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A NAGUMO TYPE VIABILITY THEOREM*†

BY

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Abstract. In this paper we prove a viability result for a class of nonlinear differential equations of the type

$$u'(t) = G(t, u(t)),$$

where X is a real Banach space, I is a nonempty and open interval, K is a nonempty, locally closed subset in X and $G : I \times K \to X$ is of the form $G(t, u) = F(t, u, u)$ with F continuous, $u \mapsto F(t, u, v)$ compact and $v \mapsto F(t, u, v)$ Lipschitz. Two applications to some nonlinear partial differential equations are included.

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1. Introduction. The goal of this paper is to prove a viability result referring to a class of nonlinear differential equations of the type

$$u'(t) = G(t, u(t)), \tag{1.1}$$

where X is a real Banach space, I is a nonempty and open interval, K is a nonempty, locally closed subset in X and $G : I \times K \to X$ is a *Lipschitz quasi-compact function*. See Definition 1.1 below. Although essentially an "ordinary differential" result, our main result, i.e. Theorem 2.1, proves useful in obtaining the existence of positive solutions to some classes of partial differential equations as well.

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We begin by recalling that a subset K in X is *locally closed* if for each $\xi \in K$ there exists $\rho > 0$ such that $D(\xi, \rho) \cap K$ is closed[‡]. Obviously, each subset in X which is either open or closed is locally closed. Moreover, each subset K in X which is *closed relative to some open subset* D, i.e., for which there exists a closed subset $C \subseteq X$ such that $K = C \cap D$, is locally closed in X (and conversely, whenever X is finite dimensional).

We also recall that K is *viable* with respect to G if for each $(\tau, \xi) \in I \times K$ there exists $T > \tau$, such that the equation (1.1) has at least one solution $u : [\,\tau, T\,] \to K$ satisfying $u(\tau) = \xi$.

Definition 1.1. *A function $G : I \times K \to X$ is Lipschitz quasi-compact if there exists a function $F : I \times K \times K \to X$ such that $G(t, u) = F(t, u, u)$ for each $(t, u) \in I \times K$, with F continuous on $I \times K \times K$ and, in addition, for each $\tau \in I$ and $\xi \in K$, there exist $r > 0$ and $T > \tau$ with $[\,\tau, T\,] \subset I$ and $D(\xi, r) \cap K$ closed and such that:*

(H_1) *each sequence $(u_n)_n$, from $D(\xi, r) \cap K$, has at least one subsequence $(u_{n_k})_k$, such that $(F(t, u_{n_k}, v))_k$ is fundamental, uniformly for $(t, v) \in [\,\tau, T\,] \times (D(\xi, r) \cap K)$, i.e.: for each $\varepsilon > 0$ there exists $k(\varepsilon) > 0$ such that, for each $k, p \geq k(\varepsilon)$, each $(t, v) \in [\,\tau, T\,] \times (D(\xi, r) \cap K)$, we have*

$$\|F(t, u_{n_k}, v) - F(t, u_{n_p}, v)\| \leq \varepsilon.$$

(H_2) *there exists $L > 0$, such that*

$$\|F(t, u, v) - F(t, u, w)\| \leq L\|v - w\|$$

for each $t \in [\,\tau, T\,]$ and each $u, v, w \in D(\xi, r) \cap K$.

(H_3) *the family $\{u \mapsto F(t, u, v)\,;\ (t, v) \in [\,\tau, T\,] \times (D(\xi, r) \cap K)\}$ is uniformly equicontinuous on $D(\xi, r) \cap K$, i.e.: given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that*

$$\|F(t, u, v) - F(t, \widetilde{u}, v)\| \leq \varepsilon$$

for each $t \in [\,\tau, T\,]$ and $u, \widetilde{u}, v \in D(\xi, r) \cap K$ with $\|u - \widetilde{u}\| \leq \delta(\varepsilon)$.

Remark 1.1. The conditions (H_1) (H_2) and (H_3) are satisfied by all functions $G(t, u) = F(t, u, u)$, $F : I \times K \times K \to X$, whenever:

[‡] As usual, $D(\xi, \rho)$ denotes the closed ball with center ξ and radius ρ.

(i) $F(t,u,v) = f(t,u) + g(t,v)$, with $f : I \times K \to X$ compact and locally uniformly continuous on K and $g : I \times K \to X$ locally Lipschitz;

(ii) $F(t,u,v) = f(t,u)g(t,v)$, where $f : I \times K \to R$ is locally uniformly continuous on K, and $g : I \times K \to X$ is locally Lipschitz.

(iii) In particular, if K is locally compact (which happens for instance whenever X is finite dimensional and K is locally closed) and G is continuous, then G is Lipschitz quasi-compact.

Moreover, the set of functions satisfying (H_1) (H_2) and (H_3) is a vector space over R.

