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A VIABILITY RESULT FOR NONLINEAR FUNCTIONAL DIFFERENTIAL INCLUSIONS*

BY

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Abstract. By using an appropriate tangency condition, we establish some sufficient conditions in order that a given locally closed subset of a Banach space be a viable domain for a nonlinear functional differential inclusion.

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1. Introduction. For any given real Banach space X and any $\sigma > 0$, we shall denote by $\mathcal{C}_\sigma = C([-\sigma, 0]; X)$ the Banach space of all continuous functions from $[-\sigma, 0]$ into X and by $\mathcal{M}_\sigma$ the Banach space

$$\mathcal{M}_\sigma = \{\varphi : [-\sigma, 0] \to X, \varphi \text{ bounded and continuous at the right}\},$$

endowed with the norm $\|\varphi\|_\sigma = \sup\{\|\varphi(s)\|; s \in [-\sigma, 0]\}$. For any function $u : [\tau - \sigma, T] \to X$ and any $t \in [\tau, T]$ we shall denote by u_t the function $u_t : [-\sigma, 0] \to X$ defined by $u_t(s) = u(t + s)$, for every $s \in [-\sigma, 0]$. Clearly, if u is continuous then $u_t \in \mathcal{C}_\sigma$, for each $t \in [\tau, T]$.

Let K be a given locally closed subset in X and let $\mathcal{K}$ be the following subset of $\mathcal{M}_\sigma$:

$$\mathcal{K} = \{\varphi \in \mathcal{M}_\sigma; \varphi(s) \in K \text{ for every } s \in [-\sigma, 0]\}.$$

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We recall that a subset $K \subset X$ is *locally closed* if for each $\xi \in K$ there exists $r > 0$ such that $K \cap B(\xi, r)$ is closed in X, where, as usual, $B(\xi, r)$ denotes the closed ball with center ξ and radius r.

We consider the following nonlinear functional differential inclusion

$$v'(t) + Av(t) \ni f(t, v(t), v_t),\ t \in [a, b), \tag{1.1}$$

where $A : D(A) \subset X \to 2^X$ is an m-accretive operator with $K \subset \overline{D(A)}$ and $f : [a, b) \times K \times \mathcal{K} \to X$ is a continuous function, and we are interested in finding sufficient conditions in order that K be a *viable domain* for (1.1), i.e. for each $\tau \in [a, b)$ and $\varphi \in \mathcal{K} \cap \mathcal{C}_\sigma$ there exists at least one solution $v : [\tau - \sigma, T] \to K$ of (1.1) satisfying the initial condition

$$v_\tau = \varphi. \tag{1.2}$$

Definition 1.1. *A continuous function $v : [\tau - \sigma, T] \to X$ is a solution of the problem* (1.1) *and* (1.2) *if*

i) $v(t) = \varphi(t - \tau)$ *for* $t \in [\tau - \sigma, \tau]$;

ii) *the function $g : [\tau, T] \to X$, defined by $g(t) = f(t, v(t), v_t)$ for every $t \in [\tau, T]$, is integrable on $[\tau, T]$;*

iii) *v coincides on $[\tau, T]$ with the mild solution of the problem*

$$\begin{cases} u'(t) + Au(t) \ni g(t),\ t \in [\tau, T], \\ u(\tau) = \varphi(0), \end{cases}$$

in the sense of Definition 1.7.5, p. 25 in Vrabie [23].

The existence of solutions for functional differential equations governed or not by linear and nonlinear operators in Banach spaces has been studied in many papers and monographs (see, for example, [7], [11], [13], [19], [20], [21] and [24]).

As far as we know, the first viability results for the equation (1.1), in the case $A = 0$, were given by Lakshmikantham, Leela and Moauro [14] and by Leela and Moauro [15]. The case when $A = 0$, X is a finite dimensional space and f is an upper semicontinuous multifunction with convex compact values has been studied by Haddad [9], [10]. The Haddad's

result has been extended by Gavioli and Malaguti [8] to the infinite dimensional setting. Also, the lower semicontinuous case has been studied by Donchev [5], [6].

The case when A is the infinitesimal generator of C_0-semigroup and f is a continuous single-valued function has been studied by Iacob and Pavel [12] and the corresponding multivalued case, with f a Carathèodory multifunction with nonconvex values, has been considered by Lupulescu and Necula [17].

In the present study, which is a nonlinear variant of the last mentioned paper, we shall use a tangency condition introduced by Vrabie [22], accordingly adapted. Also, the construction method for a sequence of approximate solutions of (1.1), defined on an apriori given interval, is closed to the methods used by Cârjă and Vrabie [3], [4] and by Lupulescu and Necula [16].

2. Preliminaries and the main result. For an easy reference, we begin by recalling some basic concepts and results concerning m-accretive operators to be used in the sequel.

Let X be a real Banach space, let $A : D(A) \subset X \to 2^X$ be an m-accretive operator and let us consider the Cauchy problem

$$\begin{cases} u'(t) + Au(t) \ni f(t)\,, \quad \tau \le t \le T\,, \\ u(\tau) = \xi\,; \end{cases} \tag{2.1}$$

where $f \in L^1(\tau, T\,; X)$ and $\xi \in \overline{D(A)}$.

Throughout this paper we shall denote by $u(\cdot, \tau, \xi, f) : [\,\tau, T] \to \overline{D(A)}$ the unique mild solution of problem (2.1). We recall that $u(\cdot, \tau, \xi, f)$ is continuous from $[\,\tau, T]$ to X, $u(\tau, \tau, \xi, f) = \xi$ and for every $f, g \in L^1(\tau, T\,; X)$ and every $\xi, \eta \in \overline{D(A)}$, we have

$$\|u(t, \tau, \xi, f) - u(t, \tau, \eta, g)\| \le \|\xi - \eta\| + \int_\tau^t \|f(\sigma) - g(\sigma)\|\, d\sigma\,, \tag{2.2}$$

for every $t \in [\,\tau, T]$. Moreover, for every $f \in L^1(\tau, T\,; X)$ and $\xi \in \overline{D(A)}$, we have

