}^{\infty} K_n) = \bigcup\limits_{n=1}^{\infty} (A_n \backslash K_n)$, there exists an increasing sequence $(B_m)_m \subset \mathcal{C}$, with $B = \bigcup\limits_{m=1}^{\infty} B_m$.

We observe that, for $n \to \infty$,

$$\bigcup_{m=1}^{n} [B_m \cap (A_m \backslash K_m)] \nearrow \bigcup_{m=1}^{\infty} [B_m \cap (A_m \backslash K_m)] = B \cap A \cap cK = B,$$

which implies that $\mu^*(B) = \lim\limits_{n\to\infty} \mu(\bigcup\limits_{m=1}^{n} [B_m \cap (A_m \backslash K_m)])$ (with respect to h), hence there exists $n_0 \in N$ so that

$$h(\mu^*(B), \mu(\bigcup_{m=1}^{n} [B_m \cap (A_m \backslash K_m)])) < \frac{\varepsilon}{2}, \text{ for every } n \geq n_0.$$

Particularly, $h(\mu^*(B), \mu(\bigcup\limits_{m=1}^{n_0}[B_m \cap (A_m \backslash K_m)])) < \frac{\varepsilon}{2}$. Then

$$\begin{aligned}|\mu^*(B)| &\le h(\mu^*(B), \mu(\bigcup_{m=1}^{n_0}[B_m \cap (A_m \backslash K_m)])) \\ &+ \sum_{m=1}^{n_0} |\mu(B_m \cap (A_m \backslash K_m))| < \frac{\varepsilon}{2} \\ &+ \sum_{m=1}^{n_0} |\mu(A_m \backslash K_m)| < \frac{\varepsilon}{2} + \sum_{m=1}^{n_0} \frac{\varepsilon}{2^{m+1}} < \varepsilon.\end{aligned}$$

Consequently, we found a set $K \in \mathcal{K}, K \subset A$ so that $|\mu^*(B)| < \varepsilon$, for every $B \in \mathcal{C}_\sigma$, with $B \subset A \backslash K$; it means that, indeed, μ^* is $\mathcal{K}$-tight.

We observe that it can also be indicated another proof, based on the fact that if μ^* is increasing convergent on $\mathcal{C}_\sigma$, then μ^* is $h-\sigma$-subadditive on $\mathcal{C}_\sigma$ (see also [5] for the general setting of an arbitrarily multisubmeasure).

Since $B = \bigcup\limits_{m=1}^{\infty} B_m \subset \bigcup\limits_{n=1}^{\infty}(A_n \backslash K_n)$, it follows that

$|\mu^*(B)| \le \sum\limits_{n=1}^{\infty} |\mu^*(A_n \backslash K_n)| = \sum\limits_{n=1}^{\infty} |\mu(A_n \backslash K_n)| < \sum\limits_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}} < \varepsilon.$

This completes the proof. □

Corollary 1.14. *If T is a locally compact space, $\mathcal{C}$ is a ring of subsets of T, X is a Banach space, $\mathcal{B}_0$ is the Baire δ-ring generated by the G_δ-compact sets of T, $\mathcal{B}'_0$ is the Baire σ-ring generated by $\mathcal{B}_0$, and $\mu : \mathcal{B}_0 \mapsto \mathcal{P}_f(X)$ is a R'_l-regular multisubmeasure, then μ uniquely extends to a R'_l-regular multisubmeasure $\mu^* : \mathcal{B}'_0 \mapsto \mathcal{P}_f(X)$.*

Proof. Let $\mathcal{K}$ be the family of all compact sets of $\mathcal{B}_0$. Obviously, $\mathcal{K}$ is closed with respect to countable intersections and the R'_l-regularity of μ on $\mathcal{B}_0$ is equivalent to its $\mathcal{K}$-tightness on $\mathcal{B}_0$.

Also, since μ is R'_l-regular, it is also exhaustive, which implies that $\mu : \mathcal{B}_0 \mapsto \mathcal{P}_{bf}(X)$. On the other hand, it is known from [6] that the R'_l-regularity of μ is equivalent to its o-continuity on $\mathcal{B}_0$; thus, we get from [5] that μ is increasing convergent on $\mathcal{B}_0$.