Remark 1.2. If X is reflexive and $f : I \times D \to X$ is weakly-strongly sequentially continuous, then it is uniformly continuous on bounded subsets in $I \times D$. Indeed, since $[\,\tau, T\,] \times D(\xi, r)$ is a weakly compact set in $R \times X$, the conclusion follows from the well-known Weierstrass Theorem.

The first necessary and sufficient condition for viability goes back to Nagumo [8] who analyzed the case when K is a closed, or merely locally closed subset in R^n and G is continuous. See Remark 1.1 above. Namely, Nagumo [8] proved

Theorem 1.1. *Let X be finite dimensional and let $G : I \times K \to X$ be continuous. Then, a locally closed subset K in X is viable with respect to G if and only, if for each $(\tau, \xi) \in I \times K$, we have*

$$\liminf_{s \downarrow 0} \frac{1}{s} d(\xi + sG(\tau, \xi); K) = 0^{\S}.$$

It is interesting to notice that Nagumo's result (or variants of it) has been rediscovered independently, in the late sixties and early seventies, by Yorke [14], [15], Bony [1], Crandall [4], Martin Jr. [7], Redheffer [9] and Hartman [6], among others. Yorke [14] considered subsets K in R^n which are closed relative to an open subset D in R^n. But, as we already pointed out, each locally closed subset is closed relative to some open subset and conversely, all his results hold true for locally closed sets as well. Crandall [4] considers the case $K \subseteq D$ is locally closed and

§Here and thereafter, $d(\eta; K)$ denotes the distance between the point $\eta \in X$ and the subset K in X.

$G : I \times D \to R^n$ is continuous. He shows that a sufficient condition for K to be viable (*forward invariant* in his terminology) with respect to G is (2.1). HARTMAN [6] proves essentially the same result for D open, K closed relative to D and $G : I \times D \to R^n$ continuous, and shows, in addition, that (2.1) is necessary for the viability of K with respect to G. BREZIS [2] analyzes the case when D is open in an arbitrary Banach space X, $K \subseteq D$ is relatively closed, G locally Lipschitz, and proves that (2.1) with "lim" instead of "lim inf" is necessary and sufficient for K to be local invariant, "*flow invariant*" in his terminology, with respect to G. For more details on this subject, we refer the reader to CÂRJĂ-VRABIE [3].

2. The main result. Our Theorem 2.1 below extends the main results in NAGUMO [8], YORKE [14], CRANDALL [4] and HARTMAN [6], as well as the existence, i.e. viability, part in BREZIS [2]. It extends also the viability theorem in VRABIE [13].

Theorem 2.1. *Let $G : I \times K \to X$ be a Lipschitz quasi-compact function. Then K is viable with respect to G if and only, if for each $(\tau, \xi) \in I \times K$, we have*

$$\liminf_{s \downarrow 0} \frac{1}{s} d(\xi + sG(\tau, \xi); K) = 0. \tag{2.1}$$

Now, a few words concerning the existence of global solutions. A function $G : I \times K \to X$ is called *positively sublinear* if there exist a norm $\|\cdot\|$ on X, equivalent with the initial one, $a, b \in L^1(R)$, and $c > 0$ such that

$$\|G(t, \xi)\| \le a(t)\|\xi\| + b(t)$$

for each $(t, \xi) \in K_+^c(G)$, where

$$K_+^c(G) = \{(t, \xi) \in I \times K;\ \|\xi\| > c \ \text{ and } [\,\xi, G(t, \xi)\,]_+ > 0\}\,.$$

We note that here $[\,\xi, \eta\,]_+$ is the right directional derivative of the norm $\|\cdot\|$ calculated at ξ in the direction η, i.e.

$$[\,\xi, \eta\,]_+ = \lim_{s \downarrow 0} \frac{1}{s}(\|\xi + s\eta\| - \|\xi\|).$$

See Definition 3.2.5, p. 95 in Vrabie [10]. Using similar arguments as those in the proof of Theorem 6.5.2 in CÂRJA-VRABIE [3], we can prove the global existence Theorem 2.2 below. It is interesting to notice that in this theorem there is no need to assume that G is Lipschitz quasi-compact.

Theorem 2.2. (Global Existence) *Let K be a closed set in X and let $G : I \times K \to X$ be a given function such that K is viable with respect to G and the latter is positively sublinear. Then, each solution of* (1.1)*, starting from $\tau \in R$, can be continued up to a global one, i.e. defined on $[\,\tau, \sup I)$.*

3. Proof of the main result. *Necessity.* This follows from the simple result below, in which neither K is assumed to be locally closed, nor G to be continuous and, in addition, "lim inf" is replaced by "lim".