$$\begin{aligned} \|u(t + h, \tau, \xi, f) - u(t, \tau, \xi, f)\| &\le \|u(s + h, \tau, \xi, f) - u(s, \tau, \xi, f)\| \\ &\quad + \int_s^t \|f(\sigma + h) - f(\sigma)\|\, d\sigma\,, \end{aligned} \tag{2.3}$$

for every $t, s \in [\,\tau, T-h\,]$ with $s < t$ and $h > 0$. Finally, if $\tau \leq \nu \leq t \leq T$ then we have

$$u(t,\tau,\xi,f) = u(t,\nu,u(\nu,\tau,\xi,f), f\,\big|_{[\nu,T]}).$$

We denote by $S(t) : \overline{D(A)} \to \overline{D(A)},\ t \geq 0$ the semigroup of nonexpansive mappings generated by $-A$, *i.e.* $S(t)\xi = u(t,0,\xi,0)$ for every $t \geq 0$ and $\xi \in \overline{D(A)}$, and we recall that the semigroup $\{S(t);\ t \geq 0\}$ is *compact* if for every $t > 0$ the operator $S(t)$ is compact, *i.e.* is continuous and carries bounded subsets in $\overline{D(A)}$ into relatively compact subsets in X. We refer the reader to Barbu [1] and Vrabie [23] for more details on the background of this theory.

We are now able to introduce the tangency condition that we are going to use in the sequel.

Definition 2.1. *A function $f : [a,b) \times K \times \mathcal{K} \to X$ is said to be A-tangent to K if*

$$\liminf_{h \downarrow 0} \frac{1}{h} d(u(\tau+h,\tau,\varphi(0), f(\tau,\varphi(0),\varphi)); K) = 0, \tag{2.4}$$

for every $(\tau,\varphi) \in [a,b) \times \mathcal{K}$.

Here and thereafter $d(x;C)$ denotes the distance from the point $x \in X$ to the subset C in X.

Remark 2.1. Since f is continuous we have

$$\lim_{h \downarrow 0} \frac{1}{h} \int_{\tau}^{\tau+h} \|f(s,\xi,\varphi) - f(\tau,\xi,\varphi)\|\, ds = 0$$

and using

$$\begin{aligned} &\|u(\tau+h,\tau,\xi,f(\tau,\xi,\varphi)) - u(\tau+h,\tau,\xi,f(\cdot,\xi,\varphi)\| \\ &\qquad \leq \int_{\tau}^{\tau+h} \|f(s,\xi,\varphi) - f(\tau,\xi,\varphi)\|\, ds, \end{aligned}$$

we obtain the following equivalent form of (2.4):

$$\liminf_{h \downarrow 0} \frac{1}{h} d(u(\tau+h,\tau,\varphi(0), f(\cdot,\varphi(0),\varphi)); K) = 0,$$

for every $(\tau,\varphi) \in [a,b) \times \mathcal{K}$.

We are now ready to state the main result of this paper.

Theorem 2.1. *Let X be a real Banach space, $A : D(A) \subset X \to 2^X$ an m-accretive operator with $-A$ the generator of a compact semigroup, K a nonempty and locally closed subset in $\overline{D(A)}$ and $f : [a, b) \times K \times \mathcal{K} \to X$ a continuous function. If the function f is A-tangent to K then K is a viable domain for* (1.1).

We shall prove, in the next section, this theorem in two steps: we shall construct a sequence of approximate solutions of (1.1), defined on an apriori given interval, and we shall show that this sequence is convergent, on a subsequence at least, to a solution of (1.1) with values in K. For the former step we shall use one general principle on ordered sets due to Brézis and Browder [2], namely:

Theorem 2.2 (Brezis-Browder). *Let $\preccurlyeq$ be a given preorder on the nonempty set $\mathcal{U}$ and $S : \mathcal{U} \to R \cup \{+\infty\}$ be an increasing function. Suppose that each increasing sequence in $\mathcal{U}$ is majorated in $\mathcal{U}$. Then, for each $\xi_0 \in \mathcal{U}$, there exists $\bar{\xi} \in \mathcal{U}$ with $\xi_0 \preccurlyeq \bar{\xi}$ such that $\bar{\xi} \preccurlyeq \xi$ implies $S(\bar{\xi}) = S(\xi)$;*

and for the latter we shall use the following compactness result due to Baras (see Theorem 2.3.3, p. 47, in Vrabie [23]):

Theorem 2.3 (Baras). *Let X be a real Banach space, $A : D(A) \subset X \to 2^X$ an m-accretive operator such that $-A$ generates a compact semigroup, let $\xi \in \overline{D(A)}$ and $\mathcal{F}$ an uniformly integrable subset in $L^1(a, b; X)$. Then the set $\{u(\cdot, a, \xi, f); f \in \mathcal{F}\}$ is relatively compact in $C([a, b]; X)$.*

Another useful tool in our forthcoming analysis is the modulus of continuity. Let u be a function defined on the interval $\mathcal{J}$ of $\mathbb{R}$ with values into X. For some $\delta > 0$, we denote by $\omega(u, \mathcal{J}_0, \delta)$ the *modulus of continuity* of u on the subinterval $\mathcal{J}_0 \subset \mathcal{J}$, defined by

$$\omega(u, \mathcal{J}_0, \delta) = \sup\{\|u(t) - u(s)\|; t, s \in \mathcal{J}_0, |t - s| \leq \delta\}.$$

It is easy to see that $\omega(u, \cdot, \delta)$ and $\omega(u, \mathcal{J}_0, \cdot)$ are increasing functions and that u is uniformly continuous on $\mathcal{J}_0$ if and only if $\lim_{\delta \downarrow 0} \omega(u, \mathcal{J}_0, \delta) = 0$.

