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AN EXAMPLE OF A RIEMANNIAN MANIFOLD SATISFYING THE CONDITION $R(X,\ Y)\cdot C = 0$ BUT div $C\neq 0$

BY

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Abstract. The object of the present paper is to study a Riemannian manifold satisfying the condition $R(X,Y)\cdot C = 0$ but div$C\neq 0$.

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1. Introduction. Let $(M^n,\ g)$ be an n-dimensional $(n > 3)$ Riemannian manifold with metric tensor g. A linear connection $\tilde{\nabla}$ on $(M^n,\ g)$ is said to be semi-symmetric [1] if its torsion tensor T satisfies the relation

$$T(X,Y) = \pi(Y)X - \pi(X)Y, \tag{1}$$

where π is a 1-form and $X,\ Y$ are any vector fields on $(M^n,\ g)$. If, in addition to (1) $\tilde{\nabla}$ satisfies the condition $\tilde{\nabla}g = 0$, then $\tilde{\nabla}$ is said to be a semi-symmetric metric connection. This paper deals with a type of semi-symmetric metric connection $\tilde{\nabla}$ whose curvature tensor k satisfies the relation

$$(\nabla_W k)(X,Y)Z = B(W)k(X,Y)Z, \tag{2}$$

where B is a non-zero 1-form and ∇ denotes the Levi-Civita connection. Denoting the curvature tensor of ∇ by R and the conformal curvature tensor of $(M^n,\ g)$ by C it is shown that if $(M^n,\ g)$ admits a semi-symmetric metric connection $\tilde{\nabla}$ for which the condition (2) is satisfied, then in (M^n, g) $R(X,Y).C = 0$ but div $C\neq 0$, where $R(X,Y)$ is considered as a derivation

of the tensor algebra at each point of $(M^n,\ g)$ for tangent vectors and divergence of C is with respect to ∇. In this connection it may be mentioned that Takagi [2] gave an example of a Riemannian manifold satisfying $R(X,Y)\cdot R = 0$ but not $\nabla R = 0$.

2. A useful result. In this section we shall derive a useful result. The conformal curvature tensor C of (M^n, g) is defined by

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z + \lambda(Y,Z)X - \lambda(X,Z)Y \\ &\quad + g(Y,Z)LX - g(X,Z)LY \end{aligned} \tag{2.1}$$

where

$$\lambda(Y,Z) = -\frac{1}{n-2}S(Y,Z) + \frac{r}{2(n-1)(n-2)}g(Y,Z) \tag{2.2}$$

and L is a (1-1) tensor field such that

$$g(LY,Z) = \lambda(Y,Z), \tag{2.3}$$

S and r being the Ricci tensor and the scalar curvature respectively of (M^n, g).

Since $\nabla g = 0$ we have $(\nabla_X g)(LY,Z) = 0$ from which we get in virtue of (2.3)

$$(\nabla_X \lambda)(Y,Z) = g[(\nabla_X L)(Y), Z)]. \tag{2.4}$$

Again, from (2.1) we have

$$\begin{aligned} (\nabla_W C)(X,Y)Z &= (\nabla_W R)(X,Y)Z + [(\nabla_W \lambda)(Y,Z)]X - [(\nabla_W \lambda)(X,Z)]Y \\ &\quad + g(Y,Z)(\nabla_W L)(X) - g(X,Z)(\nabla_W L)(Y). \end{aligned} \tag{2.5}$$

In virtue of (2.2) it follows from (2.5) that

$$(\text{div } C)(X,Y)Z = (n-3)[(\nabla_Y \lambda)(X,Z) - (\nabla_X \lambda)(Y,Z)]. \tag{2.6}$$

Using (2.5) we have

$$
\begin{aligned}
&(\nabla_W C)(X,Y)Z + (\nabla_X C)(Y,W)Z + (\nabla_Y C)(W,X)Z \\
&\quad = [(\nabla_W \lambda)(Y,Z) - (\nabla_Y \lambda)(W,Z)]X \\
&\quad +[(\nabla_X \lambda)(W,Z) - (\nabla_W \lambda)(X,Z)]Y \\
&\quad +[(\nabla_Y \lambda)(X,Z) - (\nabla_X \lambda)(Y,Z)]W \\
&\quad +g(Y,Z)[(\nabla_W L)(X) - (\nabla_X L)(W)] \\
&\quad +g(X,Z)[(\nabla_W L)(Y) - (\nabla_Y L)(X)] \\
&\quad +g(W,Z)[(\nabla_X L)(Y) - (\nabla_Y L)(X)] \\
&\quad = \frac{1}{n-3}[\{(\text{div } C)(Y,W)Z\}X \\
&\quad +\{(\text{div } C)(W,X)Z\}Y + \{(\text{div } C)(X,Y)Z\}W] \\
&\quad +g(Y,Z)[(\nabla_W L)(X) - (\nabla_X L)(W)] \\
&\quad +g(X,Z)[(\nabla_W L)(Y) - (\nabla_Y L)(W)] \\
&\quad +g(W,Z)[(\nabla_X L)(Y) - (\nabla_Y L)(X)] \quad \text{by (2.6)}
\end{aligned} \tag{2.7}
$$