Theorem 3.1. *If K is viable with respect to $G : I \times K \to X$, then, for each $(\tau, \xi) \in I \times K$, we have*

$$\lim_{s \downarrow 0} \frac{1}{s} d\left(\xi + sG(\tau, \xi); K\right) = 0.$$

For the finite-dimensional case see Theorem 3.2.1 in Cârjă-Vrabie [3], but the proof of the general case is exactly the same.
Sufficiency. The first step is concerned with the existence of approximate solutions. Let $(\tau, \xi) \in I \times K$ be arbitrary and let us choose $\rho > 0$, $M > 0$ and $T \in I$, $T > \tau$, such that $D(\xi, \rho) \cap K$ be closed,

$$\|F(t, x, y)\| \leq M \tag{3.1}$$

for every $t \in [\,\tau, T\,]$ and $x, y \in D(\xi, \rho) \cap K$, and

$$(T - \tau)(M + 1) \leq \rho. \tag{3.2}$$

The existence of these three numbers is ensured by the fact that K is locally closed (from where the existence of $\rho > 0$), by the continuity of F which implies its boundedness on $[\,\tau, T\,] \times (D(\xi, \rho) \cap K)$, provided $\rho > 0$ is small enough. So, we deduce the existence of $M > 0$ satisfying (3.1), and of $T \in I$, $T > \tau$, sufficiently close to τ, in order to have (3.2). Diminishing $T > \tau$ and/or $\rho > 0$, if necessary, we may assume that F in Definition 1.1 satisfies (H_2) on $[\,\tau, T\,] \times (D(\xi, \rho) \cap K)$.

Lemma 3.1. *For each $\varepsilon \in (0, 1)$ and $\rho > 0$, $M > 0$, $T \in I$, as above, there exist three functions $\sigma : [\,\tau, T\,] \to [\,\tau, T\,]$ nondecreasing, $g : [\,\tau, T\,] \to X$ Riemann integrable and $u : [\,\tau, T\,] \to X$ continuous, such that:*

(i) $t - \varepsilon \leq \sigma(t) \leq t$ *for each* $t \in [\,\tau, T\,]$;

(ii) $\|g(t)\| \leq \varepsilon$ *for each* $t \in [\,\tau, T\,]$;

(iii) $u(\sigma(t)) \in D(\xi,\rho) \cap K$ *for each* $t \in [\,\tau, T\,]$ *and* $u(T) \in D(\xi,\rho) \cap K$;

(iv) u *satisfies*

$$u(t) = \xi + \int_\tau^t G(s, u(\sigma(s)))\, ds + \int_\tau^t g(s)\, ds$$

for each $t \in [\,\tau, T\,]$.

The proof of Lemma 3.1 follows exactly the same lines as those in the proof of sufficiency of both Theorem 3.1.1 in Cârjă-Vrabie [3], or of Theorem 7.7.2, p. 289 in Vrabie [12], of course, with the help of the simple Proposition 3.1 below.

Proposition 3.1. *Assume that, for some* $\xi \in K$, $t \mapsto G(t,\xi)$ *is continuous from the right at* $\tau \in I$. *Then* G *satisfies* (2.1) *at* (τ,ξ) *if and only if*

$$\liminf_{s \downarrow 0} \frac{1}{s}\, d\left(\xi + \int_\tau^{\tau+s} G(t,\xi)\, dt; K\right) = 0.$$

For simplicity, in all that follows, we will say that a triple (σ, g, u), satisfying (i), (ii), (iii) and (iv) in Lemma 3.1, is an ε-*approximate solution* to the Cauchy problem

$$\begin{cases} u'(t) = G(t, u(t)) \\ u(\tau) = \xi \end{cases} \tag{3.3}$$

on the interval $[\,\tau, T\,]$.

In the second step we will prove the convergence of a suitably chosen sequence of approximate solutions. First, we state a variant of the well-known Gronwall's Lemma whose proof follows exactly the same lines as those of Lemma 10.2.1, p. 232 in Vrabie [11].

Lemma 3.2. (Gronwall) *Let* $x, k : [\,\tau, T) \to R_+$ *be measurable and* $m \geq 0$. *Let us assume that* $s \mapsto k(s)x(s)$ *is locally integrable on* $[\,\tau, T)$ *and*

$$x(t) \leq m + \int_\tau^t k(s)x(s)\, ds$$

for every $t \in [\,\tau, T)$. *Then*

$$x(t) \leq m e^{\int_\tau^t k(s)\, ds}$$

for every $t \in [\,\tau, T)$.