3. Proof of the main result. We begin by fixing an arbitrary initial date $(\tau, \varphi) \in [a, b) \times (\mathcal{K} \cap \mathcal{C}_\sigma)$. We denote $\xi = \varphi(0)$. Since K is locally

closed and f is continuous, we can choose $r > 0$, $\rho > 0$ and $M > 0$ such that $K \cap B(\xi, r)$ is closed in X and

$$\|f(t, x, \psi)\| \leq M, \tag{3.1}$$

for every $(t, x, \psi) \in [\tau, \tau + \rho] \times B(\xi, r) \times B_\sigma(\varphi, r + 3\rho)$, where

$$B(\xi, r) = \{x \in X; \|x - \xi\| \leq r\}$$

and

$$B_\sigma(\varphi, r + 3\rho) = \{\psi \in \mathcal{M}_\sigma; \|\psi - \varphi\|_\sigma \leq r + 3\rho\}.$$

We fix $T \in (\tau, \tau + \rho]$ such that

$$\omega(\varphi, [-\sigma, 0], T - \tau) + (3M + 2)(T - \tau) + \sup_{0 \leq h \leq T - \tau} \|S(h)\xi - \xi\| \leq r. \tag{3.2}$$

This choice is always possible because $\{S(t); t \geq 0\}$ is a semigroup of nonexpansive mappings and $\varphi \in \mathcal{C}_\sigma$ implies that $\lim_{\delta \downarrow 0} \omega(\varphi, [-\sigma, 0], \delta) = 0$.

We emphasize that this choice of r, ρ, M and T will be kept unmodified until the end of this proof.

Now, we define our "approximate solution" concept for the problem (1.1) and (1.2):

Definition 3.1. *Let $T > \tau$ be fixed as in* (3.2) *and let $\varepsilon \in (0, 1)$ and $\nu \in (\tau, T]$ be arbitrary. By an ε-approximate solution for the problem* (1.1) *and* (1.2)*, defined on the interval $[\tau - \sigma, \nu] \subset [\tau - \sigma, T]$, we mean a 4-tuple $(\mathcal{F}_\nu, g, v, w)$ composed by a family of pairwise disjoint intervals $\mathcal{F}_\nu = \{[t_m, s_m), m \in \mathcal{J}_\nu\}$, an integrable function $g : [\tau, \nu] \to X$ and two measurable functions $v, w : [\tau - \sigma, \nu] \to X$ such that*

(i) $\bigcup_{m \in \mathcal{J}_\nu} [t_m, s_m) = [\tau, \nu)$ *and* $s_m - t_m \leq \varepsilon$ *for each* $m \in \mathcal{J}_\nu$;

(ii) $v_\tau = \varphi$, $w_\tau = \varphi$ *and* $w(\nu) = v(\nu)$;

(iii) $v([\tau, \nu])$ *is precompact in* X;

(iv) $g(t) = f(t, v(t_m), w_{t_m})$ *and* $w(t) = v(t_m)$ *for each* $m \in \mathcal{J}_\nu$ *and each* $t \in [t_m, s_m)$;

(v) $\|v(t) - u(t, t_m, v(t_m), g)\| \leq (t - t_m)\varepsilon$ *for each* $m \in \mathcal{J}_\nu$ *and each* $t \in [t_m, s_m)$;

(vi) $\|v_t - v_{t_m}\|_\sigma \leq 3\rho\varepsilon$, *for each* $m \in \mathcal{J}_\nu$ *and each* $t \in [\,t_m, s_m)$;

(vii) $v(t_m) \in K \cap B(\xi, r)$ *and* $w_{t_m} \in \mathcal{K} \cap B_\sigma(\varphi, r + 3\rho)$ *for each* $m \in \mathcal{J}_\nu$;

(viii) $v(\nu) \in K \cap B(\xi, r)$ *and* $w_\nu \in \mathcal{K} \cap B_\sigma(\varphi, r + 3\rho)$.

Before proving the existence of ε-approximate solutions, we have to make some technical remarks concerning their properties. Let us assume in the sequel that $(\mathcal{F}_\nu, g, v, w)$ is an ε-approximate solution on $[\tau-\sigma, \nu] \subset [\tau-\sigma, T]$.

Remark 3.1. For every $t, s \in [\tau, \nu]$ we have

$$\begin{gathered}\|v_t - v_s\|_\sigma = \sup_{\alpha\in[-\sigma,0]} \|v_t(\alpha) - v_s(\alpha)\| \\ = \sup_{\alpha\in[-\sigma,0]} \|v(t+\alpha) - v(s+\alpha)\| \leq \omega(v, [\tau-\sigma, \nu], |t-s|) \\ \leq \omega(v, [\tau-\sigma, \tau], |t-s|) + \omega(v, [\tau, \nu], |t-s|).\end{gathered}$$

On the other hand, by (ii) it follows that $\omega(v, [\tau-\sigma, \tau], \delta) = \omega(\varphi, [-\sigma, 0], \delta)$. Thus

$$\|v_t - v_s\|_\sigma \leq \omega(\varphi, [-\sigma, 0], |t-s|) + \omega(v, [\tau, \nu], |t-s|). \tag{3.3}$$

Remark 3.2. From (i) we infer that there exists $m \in \mathcal{J}_\nu$ such that $t_m = \tau$ and by (v) it follows that

$$\|v(t) - u(t, \tau, \xi, g)\| \leq (t-\tau)\varepsilon, \tag{3.4}$$

for every $t \in [\tau, \nu]$. Consequently, we have

$$\|v(t+h) - v(t)\| \leq 2(\nu-\tau)\varepsilon + \|u(t+h, \tau, \xi, g) - u(t, \tau, \xi, g)\|$$

and so

$$\omega(v, [\tau, \nu], \delta) \leq 2(\nu-\tau)\varepsilon + \omega(u(\cdot, \tau, \xi, g), [\tau, \nu], \delta), \tag{3.5}$$

for every $\delta > 0$.

Remark 3.3. By (vii) and (iv) we get $\|g(s)\| \leq M$ for every $s \in [\tau, \nu]$ and therefore

$$\begin{aligned}
&\|u(t+h,\tau,\xi,g) - u(t,\tau,\xi,g)\| \\
&\quad \leq \|u(\tau+h,\tau,\xi,g) - u(\tau,\tau,\xi,g)\| + \int_\tau^t \|g(s+h) - g(s)\| ds \\
&\quad \leq \|u(\tau+h,\tau,\xi,g) - u(\tau+h,\tau,\xi,0)\| \\
&\quad + \|u(\tau+h,\tau,\xi,0) - \xi\| + 2M(\nu-\tau) \\
&\quad \leq \int_\tau^{\tau+h} \|g(s)\| ds + \|S(h)\xi - \xi\| + 2M(\nu-\tau) \\
&\quad \leq Mh + \|S(h)\xi - \xi\| + 2M(\nu-\tau).
\end{aligned}$$

We obtain

$$\omega(u(\cdot,\tau,\xi,g),[\tau,\nu],\delta) \leq 2M(\nu-\tau) + M\delta + \sup_{0\leq h\leq\delta} \|(S(h)\xi - \xi\|.$$

Hence

$$\omega(v,[\tau,\nu],\delta) \leq 2(\nu-\tau)\varepsilon + 2M(\nu-\tau) + M\delta + \sup_{0\leq h\leq\delta} \|(S(h)\xi - \xi\|, \tag{3.6}$$

for every $\delta > 0$.