Applying now theorem 1.13, we obtain the extension of μ to $\mathcal{B}'_0$, which is the R'_l-regular, increasing convergent multisubmeasure $\mu^* : \mathcal{B}'_0 \mapsto \mathcal{P}_{bf}(X)$.

Let us note that, for the case when $\mathcal{C} = \mathcal{B}'_0$, the R'_l-regularity is a consequence of the increasing convergence. Indeed, because $\mathcal{B}'_0$ is the σ-ring generated by the δ-ring $\mathcal{B}_0$, the increasing convergence of μ^* is equivalent to its o-continuity on $\mathcal{B}'_0$ (see [5]), which implies the R'_l-regularity of μ^* on $\mathcal{B}'_0$ (see [6]). $\square$

Remark 1.15. A similar result to the one of corollary 1.14 can easily be obtained, if we replace the Baire δ-ring $\mathcal{B}_0$ by the borelian δ-ring $\mathcal{B}$, and the Baire σ-ring $\mathcal{B}'_0$ by the borelian σ-ring $\mathcal{B}'$.

2. $\mathcal{K}-\mathcal{D}$-regular multisubmeasures. We remind first that in literature there appear many generalizations for the notion of regularity of set functions defined on rings of a locally compact space T. For instance, in [2], Belley and Morales introduced an abstract type of regularity for measures taking values in a topological group; this type of regularity was generalized in [12] by Maria Gonzales for measures taking values in certain topological semigroups. In [13], Precupanu adapted this concept of regularity for multimeasures and called it there (B-M)-regularity. Also, Riečanova gave in [14] and [15] general versions of regularity.

In what follows we shall introduce an abstract type of regularity for the case when T is a nonvoid, abstract set. More exactly, instead of the class $\mathcal{K}$ of compact sets and the class $\mathcal{D}$ of open sets of an arbitrary ring $\mathcal{C}$ of a locally compact space T (classes that naturally appear in the definition of R'-regularity), we shall consider two arbitrary subfamilies $\mathcal{K}\subset\mathcal{C}$ and $\mathcal{D}\subset\mathcal{C}$, $\mathcal{C}$ being a ring of subsets of T.

Thus, let us consider in the sequel $\mathcal{C}$ be a ring of subsets of a nonvoid, abstract set T and $\mathcal{K}\subset\mathcal{C},\mathcal{D}\subset\mathcal{C}$ be two arbitrary, non-empty families of $\mathcal{C}$.

Definition 2.1. *A set $A\in\mathcal{C}$ is said to be $\mathcal{K}-\mathcal{D}$-regular with respect to μ if for every $\varepsilon>0$ there exist two sets $K\in\mathcal{K}$ and $D\in\mathcal{D}$ so that $K\subset A\subset D$ and $|\mu(B)|<\varepsilon$, for every $B\in\mathcal{C}$, with $B\subset D\backslash K$.*

Definition 2.2. *μ is said to be $\mathcal{K}-\mathcal{D}$-regular (on $\mathcal{C}$) if every set $A\in\mathcal{C}$ is $\mathcal{K}-\mathcal{D}$-regular with respect to μ.*

Let us note that, obviously, the notion of $\mathcal{K}-\mathcal{D}$-regularity (for T arbitrary) generalizes the notion of R'-regularity (when T is a locally compact space). Indeed, if T is a locally compact space, $\mathcal{B}$ is the borelian δ-ring, $\mathcal{K}$ is the family of all compact sets of $\mathcal{B}$ and $\mathcal{D}$ is the family of all open sets of $\mathcal{B}$, then any R'-regular multisubmeasure on $\mathcal{B}$ is $\mathcal{K}-\mathcal{D}$-regular.

The same is true if we consider the Baire δ-ring $\mathcal{B}_0$.

In the following, let us consider $\mathcal{C}_\mu = \{A \in \mathcal{C}; A$ is $\mathcal{K}-\mathcal{D}$-regular with respect to $\mu\}$. Obviously, $\mathcal{C}_\mu \subseteq \mathcal{C}$ and the equality holds if and only if μ is $\mathcal{K}-\mathcal{D}$-regular on $\mathcal{C}$.

We shall point out some properties of $\mathcal{C}_\mu$.