So, let us consider a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $(0,1)$, decreasing to 0, and let $((\sigma_n, g_n, u_n))_{n\in\mathbb{N}}$ be a sequence of ε_n-approximate solutions of (3.3) defined on $[\,\tau, T\,]$. We will show first that $(u_n(\sigma_n))_{n\in\mathbb{N}}$ has at least one Cauchy subsequence in the sup norm. For each $n, m \in N$ and $s \in [\,\tau, T\,]$, we denote by

$$f_{n,m}(s) = F(s, u_n(\sigma_n(s)), u_m(\sigma_m(s))).$$

We have

$$\|u_n(\sigma_n(t)) - u_m(\sigma_m(t))\| \leq \left\| \int_\tau^{\sigma_n(t)} f_{n,n}(s)ds - \int_\tau^{\sigma_m(t)} f_{m,m}(s)\,ds \right\|$$

$$+ \int_\tau^{\sigma_n(t)} \|g_n(s)\|\,ds + \int_\tau^{\sigma_m(t)} \|g_m(s)\|\,ds$$

$$\leq \int_{\sigma_n(t)}^{t} \|f_{n,n}(s)\|\,ds + \int_{\sigma_m(t)}^{t} \|f_{m,m}(s)\|\,ds$$

$$+ \int_\tau^T \|f_{n,n}(s) - f_{m,m}(s)\|\,ds + \int_\tau^{\sigma_n(t)} \|g_n(s)\|\,ds + \int_\tau^{\sigma_m(t)} \|g_m(s)\|\,ds.$$

Since by (i) and (ii) in Lemma 3.1, we have both $t - \sigma_n(t) \leq \varepsilon_n$ and $\|g_n(s)\| \leq \varepsilon_n$ while, in view of (H_2), we have

$$\|f_{n,n}(s) - f_{n,m}(s)\| \leq L\|u_n(\sigma_n(s)) - u_m(\sigma_m(s))\|$$

for each $n, m \in N$, from the inequality above and (3.1), we deduce

$$\tag{3.4} \begin{gathered} \|u_n(\sigma_n(t)) - u_m(\sigma_m(t))\| \leq (T - \tau + M)(\varepsilon_n + \varepsilon_m) \\ +L\int_\tau^t \|u_n(\sigma_n(s)) - u_m(\sigma_m(s))\|\,ds + \int_\tau^T \|f_{n,m}(s) - f_{m,m}(s)\|\,ds. \end{gathered}$$

In view of (3.4) and Gronwall's Lemma 3.2, we have

$$\tag{3.5} \begin{gathered} \|u_n(\sigma_n(t)) - u_m(\sigma_m(t))\| \\ \leq \left[(T-\tau+M)(\varepsilon_n+\varepsilon_m) + \int_\tau^T \|f_{n,m}(s) - f_{m,m}(s)\|\,ds\right] e^{L(T-\tau)} \end{gathered}$$

We shall complete the proof with the following compactness lemma.

Lemma 3.3. *In the hypotheses of* Theorem 2.1, *for each sequence of approximate solutions of* (3.3), $(\sigma_n, g_n, u_n)_n$, *there exists one subsequence of* $(u_n)_n$, *denoted again by* $(u_n)_n$, *such that, for each* $\varepsilon > 0$ *there exists* $k(\varepsilon) \in N$ *such that, for each* $k, p \in N$, $k, p \geq k(\varepsilon)$ *we have*

$$\int_\tau^T \|F(s, u_k(\sigma_k(s)), v(s)) - F(s, u_p(\sigma_p(s)), v(s))\| \, ds \leq \varepsilon,$$

uniformly for all mappings $v : [\tau, T] \to D(\xi, r) \cap K$ *rendering the function* $s \mapsto F(s, u_n(\sigma_n(s)), v(s))$ *integrable for* $n \in N$.

Proof. Let $\varepsilon > 0$, let $\delta(\varepsilon) > 0$ be given by (H_3) and let P be a countable, dense subset in $[\tau, T]$ with $\tau, T \in P$. Using (H_1) and the Cantor's Diagonal Procedure, we find a subsequence of $(u_n(\sigma_n))_n$, denoted for simplicity again by $(u_n(\sigma_n))_n$, such that, for each $\theta \in P$, $(F(s, u_n(\sigma_n(\theta)), v))_n$ is fundamental, for each $s \in [\tau, T]$ and uniformly with respect to $v \in D(\xi, r) \cap K$. Let $\Delta : \tau = t_1 < t_2 < \cdots < t_{m(\varepsilon)} = T$ be a partition of $[\tau, T]$, with $t_i \in P$ for $i = 1, 2, \ldots m(\varepsilon) - 1$ and

$$t_{i+1} - t_i \leq \frac{\delta(\varepsilon)}{2(M+1)},$$

where $\delta(\varepsilon)$ is given by (H_3). So, for the very same $\varepsilon > 0$, there exists $k(\varepsilon) \in N$ such that, for each $n, m \in N$, $n, m \geq k(\varepsilon)$, each $j = 1, 2, \ldots m(\varepsilon)$ and each $(s, v) \in [\tau, T] \times (D(\xi, r) \cap K)$, we have

$$\|F(s, u_n(\sigma_n(t_j)), v) - F(s, u_m(\sigma_m(t_j)), v)\| \leq \varepsilon.$$