Remark 3.4. From (vi) we deduce that

$$\|v(t) - w(t)\| = \|v(t) - v(t_m)\| = \|v_t(0) - v_{t_m}(0)\| \leq \|v_t - v_{t_m}\|_\sigma \leq 3\rho\varepsilon,$$

for every $m \in \mathcal{J}_\nu$ and every $t \in [\,t_m, s_m)$. For $t \in [\tau-\sigma,\tau] \cup \{\nu\}$ we have $v(t) = w(t)$ and therefore it follows that $\|v(t) - w(t)\| \leq 3\rho\varepsilon$ for every $t \in [\tau-\sigma,\nu]$. So, we have

$$\|v_t - w_t\|_\sigma \leq 3\rho\varepsilon, \tag{3.7}$$

for every $t \in [\tau,\nu]$.

Remark 3.5. Let $\bar{g} \in L^1(\tau,\bar{\nu},X)$, $\nu \in (\tau,\bar{\nu})$ and $g = \bar{g}_{|_{[\tau,\nu]}}$. Since $u(t,\tau,\xi,g)$ depends only by the values of g on the interval $[\tau,t]$, we deduce that

$$u(\cdot,\tau,\xi,g) = u(\cdot,\tau,\xi,\bar{g})_{|_{[\tau,\nu]}}$$

and therefore

$$\omega(u(\cdot,\tau,\xi,g),[\tau,\nu],\delta)=\omega(u(\cdot,\tau,\xi,\bar{g}),[\tau,\nu],\delta)\leq\omega(u(\cdot,\tau,\xi,\bar{g}),[\tau,\bar{\nu}],\delta),$$

for every $\delta\geq 0$.

Remark 3.6. Our choice of T ensure that, if a 4-tuple $(\mathcal{F}_\nu,g,v,w)$ satisfies (i)-(vii) on the interval $[\tau-\sigma,\nu]$ with $\nu\in(\tau,T]$, then we have $v(\nu)\in B(\xi,r)$ and $w_\nu\in B_\sigma(\varphi,r+3\rho)$. Indeed, in view of (3.1), (3.2) and (3.6) we have

$$\begin{aligned}&\|v_\nu-\varphi\|_\sigma=\|v_\nu-v_\tau\|_\sigma\leq\omega(\varphi,[-\sigma,0],\nu-\tau)+\omega(v,[\tau,\nu],\nu-\tau)\\&\leq\omega(\varphi,[-\sigma,0],\nu-\tau)+(\nu-\tau)\varepsilon+3M(\nu-\tau)+\sup_{0\leq h\leq T-\tau}\|S(h)\xi-\xi\|\\&\leq\omega(\varphi,[-\sigma,0],T-\tau)+(3M+2)(T-\tau)+\sup_{0\leq h\leq T-\tau}\|S(h)\xi-\xi\|\leq r,\end{aligned}$$

and therefore

$$||v(\nu)-\xi||=||v_\nu(0)-\varphi(0)||\leq\|v_\nu-\varphi\|_\sigma\leq r.$$

Further, by (3.7), we get

$$\|w_\nu-\varphi\|_\sigma\leq\|w_\nu-v_\nu\|_\sigma+||v_\nu-\varphi||\leq r+3\rho.$$

In the next lemma we show how to construct, for every $\varepsilon\in(0,1)$, an ε-approximate solution on $[\tau-\sigma,T]$.

Lemma 3.1. *Assume that the hypotheses of Theorem* 2.1 *are satisfied and let T be fixed as in* (3.2)*. Then, for every $\varepsilon\in(0,1)$, the problem* (1.1) *and* (1.2) *has at least one ε-approximate solution on $[\tau-\sigma,T]$.*

Proof. We denote by $\mathcal{H}_T$ the set of all ε-approximate solutions $(\mathcal{F}_\nu,g,v,w)$ defined on subintervals $[\tau-\sigma,\nu]\subset[\tau-\sigma,T]$ and we prove first that $\mathcal{H}_T$ is a nonempty set.

By using the tangential condition (2.4) we obtain that there exist the sequences $(h_n)_n$ in $\mathbb{R}_+$ with $h_n\downarrow 0$ and $(p_n)_n$ in X with $p_n\to 0$ such that

$$u(\tau+h_n,\tau,\xi,f(\cdot,\xi,\varphi))+h_np_n\in K,\tag{3.8}$$

for every $n \in \mathbb{N}$.

We define $\overline{g} : [\tau, T] \to X$ by $\overline{g}(t) = f(t, \xi, \varphi)$ for every $t \in [\tau, T]$. Since φ and $\overline{g}$ are continuous functions and $p_n \to 0$, we can fix $n_0 \in \mathbb{N}$ such that $h_{n_0} \in (0, T - \tau]$, $\|p_{n_0}\| \leq \varepsilon$ and

$$\omega(\varphi, [-\sigma, 0], h_{n_0}) + \omega(u(\cdot, \tau, \xi, \overline{g}), [\tau, T], h_{n_0}) \leq \rho\varepsilon.$$

For n_0 fixed above, we define: $\nu_0 := \tau + h_{n_0}$, $\mathcal{F}_{\nu_0} = \{[\tau, \nu_0]\}$, $g(t) = \overline{g}(t)$ on $[\tau, \nu_0]$,

$$v(t) = \begin{cases} \varphi(t - \tau) & \text{for } t \in [\tau - \sigma, \tau), \\ u(t, \tau, \xi, g) + (t - \tau)p_{n_0} & \text{for } t \in [\tau, \nu_0], \end{cases}$$

and

$$w(t) = \begin{cases} \varphi(t - \tau) & \text{for } t \in [\tau - \sigma, \tau), \\ \xi & \text{for } t \in [\tau, \nu_0), \\ v(\nu_0) & \text{for } t = \nu_0. \end{cases}$$