Theorem 2.3. i) *If the family $\mathcal{K}$ is closed with respect to countable intersections, $\mathcal{D}$ is closed with respect to finite intersections and μ is o-continuous on $\mathcal{K}$, then $\mathcal{C}_\mu$ is a monotone decreasing class, that is, $\bigcap\limits_{n=1}^{\infty} A_n \in \mathcal{C}_\mu$, for every decreasing sequence of sets $(A_n)_n \subset \mathcal{C}_\mu$.*

ii) *If the family $\mathcal{K}$ is closed with respect to countable intersections, $\mathcal{D}$ is closed with respect to countable unions and μ is o-continuous on $\mathcal{K}$ and $\mathcal{D}$, then $\mathcal{C}_\mu$ is a monotone increasing class, that is, $\bigcup\limits_{n=1}^{\infty} A_n \in \mathcal{C}_\mu$, for every increasing sequence of sets $(A_n)_n \subset \mathcal{C}_\mu$.*

Proof. i) Let $\varepsilon > 0$ and $(A_n)_n \subset \mathcal{C}_\mu$, with $A_n \searrow A$. We prove that $A \in \mathcal{C}_\mu$.

Because $A_n \in \mathcal{C}_\mu$, for every $n \in N$, there exist $K_n \in \mathcal{K}$ and $D_n \in \mathcal{D}$ such that $K_n \subset A_n \subset D_n$ and $|\mu(B)| < \frac{\varepsilon}{2^{n+3}}$, for every $B \in \mathcal{C}$, with $B \subset D_n \backslash K_n$. Then, also, $|\mu(B)| < \frac{\varepsilon}{2^{n+2}}$, for every $B \in \mathcal{C}$, with $B \subset D_n \backslash A_n$.

By induction, we easily obtain the existence of a decreasing sequence of sets $(K'_n)_n \subset \mathcal{K}$ so that $K'_n \subset A_n \subset D_n$, for every $n \in N$ and $|\mu(B)| < \frac{\varepsilon}{4}$, for every $B \in \mathcal{C}$, with $B \subset A_n \backslash K'_n$.

It follows then that $|\mu(B)| < \frac{\varepsilon}{4} + \frac{\varepsilon}{2^{n+2}} < \frac{\varepsilon}{2}$, for every $B \in \mathcal{C}$, with $B \subset D_n \backslash K'_n = (D_n \backslash A_n) \cup (A_n \backslash K'_n)$.

Let $K = \bigcap\limits_{n=1}^{\infty} K'_n$. Obviously, $K \subset A$ and $K \in \mathcal{K}$.

Since $K'_n \searrow K$ and μ is o-continuous on $\mathcal{K}$, there exists $n_0(\varepsilon) \in N$ such that $|\mu(K'_{n_0} \backslash K)| < \frac{\varepsilon}{2}$.

Let us consider $D_{n_0} = \bigcap\limits_{i=1}^{n_0} D_i$. We observe that $D_{n_0} \in \mathcal{D}$ and

$A = \bigcap\limits_{n=1}^{\infty} A_n \subset \bigcap\limits_{i=1}^{n_0} A_i \subset \bigcap\limits_{i=1}^{n_0} D_i = D_{n_0}$.

Thus, we found two sets $K \in \mathcal{K}$ and $D_{n_0} \in \mathcal{D}$, with $K \subset A \subset D_{n_0}$ so that, if $B \in \mathcal{C}$, with $B \subset D_{n_0} \backslash K = (D_{n_0} \backslash K'_{n_0}) \cup (K'_{n_0} \backslash K)$, then $|\mu(B)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Consequently, $A \in \mathcal{C}_\mu$.

ii) Let $\varepsilon > 0$ and $(A_n)_n \subset \mathcal{C}_\mu$ be a sequence of sets so that $A_n \nearrow A$. We prove that $A \in \mathcal{C}_\mu$.

Using the same sets $(K_n)_n$ and $(D_n)_n$ as in i), we consider here $K'_n = \bigcap_{i=1}^{n} K_i \in \mathcal{K}$, $K = \bigcap_{n=1}^{\infty} K'_n \in \mathcal{K}$, $D'_n = \bigcup_{i=1}^{n} D_i \in \mathcal{D}$ and $D = \bigcup_{n=1}^{\infty} D'_n \in \mathcal{D}$.

Then $K \subset A \subset D$, $(K'_n)_n \subset \mathcal{K}$ is a decreasing sequence of sets and $(D'_n)_n \subset \mathcal{D}$ is an increasing one.