Taking a greater $k(\varepsilon)$ if necessary, we may assume that for each $n \in N$, $n \geq k(\varepsilon)$, we have

$$\varepsilon_n \leq \frac{\delta(\varepsilon)}{4(M+1)}.$$

Accordingly, in view of (i) and (iii) in Lemma 3.1 and (3.1), we deduce that for each $n \in N$, $n \geq k(\varepsilon)$,

$$\|u_n(\sigma_n(s)) - u_n(\sigma_n(t_i))\| \leq (M+1)|\sigma_n(s) - \sigma_n(t_i)|$$

$$\leq (M+1)(|\sigma(s) - s| + |s - t_i| + |t_i - \sigma(t_i)|) \leq \delta(\varepsilon)$$

for each $i = 1, \dots m(\varepsilon) - 1$ and each $s \in [\, t_i, t_{i+1}\,]$. Accordingly, for each $k, p \in N$, $k, p \geq k(m, \varepsilon) = k(\varepsilon)$ and each $v : [\,\tau, T\,] \to D(\xi, r) \cap K$ rendering the function $s \mapsto F(s, u_n(\sigma_n(s)), v(s))$ integrable for each $n \in N$, we have

$$\int_\tau^T \|F(s, u_k(\sigma_k(s)), v(s)) - F(s, u_p(\sigma_p(s)), v(s))\| \, ds$$

$$\leq \sum_{i=1}^{m(\varepsilon)-1} \int_{t_i}^{t_{i+1}} \|F(s, u_k(\sigma_k(s)), v(s)) - F(s, u_k(\sigma_k(t_i)), v(s))\| \, ds$$

$$+ \sum_{i=1}^{m(\varepsilon)-1} \int_{t_i}^{t_{i+1}} \|F(s, u_k(\sigma_k(t_i)), v(s)) - F(s, u_p(\sigma_p(t_i)), v(s))\| \, ds$$

$$+ \sum_{i=1}^{m(\varepsilon)-1} \int_{t_i}^{t_{i+1}} \|F(s, u_p(\sigma_p(t_i)), v(s)) - F(s, u_p(\sigma_p(s)), v(s))\| \, ds$$

$$\leq (T - \tau)\varepsilon + (T - \tau)\varepsilon + (T - \tau)\varepsilon = 3(T - \tau)\varepsilon.$$

Consequently, the Cauchy condition holds true uniformly for each function $v : [\,\tau, T\,] \to D(\xi, r) \cap K$ with $s \mapsto F(s, u_p(\sigma_p(s)), v(s))$ integrable for each $p \in N$. The proof is complete. $\square$

Proof. Continued. In view of Lemma 3.3 we may assume with no loss of generality that for each $\varepsilon > 0$ there exists $n_0(\varepsilon) \in N$ such that, for each $n, m \in N$, $n, m \geq n_0(\varepsilon)$, we have

$$\int_\tau^T \|f_{n,m}(s) - f_{m,m}(s)\| \, ds \leq \varepsilon.$$

Moreover, since $\lim_{n\to\infty} \varepsilon_n = 0$, for the same $\varepsilon > 0$, there exists $n_1(\varepsilon)$ such that, for each $n, m \in N$, $n, m \geq n_1(\varepsilon)$, we have

$$(T - \tau + M)(\varepsilon_n + \varepsilon_m) \leq \varepsilon.$$

So, if $n, m \in N$, $n, m \geq n(\varepsilon) = \max\{n_0(\varepsilon), n_1(\varepsilon)\}$, in view of (3.5), we conclude

$$\|u_n(\sigma_n(t)) - u_m(\sigma_m(t))\| \leq 2\varepsilon e^{L(T-\tau)},$$

which means that $(u_n(\sigma_n(\cdot)))_n$ is a Cauchy sequence with respect to the usual sup-norm on $C([\,\tau, T\,]; X)$. Therefore there exists

$$u(t) = \lim_{n\to\infty} u_n(\sigma_n(t))$$

uniformly for $t \in [\tau, T]$ which, in view of (i) in Lemma 3.1 and of the continuity of G, implies successively

$$\lim_{n\to\infty} u_n(s) = u(s)$$

and

$$\lim_{n\to\infty} G(s, u_n(\sigma_n(s))) = G(s, u(s))$$

uniformly for $s \in [\tau, T]$. So, we can pass to the limit for $n \to \infty$ in (iv) in Lemma 3.1 to obtain

$$u(t) = \xi + \int_\tau^t G(s, u(s))\, ds$$

and $u(t) \in D(\xi, \rho) \cap K$ for each $s \in [\tau, T]$. Thus u is a solution of (1.1) on $[\tau, T]$ satisfying $u(\tau) = \xi$ and this completes the proof. □