It is easy to see that $(i) - (vi)$ and (vii) are fulfilled. In view of (3.1), (3.5), Remark 3.5 and the choice of h_{n_0} we have

$$\begin{aligned} \|v_t - v_\tau\|_\sigma &\leq \omega(\varphi, [-\sigma, 0], t - \tau) + \omega(v, [\tau, \nu_0], t - \nu) \\ &\leq \omega(\varphi, [-\sigma, 0], t - \tau) + \omega(u(\cdot, \tau, \xi, g), [\tau, \nu_0], t - \nu) + 2(\nu_0 - \tau)\varepsilon \\ &\leq \omega(\varphi, [-\sigma, 0], h_{n_0}) + \omega(u(\cdot, \tau, \xi, g), [\tau, \nu_0], h_{n_0}) + 2\rho\varepsilon \leq 3\rho\varepsilon, \end{aligned}$$

for every $t \in [\tau, \nu_0]$ and so (vi) holds too. From (3.8) we get $v(\nu_0) \in K$, hence by (i), (ii), (iv) and (vii) it follows that $w(t) \in K$ for every $[\tau - \sigma, \nu_0]$ and therefore $w_{\nu_0} \in \mathcal{K}$. By virtue of Remark 3.6 we have $v(\nu) \in K \cap B(\xi, r)$ and $w_\nu \in \mathcal{K} \cap B_\sigma(\varphi, r + 3\rho)$ and so $(viii)$ is also fulfilled. Therefore, $(\mathcal{F}_{\nu_0}, g, v, w)$ is an ε-approximative solution on $[\tau - \sigma, \nu_0]$ and thus we have proved that $\mathcal{H}_T$ is a nonempty set.

Next, we shall show that there exists at least one ε-approximate solution of (1.1) and (1.2), defined on the whole interval $[\tau - \sigma, T]$. For this aim we shall apply Theorem 2.2 to the set $\mathcal{H}_T$ endowed with the following preorder:

$$(\mathcal{F}_{\mu_1}, g^1, v^1, w^1) \preccurlyeq (\mathcal{F}_{\mu_2}, g^2, v^2, w^2)$$

if $\mu_1 \leq \mu_2$, $\mathcal{F}_{\mu_1} \subset \mathcal{F}_{\mu_2}$ and $v^1(s) = v^2(s)$ for every $s \in [\tau, \mu_1]$. Here $(\mathcal{F}_{\mu_1}, g^1, v^1, w^1)$ and $(\mathcal{F}_{\mu_2}, g^2, v^2, w^2)$ are two ε-approximate solutions defined on $[\tau - \sigma, \mu^1]$ and respectively on $[\tau - \sigma, \mu^2]$.

It is obvious that $\preccurlyeq$ is a preorder on $\mathcal{H}_T$. Moreover, let us observe that if $(\mathcal{F}_{\mu_1}, g^1, v^1, w^1) \preccurlyeq (\mathcal{F}_{\mu_2}, g^2, v^2, w^2)$ then we have $w^1(s) = w^2(s)$ for every $t \in [\tau - \sigma, \mu_1]$ and $g^1(s) = g^2(s)$ for every $t \in [\tau, \mu_1)$.

Let us define the function $\mathcal{S} : \mathcal{H}_T \to \mathbb{R}$ by

$$\mathcal{S}((\mathcal{F}_\mu, g, v, w)) = \mu,$$

for every ε-approximate solution defined on $[\tau - \sigma, \mu] \subset [\tau - \sigma, T]$. It is clear that $\mathcal{S}$ is increasing on $\mathcal{H}_T$.

Let us show now that each increasing sequence $((\mathcal{F}_{\mu_j}, g^j, v^j, w^j))_{j\in\mathbb{N}}$ in $\mathcal{H}_T$ is majorated in $\mathcal{H}_T$. First, we set

$$\mu^* = \sup\{\mu_j; j \in \mathbb{N}\} \tag{3.9}$$

and we observe that $\nu^* \in (\tau, T]$.

If $\mu_{j^*} = \mu^*$ for some $j^* \in \mathbb{N}$ then $(\mathcal{F}_{\mu_{j^*}}, g^{j^*}, v^{j^*}, w^{j^*})$ is obviously a majorant. If $\mu_j < \mu^*$ for every $j \in \mathbb{N}$, we define a majorant as follows. Let us observe first that for each $j \in \mathbb{N}$, $\mathcal{F}_{\mu_j}$ is a family of pairwise disjoint intervals, so each family $\mathcal{F}_{\mu_j}$ is at most countable and therefore the family $\mathcal{F}_{\mu_{j^*}} = \{[t_m^j, s_m^j); j \in \mathbb{N}, m \in \mathcal{J}_j\}$ is also at most countable. Clearly, we may assume that $(\mu_j)_{j\in\mathbb{N}}$ is strictly increasing and convergent to μ^*. Since $\mathcal{F}_{\mu_j} \subset \mathcal{F}_{\mu_{j+1}}$ and $\mu_j < \mu_{j+1}$ imply then there exists $\delta > 0$ such that $[\mu_j, \mu_j + \delta) \in \mathcal{F}_{\mu_{j+1}} \setminus \mathcal{F}_{\mu_j}$, we infer that in fact the family $\mathcal{F}_{\mu_{j^*}}$ is even countable and it can be rewritten in the form $\mathcal{F}_{\mu_{j^*}} = \{[t_m, s_m); m \in \mathbb{N}\}$.

We define $g^*(t) = g^j(t), v^*(t) = v^j(t)$ and $w^*(t) = w^j(t)$ for every $j \in \mathbb{N}$ and $t \in [\tau, \mu_j]$. For $g^*(\mu^*)$ we may choose any value in X, but this fact is not valid for $v^*(\mu^*)$ and either for $w^*(\mu^*)$. We will show how to define $v^*(\mu^*)$ and $w^*(\mu^*)$ in order to $(\mathcal{F}_{\mu^*}, g^*, v^*, w^*) \in \mathcal{H}_T$.