Let $B \in \mathcal{C}$, with $B \subset D \backslash K$.

Because $D'_n \nearrow D$ and μ is o-continuous on $\mathcal{D}$, there exists $n_1(\varepsilon) \in N$ so that $|\mu(D \backslash D'_n)| < \frac{\varepsilon}{3}$, for every $n \geq n_1$.

Also, since $K'_n \searrow K$ and μ is o-continuous on $\mathcal{K}$, there exists $n_2(\varepsilon) \in N$ so that $|\mu(K'_n \backslash K)| < \frac{\varepsilon}{3}$, for every $n \geq n_2$.

Then $|\mu(D \backslash D'_{n_0})| < \frac{\varepsilon}{3}$ and $|\mu(K'_{n_0} \backslash K)| < \frac{\varepsilon}{3}$, where $n_0 = \max(n_1, n_2)$.

Since $B \subset D \backslash K = (D \backslash D'_{n_0}) \cup (D'_{n_0} \backslash K'_{n_0}) \cup (K'_{n_0} \backslash K)$, we get that

$$|\mu(B)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \sum_{i=1}^{n_0} \frac{\varepsilon}{2^{i+2}} < \varepsilon,$$

because $D'_{n_0} \backslash K'_{n_0} = (\bigcup_{i=1}^{n_0} D_i) \backslash (\bigcap_{i=1}^{n_0} K_i) = \bigcup_{i=1}^{n_0} (D_i \backslash K_i)$.

Consequently, $A \in \mathcal{C}_\mu$, which completes the proof. □

Remark 2.4. If $\mu : \mathcal{C} \mapsto \mathcal{P}_f(X)$ is a $\mathcal{K} - \mathcal{D}$-regular multisubmeasure, then μ is $\mathcal{K}$-tight. For some particular cases, it also holds the converse, as we shall see in the following theorem.

Theorem 2.5. *Let $\mathcal{C}$ be an algebra, $\mathcal{K} \subset \mathcal{C}$, $\mathcal{D} \subset \mathcal{C}$ two arbitrary subfamilies of $\mathcal{C}$ and $\mu : \mathcal{C} \mapsto \mathcal{P}_f(X)$ a $\mathcal{K}$-tight multisubmeasure. Then μ is $\mathcal{K} - \mathcal{D}$-regular, where $\mathcal{D} = \{A \in \mathcal{C}; cA \in \mathcal{K}\}$.*

Proof. Let $A \in \mathcal{C}$ and $\varepsilon > 0$. Because μ is $\mathcal{K}$-tight, there exists $K \in \mathcal{K}$ so that $K \subset A$ and $|\mu(B)| < \frac{\varepsilon}{2}$, for every $B \in \mathcal{C}$, with $B \subset A \backslash K$. But $cA \in \mathcal{C}$ and, therefore, there exists $K_1 \in \mathcal{K}$ such that $K_1 \subset cA$ and $|\mu(B)| < \frac{\varepsilon}{2}$, for every $B \in \mathcal{C}, B \subset cA \backslash K_1 = cA \cap cK_1$.

Let us denote $cK_1 = D \in \mathcal{D}$. Then $A \subset D$ and $|\mu(B)| < \frac{\varepsilon}{2}$, for every $B \in \mathcal{C}$, $B \subset D \backslash A = cK_1 \cap cA$.

Consequently, there exist $K \in \mathcal{K}$ and $D \in \mathcal{D}$ so that $K \subset A \subset D$ and $|\mu(B)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, for every $B \in \mathcal{C}, B \subset D \backslash K = (D \backslash A) \cup (A \backslash K)$, which means that μ is $\mathcal{K} - \mathcal{D}$-regular (on $\mathcal{C}$). □

Corollary 2.6. *If $\mathcal{C}$ is an algebra of subsets of a compact space T and $\mu : \mathcal{C} \mapsto \mathcal{P}_f(X)$ is an arbitrary multisubmeasure, then μ is R'_l-regular if and only if it is R'_r-regular (on $\mathcal{C}$).*

Proof. For the if part, we apply theorem 2.5 for the class $\mathcal{K}$ of all compact sets and the class $\mathcal{D}$ of all open sets of $\mathcal{C}$. We also use (see [5]) the fact that R'-regularity implies R'_r-regularity.