Hence in virtue of (2.4), (2.6) and (2.7) we get

$$
\begin{aligned}
&g((\nabla_W C)(X,Y)Z + (\nabla_X C)(Y,W)Z + (\nabla_Y C)(W,X)Z, U) \\
&\quad = \frac{1}{n-3}[g(X,U)(\text{div } C)(Y,W)Z + g(Y,U)(\text{div } C)(W,X)Z \\
&\quad +g(W,U)(\text{div } C)(X,Y)Z + g(Y,Z)(\text{div } C)(X,W)U \\
&\quad +g(X,Z)(\text{div } C)(Y,W)U + g(W,Z)(\text{div } C)(Y,X)U]
\end{aligned} \tag{2.8}
$$

We shall use the result (2.7) in section 3.

3. A special type of semi-symmetric metric connection. In this section we consider a semi-symmetric metric connection on (M,g) whose curvature tensor k satisfies the condition

$$(\nabla_W k)(X,Y)Z = B(W)k(X,Y)Z, \tag{3.1}$$

It is known [3] that if a Riemannian manifold (M^n, g) $(n > 3)$ admits a semi-symmetric connection whose curvature tensor is a recurrent tensor with respect to the Levi-Civita connection having B as a 1-form of recurrence.

Hence

$$(\nabla_W C)(X,Y)Z = B(W)C(X,Y)Z. \tag{3.2}$$

We now define a function f on (M,g) by

$$f^2 = g(C,C) \tag{3.3}$$

where the Riemannian metric g is extended to the inner product between the tensor fields in the standard fashion [4], [5].

Using the fact that $\nabla_Y g = 0$ it follows from (3.3) that

$$2f(Yf) = 2f^2 B(Y)$$

$$\text{or,} \quad Yf = fB(Y) \quad (\text{because } f \neq 0.) \tag{3.4}$$

From (3.4) we get

$$X(Yf) = \frac{1}{f}(Xf)(Yf) + X(B(Y))f.$$

Hence

$$X(Yf) - Y(Xf) = \{X(B(Y)) - Y(B(X))\}f.$$

Therefore

$$\begin{aligned}(\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})f &= \{X(B(Y)) - Y(B(X)) - B([X,Y])\}f \\ &= \{dB(X,Y)\}f.\end{aligned} \tag{3.5}$$

Since the left hand side of (3.5) is zero and $f \neq 0$, it follows from (3.5) that

$$dB(X,Y) = 0 \tag{3.6}$$

This means that the 1-form B is closed.

Now from (3.2) we

$$(\nabla_X\nabla_Y C)(U,V)W = \{XB(Y) + B(X)B(Y)\}C(U,V)W.$$

Hence

$$(R(X,Y)\cdot C)(U,V)W = \{(dB)(X,Y)\}C(U,V)W = 0, \text{by (3.6)}$$

Therefore

$$R(X,Y)\cdot C = 0. \tag{3.7}$$

Next we show that div $C\neq 0$.
If possible, let

$$div\ C = 0 \tag{3.8}$$

Then from (2.6) it follows that

$$g[(\nabla_W C)(X,Y)Z,U] + g[(\nabla_X C)(Y,W)Z,U] + g[(\nabla_Y C)(W,X)Z,U] = 0$$

or,
(3.9)

$$B(W)\,'C(X,Y,Z,U)+B(X)\,'C(Y,W,Z,U)+B(Y)\,'C(W,X,Z,U)=0$$

where $'\,C(X,Y,Z,U) = g(C(X,Y)Z,U)$.
Let σ be the vector field such that

$$g(X,\sigma) = B(X) \quad \text{for any vector field } X. \tag{3.10}$$

Putting $W = \sigma$ in (3.9) we get

$$B(\sigma)\,'C(X,Y,Z,U)+B(X)\,'C(Y,\sigma,Z,U)+B(Y)\,'C(\sigma,X,Z,U)=0. \tag{3.11}$$

In vertue of (3.8) and (3.2) we get

$$B(C(X,Y)Z) = 0 \quad \text{or} \quad 'C(X,Y,Z,\sigma) = 0. \tag{3.12}$$

Hence, it follows from (3.11) that

$$B(\sigma)\,'C(X,Y,Z,U) = 0. \tag{3.13}$$

Since $C\neq 0$, from (3.13) we get $B(\sigma) = 0$,
or $g(\sigma,\sigma) = 0$. This implies that $\sigma = 0$, because g is positive definite.

Hence from (3.10) we have $B(X) = 0$. But this is contrary to the assumption that B is a non-zero 1-form.
Therefore div $C\neq 0$.

We can therefore state the following

Theorem 1. *If a non-conformally flat n-dimensional Riemannian manifold* $(M^n,\ g)$, $(n > 3)$ *admits a semi-symmetric metric connection whose*

curvature tensor is recurrent with respect to the Levi-Civita connection ∇*, then in* $(M^n,\ g)$ *we have*

$$R(X,Y)\cdot C = 0, \quad \text{but} \quad \text{div } C \neq 0.$$

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