It is easy to see that $(\mathcal{F}_{\mu^*}, g^*, v^*, w^*)$ satisfies $(i), (ii), (iv), (vi)$ and (vii). Let us prove that $v^*([\tau, \mu^*))$ is precompact. Since v^* coincides with v^j on $[\tau, \mu_j]$, we know that the set $D_j = v^*([\tau, \mu_j])$ is precompact, for each $j \in \mathbb{N}$. From (iv) we have that $g^* \in L^1(\tau, \mu^*; X)$ and so, for each $j \in \mathbb{N}$, the function $u(\cdot, \mu_j, v^*(\mu_j), g^*)$ is continuous on $[\mu_j, \mu^*]$. It follows that the set $C_j = u([\mu_j, \mu^*], \mu_j, v^*(\mu_j), g^*)$ is also precompact in X. Taking into account that condition (v) is obviously satisfied on $[\tau, \mu^*)$, we obtain

$$v^*([\tau, \mu^*)) \subset C_j \cup D_j + \varepsilon(\mu^* - \mu_j)B(0, 1) \tag{3.10}$$

for every $j \in \mathbb{N}$. Let $\eta > 0$ be arbitrary fixed. We may choose $j^* \in \mathbb{N}$ such that $\varepsilon(\mu^* - \mu_j) \leq \eta$. From the inclusion above we deduce that, for every

$x \in v^*([\tau,\mu^*)$ there exists $\widetilde{x} \in C_{j^*} \cup D_{j^*}$ such that $||x-\widetilde{x}|| \leq \eta$. Since $C_{j^*} \cup D_{j^*}$ is precompact, there exists a finite family $\{x_1, x_2, \ldots, x_{n_\eta}\} \subset X$ such that, for every $\widetilde{x} \in C_{j^*} \cup D_{j^*}$, there exists $\widetilde{k} \in \{1,2,\ldots,n_\eta\}$ such that $||\widetilde{x} - x_{\widetilde{k}}|| \leq \eta$ and hence

$$||x - x_{\widetilde{k}}|| + ||x - \widetilde{x}|| \leq 2\eta.$$

Finally, we obtain that $v^*([\tau,\mu^*) \subset \cup_{k=1}^{n_\eta} B(x_k, 2\eta)$ and therefore $v^*([\tau,\mu^*))$ is precompact, so (iii) is checked.

Now we are able to complete the definition of v^*. We take any limit point v_0^* of $v^*(\mu_j)$ as j tends to $+\infty$ and we set $v^*(\mu^*) = w^*(\mu^*) = v_0^*$. By the choice of v_0^* there exists a subsequence $(\mu_{j_k})_{k\in\mathbb{N}}$ such that $\lim\limits_k \mu_{j_k} = \mu^*$ and $\lim\limits_k v^*(\mu_{j_k}) = v_0^*$. Since

$$v^*(\mu_{j_k}) = v^{j_k}(\mu_{j_k}) \in K \cap B(\xi, r)$$

for every $k \in \mathbb{N}$, keeping in mind that the set $K \cap B(\xi, r)$ is closed, we obtain that $v^*(\mu^*) = v_0^* \in K \cap B(\xi, r)$. From Remark 3.6 we have $v_{\mu^*}^* \in B_\sigma(\varphi, r)$ and from (3.7) it follows that $w_{\mu^*}^* \in \mathcal{K} \cap B_\sigma(\varphi, r+3\rho)$ and thus we have $v_{\mu^*}^* \in \mathcal{K}$, so $(viii)$ holds. Finally, the condition (v) for $t = \mu^*$ is proven by fixing any $m \in \mathbb{N}$, taking $t = \mu_{j_k}$ with $\mu_{j_k} > t_m$ in (v) written for $(\mathcal{F}_{\mu_{j_k}}, g^{j_k}, v^{j_k}, w^{j_k})$ and passing to the limit for k tending to $+\infty$ both sides.

Therefore, $(\mathcal{F}_{\mu^*}, g^*, v^*, w^*) \in \mathcal{H}_T$ and since $(\mathcal{F}_{\mu^*}, g^*, v^*, w^*)$ is clearly a majorant for $((\mathcal{F}_{\mu_j}, g^j, v^j, w^j))_{j\in\mathbb{N}}$, we have shown that the hypotheses of Theorem 2.2 are satisfied.

Before using the conclusion of Theorem 2.2, we show that every element $(\mathcal{F}_\nu, g, v, w) \in \mathcal{H}_T$ with $\mathcal{S}((\mathcal{F}_\nu, g, v, w)) < T$ is majorated in $(\mathcal{H}_T, \preceq)$ by an element $(\mathcal{F}_{\widetilde{\nu}}, \widetilde{g}, \widetilde{v}, \widetilde{w}) \in \mathcal{H}_T$ with $\mathcal{S}((\mathcal{F}_\nu, g, v, w)) < \mathcal{S}((\mathcal{F}_{\widetilde{\nu}}, \widetilde{g}, \widetilde{v}, \widetilde{w}))$. To this aim let us consider $(\mathcal{F}_\nu, g, v, w)$ an ε-approximate solution defined on $[\tau-\sigma, \nu]$ with $\nu \in (\tau, T)$. Since $w_\nu(0) = w(\nu) = v(\nu) \in K \cap B(\varphi, r)$ and $w_\nu \in \mathcal{K} \cap B_\sigma(\varphi, r+3\rho)$, we can apply the tangential condition (2.4) with (ν, w_ν) instead of (τ, φ) and we obtain that there exist $(h_n)_n$ in $\mathbb{R}_+$ with $h_n \downarrow 0$ and $(p_n)_n$ in X with $p_n \to 0$ such that

$$u(\nu + h_n, \nu, v(\nu), f(\cdot, v(\nu), w_\nu)) + h_n p_n \in K, \tag{3.11}$$

for every $n \in \mathbb{N}$. We define $\overline{g} : [\tau, T] \to X$ by

$$\overline{g}(t) = \begin{cases} g(t), & \text{for } t \in [\tau, \nu) \\ f(t, v(\nu), w_\nu), & \text{for } t \in (\nu, T]. \end{cases}$$