The only if part follows immediately from the definitions of R'_l-regularity and R'_r-regularity, taking into account that T is a compact space. □

In the following we give another example of a $\mathcal{K}-\mathcal{D}$-regular multisubmeasure.

Theorem 2.7. *Let $\mu : \sigma(\mathcal{C}) \mapsto \mathcal{P}_f(X)$ be an o-continuous multisubmeasure. Then μ is $\mathcal{C}_\delta - \mathcal{C}_\sigma$-regular, that is, for every $A \in \sigma(\mathcal{C})$ and for every $\varepsilon > 0$, there exist $K \in \mathcal{C}_\delta$ and $D \in \mathcal{C}_\sigma$ such that $K \subset A \subset D$ and $|\mu(B)| < \varepsilon$, for every $B \in \sigma(\mathcal{C})$, with $B \subset D \backslash K$.*

Proof. We note that it is sufficient to prove that $|\mu(D\backslash K)| < \varepsilon$.

Let $\mathcal{C}_\mu = \{A \in \sigma(\mathcal{C}); A$ is $\mathcal{C}_\delta - \mathcal{C}_\sigma$-regular$\}$. It is easy to establish that $\mathcal{C} \subset \mathcal{C}_\mu$ and $\mathcal{C}_\mu$ is a ring. We demonstrate that, moreover, $\mathcal{C}_\mu$ is a σ-ring; it will imply that $\mathcal{C}_\mu = \sigma(\mathcal{C})$.

Let $\varepsilon > 0$ and $(A_n)_n \subset \mathcal{C}_\mu$ so that $A_n \nearrow A$. We have to prove that $A \in \mathcal{C}_\mu$.

Because μ is o-continuous, there exists $n_0(\varepsilon) \in N$ such that $|\mu(A\backslash A_{n_0})| < \frac{\varepsilon}{3}$.

Since $A_{n_0} \in \mathcal{C}_\mu$, there exists $K \in \mathcal{C}_\delta, K \subset A_{n_0}$, with $|\mu(A_{n_0}\backslash K)| < \frac{\varepsilon}{3}$.

On the other hand, by the definition of $\mathcal{C}_\mu$, for every $n \in N^*$, $n = n_0 + k, k \in N^*$, there is $D_n \in \mathcal{C}_\sigma$ so that $A_n \subset D_n$ and $|\mu(D_{n_0+k}\backslash A_{n_0+k})| < \frac{\varepsilon}{3^{k+1}}$.

Let $D = \bigcup\limits_{k=1}^{\infty} D_{n_0+k}$. Because $(\mathcal{C}_\sigma)_\sigma = \mathcal{C}_\sigma$, it follows that $D \in \mathcal{C}_\sigma$.

Also, $K \subset A \subset A_{n_0} \cup (\bigcup\limits_{k=n_0+1}^{\infty} D_k) = A_{n_0} \cup D \subset D_{n_0+1} \cup D = D$ and, since $D\backslash K = (D\backslash A) \cup (A\backslash A_{n_0}) \cup (A_{n_0}\backslash K)$, we get that

$$|\mu(D\backslash K)| \leq |\mu(D\backslash A)| + |\mu(A\backslash A_{n_0})| + |\mu(A_{n_0}\backslash K)| < \frac{2\varepsilon}{3} + |\mu(D\backslash A)|.$$

But,

$$D\backslash A = (\bigcup_{k=n_0+1}^{\infty} D_k)\backslash(\bigcup_{k=1}^{\infty} A_k) \subset (\bigcup_{k=n_0+1}^{\infty} D_k)\backslash(\bigcap_{k=n_0+1}^{\infty} A_k) = \bigcup_{k=n_0+1}^{\infty} (D_k\backslash A_k),$$

and, using the o-continuity, equivalently, the $h-\sigma$-subadditivity (see [5]) of μ on $\sigma(\mathcal{C})$, we obtain that

$$|\mu(D\backslash A)| \leq \sum_{k=n_0+1}^{\infty} |\mu(D_k\backslash A_k)| < \sum_{k=n_0+1}^{\infty} \frac{\varepsilon}{3^{k+1}} < \frac{\varepsilon}{3}.$$

Finally, $|\mu(D\backslash K)| < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$, as claimed. □

In the sequel, we establish a result which is related to theorem 1.13. More precisely, we shall prove that, under some supplementary hypothesis, if μ is $\mathcal{K}-\mathcal{D}$-regular, then the multisubmeasure μ^* from theorem 1.13 is also $\mathcal{K}-\mathcal{D}$-regular.