4. Application to a pseudoparabolic equation. Let Ω be a bounded domain in R^3, with smooth boundary Σ and let us consider the following semilinear pseudoparabolic partial differential equation

$$\begin{cases} u_t = \Delta u_t + f(u, \Delta u) & (t,x) \in Q_T \\ u = 0 & (t,x) \in \Sigma_T \\ u(0,x) = \eta(x) & x \in \Omega. \end{cases} \tag{4.1}$$

Here and thereafter $Q_T = (0,T) \times \Omega$ and $\Sigma_T = (0,T) \times \Sigma$. We notice that, for the specific case $f(u, \Delta u) = g(u) - \Delta u$, (4.1) describes the flow of a slow fluid within Ω. See for instance Galdi-Padula-Rajagopal [5].

Theorem 4.1. *Let $f : R\times R \to R$ be a Lipschitz quasi-compact function satisfying the growth condition*

$$|f(u,v)| \le a(u)|v| + b(u), \tag{4.2}$$

where $a, b : R \to R$ are two continuous functions. Assume also that

$$u + f(u,v) \ge 0 \tag{4.3}$$

for each $u \ge 0$ and each $v \le 0$. Then, for each $\eta \in H^2(\Omega) \cap H_0^1(\Omega)$ with $\eta - \Delta\eta \ge 0$ a.e. on Ω, there exists $T > 0$ such that the problem (4.1) *has at least one solution $u \in C^1([0,T]; H^2(\Omega) \cap H_0^1(\Omega))$ satisfying $u(t) \ge 0$ and $u(t) - \Delta u(t) \ge 0$ for each $t \in [0,T]$ and a.e for $x \in \Omega$. If, in addition f is positively sublinear, then each solution of* (4.1) *can be continued up to a global one.*

Proof. Clearly (4.1) is equivalent to:

$$\begin{cases} (u-\Delta u)_t = f(u,\Delta u) & (t,x)\in Q_T \\ u=0 & (t,x)\in \Sigma_T \\ u(0,x)=\eta(x) & x\in\Omega, \end{cases} \tag{4.4}$$

Let $A : D(A) \subseteq L^2(\Omega) \to L^2(\Omega)$ be defined by

$$\begin{cases} D(A) = H_0^1(\Omega)\cap H^2(\Omega) \\ Au = \Delta u \text{ for each } u \in D(A), \end{cases}$$

let $J = (I-A)^{-1}$ and let us denote by $u = Jv$. Since $AJ = J - I$, (4.4) can be rewritten as an abstract differential equation in the space $L^2(\Omega)$, i.e.

$$\begin{cases} v' = G(v) \\ v(0) = \xi, \end{cases} \tag{4.5}$$

where $G(v) = F(v,v)$ and $F : L^2(\Omega)\times L^2(\Omega) \to L^2(\Omega)$ is defined by

$$F(v,w)(x) = f(Jv, Jw - w)(x))$$

for each $v, w \in L^2(\Omega)$ and a.e. for $x \in \Omega$ and $\xi = (I-A)\eta$.

Since the operator J is continuous from $L^2(\Omega)$ to $H^2(\Omega)\cap H_0^1(\Omega)$, and, in our specific case, i.e. $n = 3$, $H^2(\Omega)$ is compactly imbedded in $C(\overline{\Omega})$ (see (iii) in Theorem 1.5.4, p. 18 in Vrabie [11]), in view of the growth condition that f satisfies, it follows that F is well-defined and Lipschitz quasi-compact.

At this point, let us recall that $\xi = (I-A)\eta \geq 0$ a.e. on Ω. Let $K = \{v \in L^2(\Omega)\,;\ v \geq 0 \text{ a.e. on } \Omega\}$. In view of (4.3), we deduce that F and K satisfy the tangency condition (2.1). Indeed, to prove that

$$\liminf_{s\downarrow 0} \frac{1}{s} d(\xi + sG(\xi); K) = 0,$$

for each $\xi \in K$, it suffices to show that, for each $\xi \in R\times K$ and each $s \in (0,1)$, we have

$$\xi - sJ\xi + s[J\xi + f(J\xi, J\xi - \xi)] \geq 0 \tag{4.6}$$

a.e. on Ω. But this is certainly the case, because $\xi \geq 0$ a.e. on Ω, along with the maximum principle for elliptic equations, implies both $\xi - sJ\xi \geq 0$