Since $\lim_{\delta\downarrow 0}\omega(\varphi,[-\sigma,0],\delta)=0$ and $\lim_{\delta\downarrow 0}\omega(u(\cdot,\tau,\xi,\overline{g}),[\tau,T],\delta)=0$ we can fix $\widetilde{n}\in\mathbb{N}$ such that $h_{\widetilde{n}}\in(0,T-\nu]$, $\|p_{\widetilde{n}}\|\leq\varepsilon$ and

$$\omega(\varphi,[-\sigma,0],h_{\widetilde{n}})+\omega(u(\cdot,\tau,\xi,\overline{g}),[\tau,T],h_{\widetilde{n}})\leq\rho\varepsilon.$$

For $\widetilde{n}$ fixed as above, we define: $\widetilde{\nu}:=\tau+h_{\widetilde{n}}\in(\nu,T]$, $\mathcal{F}_{\widetilde{\nu}}=\mathcal{F}_{\nu}\cup\{[\nu,\widetilde{\nu}]\}$, $\widetilde{g}(t)=\overline{g}(t)$ on $[\tau,\widetilde{\nu}]$,

$$\widetilde{v}(t)=\begin{cases} v(t), & \text{for } t\in[\tau-\sigma,\nu)\\ u(t,\nu,v(\nu),\widetilde{g})+(t-\nu)p_{\widetilde{n}}, & \text{for } t\in[\nu,\widetilde{\nu}]\end{cases}$$

and

$$\widetilde{w}(t)=\begin{cases} w(t), & \text{for } t\in[\tau-\sigma,\nu)\\ v(\nu), & \text{for } t\in[\nu,\widetilde{\nu})\\ \widetilde{v}(\widetilde{\nu}) & \text{for } t=\widetilde{\nu}.\end{cases}$$

We show now that $(\mathcal{F}_{\widetilde{\nu}},\widetilde{g},\widetilde{v},\widetilde{w})\in\mathcal{H}_T$. Let us observe that $(\mathcal{F}_{\widetilde{\nu}},\widetilde{g},\widetilde{v},\widetilde{w})$ satisfies $(i)-(iv)$ and (vii). Also, it is easy to see that (v) is satisfied for $t_m<\nu$ and $t\in[t_m,\nu]$. If $t_m\leq\nu<t\leq\nu+\delta$ then, from (2.2) and (v), we have

$$\begin{aligned}||\widetilde{v}(t)\ &-u(t,t_m,\widetilde{v}(t_m),\widetilde{g})||\\ &=||u(t,\nu,\widetilde{v}(\nu),\widetilde{g})+(t-\nu)p_{\widetilde{n}}-u(t,\nu,u(\nu,t_m,\widetilde{v}(t_m),\widetilde{g}),\widetilde{g})||\\ &\leq(\nu-t_m)\varepsilon+(t-\nu)\varepsilon=(t-t_m)\varepsilon\end{aligned}$$

and therefore (v) is fulfilled.

We have to check (vi) only on the subinterval $[t_m,s_m)=[\nu,\widetilde{\nu})$. Using (3.3), (3.5), Remark 3.5 and our choice for $h_{\widetilde{n}}$, we have

$$\begin{aligned}||\widetilde{v}_t\ &-\widetilde{v}_\nu||_\sigma\leq\omega(\varphi,[-\sigma,0],t-\nu)+\omega(\widetilde{v},[\tau,\widetilde{\nu}],t-\nu)\\ &\leq\omega(\varphi,[-\sigma,0],t-\nu)+\omega(u(\cdot,\tau,\xi,\widetilde{g}),[\tau,\widetilde{\nu}],t-\nu)+2(\widetilde{\nu}-\tau)\varepsilon\\ &\leq\omega(\varphi,[-\sigma,0],h_{\widetilde{n}})+\omega(u(\cdot,\tau,\xi,\widetilde{g}),[\tau,\widetilde{\nu}],h_{\widetilde{n}})+2\rho\varepsilon\leq 3\rho\varepsilon,\end{aligned}$$

for every $t\in[\nu,\widetilde{\nu})$.

Finally, from (3.11) we have $\widetilde{v}(\widetilde{\nu})\in K$ and thus, by (i), (ii) , (iv) and (vii), it follows that $\widetilde{w}(t)\in K$ for every $t\in[\tau-\sigma,\widetilde{\nu}]$. Hence $\widetilde{w}_{\widetilde{\nu}}\in\mathcal{K}$ and by virtue of Remark 3.6 we conclude that $\widetilde{v}(\widetilde{\nu})\in K\cap B(\xi,r)$ and $\widetilde{w}_{\widetilde{\nu}}\in\mathcal{K}\cap B_\sigma(\varphi,r+3\rho)$.

Therefore, $(\mathcal{F}_{\widetilde{\nu}},\widetilde{g},\widetilde{v},\widetilde{w})$ is an ε-approximate solution defined on $[\tau-\sigma,\widetilde{\nu}]$. Moreover, by construction, we have that $(\mathcal{F}_\nu,g,v,w)\preccurlyeq(\mathcal{F}_{\widetilde{\nu}},\widetilde{g},\widetilde{v},\widetilde{w})$ and $\mathcal{S}((\mathcal{F}_\nu,g,v,w))=\nu<\widetilde{\nu}=\mathcal{S}((\mathcal{F}_{\widetilde{\nu}},\widetilde{g},\widetilde{v},\widetilde{w}))$.

Now, let $(\mathcal{F}_{\nu^0}, g^0, v^0, w^0)$ be arbitrary fixed in $\mathcal{H}_T$. By Theorem 2.2 we deduce that there exists a maximal element $(\mathcal{F}_\nu, g, v, w) \in \mathcal{H}_T$, with

$$(\mathcal{F}_{\nu^0}, g^0, v^0, w^0) \preccurlyeq (\mathcal{F}_\nu, g, v, w),$$

such that $\mathcal{S}((\mathcal{F}_\nu, g, v, w)) = \mathcal{S}((\mathcal{F}_{\widetilde{\nu}}, \widetilde{g}, \widetilde{v}, \widetilde{w}))$, for each $(\mathcal{F}_{\widetilde{\nu}}, \widetilde{g}, \widetilde{v}, \widetilde{w}) \in \mathcal{H}_T$ with $(\mathcal{F}_\nu, g, v, w) \preccurlyeq (\mathcal{F}_{\widetilde{\nu}}, \widetilde{g}, \widetilde{v}, \widetilde{w})$.