Theorem 2.8. *Let $\mathcal{C}$ be a ring of sets of T, X be a Banach space, $\mu : \mathcal{C} \mapsto \mathcal{P}_f(X)$ a multisubmeasure and $\mathcal{K} \subset \mathcal{C}$, $\mathcal{D} \subset \mathcal{C}$ be two subfamilies of $\mathcal{C}$ such that $\mathcal{K}$ is closed with respect to countable intersections and $\mathcal{D}$ is closed with respect to countable unions. If μ is $\mathcal{K}-\mathcal{D}$-regular and increasing convergent, then the multisubmeasure μ^*: $C_\sigma \mapsto P_f(X)$ defined by:*

$$\mu^*(A) = \lim_{n\to\infty} \mu(A_n) \text{ (with respect to } h\text{), for every } A \in C_\sigma,$$
$$\text{where } (A_n) \subset \mathcal{C}, A_n \nearrow A$$

is also $\mathcal{K}-\mathcal{D}$-regular and increasing convergent.

Proof. Let $\varepsilon > 0$. Since μ is $\mathcal{K}-\mathcal{D}$-regular, it is also $\mathcal{K}$-tight, so, taking into account its increasing convergence, from theorem 1.13 it uniquely extends to an increasing convergent and $\mathcal{K}$-tight multisubmeasure $\mu^* : \mathcal{C}_\sigma \mapsto \mathcal{P}_f(X)$.

Therefore, for every $A \in \mathcal{C}_\sigma$, there exists $K \in \mathcal{K}, K \subset A$ such that $|\mu^*(B)| < \frac{\varepsilon}{2}$, for every $B \in \mathcal{C}_\sigma$, with $B \subset A\backslash K$.

Thus, in order to prove that μ^* is $\mathcal{K}-\mathcal{D}$-regular, it is sufficient to establish the existence of a set $D \in \mathcal{D}$, with $A \subset D$ such that $|\mu^*(B)| < \frac{\varepsilon}{2}$, for every $B \in \mathcal{C}_\sigma$, $B \subset D\backslash A$ (then, $|\mu^*(B)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, for every $B \in \mathcal{C}_\sigma$, with $B \subset D\backslash K = (D\backslash A) \cup (A\backslash K)$).

Because $A \in \mathcal{C}_\sigma$, there exists an increasing sequence of sets $(A_n)_n \subset \mathcal{C}$, with $A = \bigcup\limits_{n=1}^{\infty} A_n$, so that $\mu^*(A) = \lim\limits_{n\to\infty} \mu(A_n)$.

Since $(A_n)_n \subset \mathcal{C}$ and μ is $\mathcal{K}-\mathcal{D}$-regular on $\mathcal{C}$, for every $n \in N$ there exists $D_n \in \mathcal{C}$, $A_n \subset D_n$, such that $|\mu(B)| < \frac{\varepsilon}{2^{n+2}}$, for every $B \in \mathcal{C}, B \subset D_n\backslash A_n$. Consequently, $|\mu(D_n\backslash A_n)| < \frac{\varepsilon}{2^{n+1}}$, for every $n \in N$.

Let $D = \bigcup_{n=1}^{\infty} D_n$. Then $D \in \mathcal{D}$, $A \subset D$ and if $B \in \mathcal{C}_\sigma, B \subset D \backslash A = (\bigcup_{n=1}^{\infty} D_n) \backslash (\bigcup_{n=1}^{\infty} A_n) \subseteq \bigcup_{n=1}^{\infty} (D_n \backslash A_n)$, taking into account that any increasing convergent multisubmeasure on $\mathcal{C}_\sigma$ is $h-\sigma$-subadditive (see [5]), we get that

$$|\mu^*(B)| \leq \sum_{n=1}^{\infty} |\mu^*(D_n \backslash A_n)| = \sum_{n=1}^{\infty} |\mu(D_n \backslash A_n)| < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}} < \frac{\varepsilon}{2},$$

as claimed. □

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