for each $s \in (0,1)$ and $J\xi - \xi \leq 0$ a.e. on Ω. By virtue of (4.3), it follows that, for each $\xi \in K$, $J\xi + f(\xi, J\xi - \xi) \geq 0$ a.e. on Ω, and thus (4.6) holds. So, we are in the hypotheses of Theorem 2.1. Therefore K is viable with respect to G, and this implies that there exists at least one solution, $v : [0,T] \to L^2(\Omega)$, of (4.5), with $v(t) \in K$ for each $t \in [0,T]$. But $u(t) = Jv(t)$, and consequently $u(t) - \Delta u(t) \geq 0$ for each $t \in [0,T]$ and a.e. on Ω. Using again the maximum principle for elliptic equations, we conclude that $u(t) \geq 0$ for each $t \in [0,T]$ and a.e. on Ω. Since $v \in C^1([0,T]; L^2(\Omega))$, and $u = Jv$, with J linear continuous from $L^2(\Omega)$ to $H^2(\Omega) \cap H_0^1(\Omega)$, we conclude that $u \in C^1([0,T]; H^2(\Omega) \cap H_0^1(\Omega))$ and the proof is complete. □

5. Application to a nonlinear hyperbolic equation. Next we shall present another interesting application of Theorem 2.1. We begin with a corollary of Theorem 2.1 referring to the simplest case when K is open, corollary which, as far as we know, is new and interesting by itself.

Theorem 5.1. *Let D be a nonempty and open subset in X and let $G : I \times D \to X$ be a Lipschitz quasi-compact function. Then D is viable with respect to G.*

Proof. Since D is open, the tangency condition (2.1) is automatically satisfied, and the conclusion follows from Theorem 2.1. □

Let $A : D(A) \subset X \to X$ the generator of a C_0-group of isometries $\{S(t)\,;\ t \in R\}$, let $g : X \times X \to X$ be a continuous function of the form $g(v) = f(v,v)$, where $f : X \times X \to X$, and let us consider the problem

$$\begin{cases} u' = Au + g(u) \\ u(0) = \xi. \end{cases} \tag{5.1}$$

We begin with the following useful simple variant of Lemma 10.4.1, p. 242 in Vrabie [11].

Lemma 5.4. *A function $u : [\tau,T] \to X$ is a C^0-solution for the problem* (5.1) *if and only if the function $v : [\tau,T] \to X$, defined by*

$$v(t) = S(-t)u(t)$$

for each $t \in [\tau,T]$, is a C^1-solution of the problem

$$\begin{cases} v' = G(t,v) \\ v(0) = \xi, \end{cases} \tag{5.2}$$

where $G(t,v)=F(t,v,v)$*,* $F:R\times X\times X\to X$ *being defined by*

$$F(t,v,w)=S(-t)f(S(t)v,S(t)w)$$

for each $(t,v,w)\in R\times X\times X$*.*

Proof. First, let us recall that $u:[\,\tau,T\,]\to X$ is a C^0-solution of (5.1) if

$$u(t)=S(t)\xi+\int_a^t S(t-s)f(u(s),u(s))\,ds$$

for each $t\in[\,\tau,T\,]$. Consequently we get

$$S(-t)u(t)=\xi+\int_a^t S(-s)f(u(s),u(s))\,ds$$

for $t\in[\,\tau,T\,]$. Let $v:[\,\tau,T\,]\to X$ be defined by $v(t)=S(-t)u(t)$ for $t\in[\,\tau,T\,]$. Obviously it satisfies

$$v(t)=\xi+\int_a^t S(-s)f(S(s)v(s),S(s)v(s))\,ds$$

for each $t\in[\,\tau,T\,]$, and this achieves the proof. □

Theorem 5.2. *Let* $A:D(A)\subset X\to X$ *be the generator of a* C_0*-group of isometries,* $\{S(t)\,;\ t\in R\}$*, and let* $g:X\to X$ *be a Lipschitz quasi-compact function. Then, for each* $\xi\in X$*, there exists* $T>0$ *such that* (5.1) *has at least one* C^0*-saturated solution defined on* $[\,\tau,T\,]$*.*

Proof. As g is Lipschitz quasi-compact and the group generated by A is of isometries, we deduce that the function G, defined in Lemma 5.4, is Lipschitz quasi-compact too. Since, in our case $D=X$, which is open, we are in the hypotheses of Theorem 5.1. The proof is complete. □