It follows that $\mathcal{S}((\mathcal{F}_\nu, g, v, w)) = T$ because, if we assume the contrary, by the precedent step we know that there exists $(\mathcal{F}_{\widetilde{\nu}}, \widetilde{g}, \widetilde{v}, \widetilde{w}) \in \mathcal{H}_T$ with $(\mathcal{F}_\nu, g, v, w) \preccurlyeq (\mathcal{F}_{\widetilde{\nu}}, \widetilde{g}, \widetilde{v}, \widetilde{w})$ and such that $\mathcal{S}((\mathcal{F}_\nu, g, v, w)) < \mathcal{S}((\mathcal{F}_{\widetilde{\nu}}, \widetilde{g}, \widetilde{v}, \widetilde{w}))$, that is in contradiction with our choice of $(\mathcal{F}_\nu, g, v, w)$.

Thus we have proved that there exists at least an ε-approximate solution of (1.1) and (1.2) defined on the whole interval $[\tau - \sigma, T]$. □

We are now prepared to complete the proof of Theorem 2.1.

End of proof. For $\tau \in [a, b)$, $\varphi \in \mathcal{K}_0 \cap \mathcal{C}_\sigma$ and $T \in (\tau, \tau + \rho]$ already fixed, we apply Lemma 3.1 with $\varepsilon_n = 1/n$, for each $n \in \mathbb{N}^*$. Consequently, for each $n \in \mathbb{N}^*$, there exists at least an ε_n-approximative solution $(\mathcal{F}_T^n, g^n, v^n, w^n)$, on $[\,\tau, T]$. We denote

$$\mathcal{F}_T^n = \{[\,t_m^n, s_m^n), m \in \mathcal{J}_n\},$$

for each $n \in \mathbb{N}^*$.

We have that, for every $n \in \mathbb{N}^*$, $\|g^n(s)\| \leq M$ on $[\,\tau, T]$, and therefore the set $\{g^n;\ n \in \mathbb{N}^*\}$ is uniformly integrable in $L^1(\tau, T\,; X)$. Since the semigroup generated by $-A$ is compact, by virtue of Theorem 2.3 we conclude that there exists $v \in C([\,\tau, T]; X)$ such that, on a subsequence at least, we have

$$\lim_n u(t, \tau, \xi, g^n) = v(t) \tag{3.12}$$

uniformly for $t \in [\,\tau, T\,]$.

From (3.4) we get

$$\|v^n(t) - u(t, \tau, \xi, g^n)\| \leq \frac{t - \tau}{n},$$

for every $n \in \mathbb{N}^*$ and every $t \in [\,\tau, T]$ and thus

$$\lim_n v^n(t) = v(t) \tag{3.13}$$

uniformly on $[\tau, T]$.

Let us define $v(t) = \varphi(t-\tau)$ for $t \in [\tau-\sigma, \tau]$. Since $v^n(t) = v(t)$ for every $t \in [\tau-\sigma, \tau]$ and every $n \in \mathbb{N}^*$, by (3.13) it follows that

$$\lim_n v_t^n = v_t \tag{3.14}$$

in the norm $\|\cdot\|_\sigma$.

Next, we shall prove that $v(t) \in K \cap B(\xi, r)$, for every $t \in [\tau, T)$. Let t be fixed in $[\tau, T)$. From (i) we deduce that for each $n \in \mathbb{N}^*$ there exists $m_n \in \mathcal{J}_n$, such that $t \in [t_{m_n}^n, s_{m_n}^n)$. We denote $\theta_n = t_{m_n}^n$ (for the sake of simplicity) and we have $|t-\theta_n| \leq 1/n$. From (vi) it follows

$$\|v_t^n - v_{\theta_n}^n\| \leq 3\rho\varepsilon_n,$$

for every $n \in \mathbb{N}^*$, and by (3.14) we obtain

$$\lim_n v_{\theta_n}^n = v_t \tag{3.15}$$

in the norm $\|\cdot\|_\sigma$ and hence

$$\lim_n v^n(\theta_n) = v(t) \tag{3.16}$$

in X. Since $v^n(\theta_n) \in K \cap B(\xi, r)$ and $K \cap B(\xi, r)$ is closed in X, we infer that $v(t) \in K \cap B(\xi, r)$, for every $t \in [\tau, T)$.

In order to complete the proof, it remains to show that v is a solution of (1.1) and (1.2). To this aim, let us observe first that, for every $t \in [\tau, T)$, we have

$$g^n(t) = f(t, v^n(\theta_n), w_{\theta_n}^n)$$

where θ_n is chosen as above. From (3.7) we have $\|v_{\theta_n}^n - w_{\theta_n}^n\|_\sigma \leq 3\rho\varepsilon_n$, for every $n \in \mathbb{N}^*$, and by (3.14) we infer that

$$\lim_n w_{\theta_n}^n = v_t \tag{3.17}$$

in the norm $\|\cdot\|_\sigma$. Since f is continuous, from (3.16) and (3.17) it follows that

$$\lim_n g^n(t) = f(s, v(t), v_t),$$

for every $t \in [\tau, T)$. We define $g(t) = f(t, v(t), v_t)$, for every $t \in [\tau, T)$. Taking into account that $\|g^n(s)\| \leq M$, by virtue of Lebesgue Dominated Convergence Theorem we deduce that

$$\lim_n g^n = g, \tag{3.18}$$

in $L^1(\tau, T\,; X)$. From (2.2) we have, for every $t \in [\,\tau, T]$

$$\|u(t,\tau,\xi,g^n) - u(t,\tau,\xi,g))\| \leq \int_\tau^t \|g^n(s) - g(s))\|\, ds\,,$$

and hence, by (3.18) it follows that

$$\lim_n u(t,\tau,\xi,g^n) = u(t,\tau,\xi,g),$$

for every $t \in [\,\tau, T]$. In view of (3.12), we conclude that $v(t) = u(t,\tau,\xi,g)$, on $t \in [\,\tau, T]$, with $g(t) = f(t, v(t), v_t)$. But this means that v is a solution of (1.1) and (1.2), and this completes the proof. □

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