We present next a simple application of Theorem 5.2 improving Theorem 10.4.2, p. 243, in Vrabie [11]. Let Ω be a nonempty, bounded and open subset in R^n whose boundary Γ is of class C^2, let $Q_T=(0,T)\times\Omega$ and $\Sigma_T=(0,T)\times\Gamma$. Let $\lambda>0$ be defined by

$$\lambda=\inf\left\{\|\nabla u\|^2_{L^2(\Omega)}\,;\ u\in H^1_0(\Omega),\ \|u\|_{L^2(\Omega)}=1\right\},$$

i.e. the first eigenvalue of $-\Delta$ on $H^{-1}(\Omega)$, and let $m > -\lambda$. We consider the nonlinear hyperbolic equation

$$\tag{5.3} \begin{cases} u_{tt} = \Delta u - mu + h(u, u_t) & (t, x) \in Q_T \\ u(t, x) = 0 & (t, x) \in \Sigma_T \\ u(0, x) = u_0(x) & x \in \Omega, \\ u_t(0, x) = v_0(x) & x \in \Omega \end{cases}$$

where $h : R \times R \to R$ is a given function. In the specific case when h does not depend on the "damping" term u_t, this equation is known as the *semilinear Klein-Gordon equation.*

Theorem 5.3. *Let $h : R \times R \to R$ be a continuous function for which there exist $c, d > 0$ and $\alpha \in R$, and such that*

$$\tag{5.4} |h(u, v)| \leq c(|u|^\alpha + |v|) + d$$

for $(u, v) \in R \times R$, where $\alpha \geq 0$ if $n = 2$, and $\alpha < n/(n-2)$ if $n \geq 3$. Assume further that the family $\{u \mapsto h(u, v);\ v \in R\}$ is uniformly equicontinuous on R and there exists $L > 0$ such that

$$\tag{5.5} |h(u, v) - h(u, w)| \leq L|v - w|$$

for each $(v, w) \in X \times X$. Then, for each $u_0 \in H_0^1(\Omega)$ and each $v_0 \in L^2(\Omega)$, there exists $T > 0$ such that the problem (5.3) has at least one saturated solution u satisfying

(i) $u \in C([\,0, T); H_0^1(\Omega))$

(ii) $u_t \in C([\,0, T); L^2(\Omega))$.

In addition, if $T < +\infty$, then

$$\lim_{s \uparrow T} \Big(\|u(s)\|_{H_0^1(\Omega)} + \|u_t(s)\|_{L^2(\Omega)} \Big) = +\infty.$$

If $n = 1$, the conclusion remains valid without condition (5.4) and assuming only that the family $\{u \mapsto h(u, v);\ v \in R\}$ is uniformly equicontinuous on bounded subsets in R.

Proof. First, we observe that (5.3) can be rewritten as a first-order ordinary differential equation in an infinite-dimensional Hilbert space. Let

$$H = H_0^1(\Omega) \times L^2(\Omega)$$

which, endowed with the inner product $\langle\cdot,\cdot\rangle$, defined by

$$\langle (u,v),(\tilde u,\tilde v)\rangle = \int_\Omega u'(x)\tilde u'(x)\,dx + m\int_\Omega u(x)\tilde u(x)\,dx + \int_\Omega v(x)\tilde v(x)\,dx$$

for each $(u,v),(\tilde u,\tilde v)\in H$, is a real Hilbert space. We define the operator $A : D(A)\subset H\to H$ by

$$\begin{cases} D(A) = (H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega) \\ A(u,v) = (v,\Delta u - mu) \ \text{ for each } (u,v)\in D(A). \end{cases}$$

Furthermore, let us define $f : H\times H\to H$ by

$$f((u,v),(\widetilde u,\widetilde v))(x) = (0, h(x,u(x),\widetilde v(x)))$$

for each $(u,v),(\widetilde u,\widetilde v)\in H\times H$ and a.e. for $x\in\Omega$. At this point, let us observe that the problem (5.3) can be rewritten under the equivalent form

$$\begin{cases} z' = Az + g(z) \\ z(a) = \xi, \end{cases}$$

where $g(z) = f(z,z)$, A and f are as above, $z(t)(x) = (u(t,x),v(t,x))$ a.e. for (t,x) in $(0,T)\times\Omega$ and $\xi = (u_0,v_0)$. In order to prove that A and f satisfy the general conditions in Theorem 5.4, let us remark first that, by virtue of Theorem 4.3.6, p. 94 in Vrabie [11], A generates a C_0-group of isometries.

As concerns f, from (5.4), Theorem 1.5.4, p. 17 and Lemma A.6.1, p. 313 in Vrabie [11], it follows that it is well-defined and continuous on $H\times H$. Again from Theorem 1.5.4 *loc.cit*, we know that $H_0^1(\Omega)$ is compactly imbedded in: $C(\overline\Omega)$ if $n=1$, in $L^q(\Omega)$ for each $q\ge 1$ if $n=2$, and $q<2n/(n-2)$ if $n\ge 3$. From this remark, the uniform equicontinuity assumption and (5.5), it follows that f is Lipschitz quasi-compact and thus we are in the hypotheses of Theorem 5.2. The proof is complete. □

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