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NEW OSTROWSKI AND GRÜSS TYPE INEQUALITIES

BY

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Abstract. In the present paper , we establish new Ostrowski and Grüss type inequalities involving three functions and their derivatives. The discrete analogues of the results are also given.

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1. Introduction. In 1938, OSTROWSKI [6] proved the following useful inequality (see also [5, p.469]).

Let $f:[a,b]\to R$ be continuous on $[a,b]$ and differentiable on (a,b) whose derivative $f':(a,b)\to R$ is bounded on (a,b), i.e., $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$. Then

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \left\| f' \right\|_\infty , \tag{1.1}$$

for all $x \in [a,b]$.

Another useful inequality proved by GRÜSS [3] in 1935 can be stated as follows (see also [4, p.296]).

Let $f, g : [a,b] \to R$ be two integrable functions such that $p \le f(x) \le P$ and $q \le g(x) \le Q$ for all $x \in [a,b]$, where p, P, q, Q are constants. Then

$$\left| \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right) \right| \le \frac{1}{4}(P-p)(Q-q). \tag{1.2}$$

In the past few years, these inequalities have renewed interests and several studies have been dedicated to obtain various generalizations, variants, extensions and applications, see [1, 2, 4, 5, 7-10]. The main objective of the present paper is to establish new Ostrowski and Grüss type inequalities involving three functions and their derivatives by using a fairly elementary analysis. The discrete versions of the main results are also given.

2. Statement of results. In what follows R and $'$ denotes the set of real numbers and the derivative of a function and for $a < b$, $[a,b]$ be a given subset of R. Let N be the set of natural numbers and define $N_{a,b} = \{a, a+1, ..., a+n = b\}$ for $a \in R, n \in N$. For any function $u(t)$, $t \in N_{a,b}$, we define the operator Δ by $\Delta u(t) = u(t+1) - u(t)$. We assume that $u(t) = 0$ for $t \notin N_{a,b}$ and also use the usual convention that empty sum is taken to be zero.

Our main results are given in the following theorems.

Theorem 1. *Let $f, g, h : [a,b] \to R$ be continuous functions on $[a,b]$ and differentiable on (a,b) whose derivatives $f', g', h' : (a,b) \to R$ are bounded on (a,b), i.e. $\|f'\| = \sup_{t\in(a,b)} |f'(t)| < \infty$, $\|g'\| = \sup_{t\in(a,b)} |g'(t)| < \infty$, $\|h'\| = \sup_{t\in(a,b)} |h'(t)| < \infty$. Then*

$$\begin{aligned} &\left| f(x) g(x) h(x) - \frac{1}{3(b-a)} \left[g(x) h(x) \int_a^b f(y)\, dy \right.\right. \\ &\left.\left. + h(x) f(x) \int_a^b g(y)\, dy + f(x) g(x) \int_a^b h(y)\, dy \right] \right| \\ &\le \frac{1}{3} \left[|g(x)| |h(x)| \left\|f'\right\|_\infty + |h(x)| |f(x)| \left\|g'\right\|_\infty \right. \\ &\left. + |f(x)| |g(x)| \left\|h'\right\|_\infty \right] A(x), \end{aligned} \tag{2.1}$$

for all $x \in [a,b]$, where

$$A(x) = \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a). \tag{2.2}$$

Remark 1. If we take $h(t) = 0$ and hence $h'(x) = 0$ in Theorem 1, then by elementary calculations we get

$$\left| f(x) g(x) - \frac{1}{2(b-a)} \left[g(x) \int_a^b f(y)\, dy + f(x) \int_a^b g(y)\, dy \right] \right|$$

$$\leq \frac{1}{2}\left[|g(x)|\left\|f'\right\|_\infty + |f(x)|\left\|g'\right\|_\infty\right] A(x), \tag{2.3}$$

for all $x \in [a,b]$. Further, if we take $g(x) = 1$ and hence $g'(x) = 0$ in (2.3), then by simple calculations we recapture the well known Ostrowski's inequality in (1.1).

Theorem 2. *Let f, g, h, f', g', h' be as in Theorem 1. Then*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,h(x)\,dx\right. \\
&\quad -\frac{1}{3}\left[\left(\frac{1}{b-a}\int_a^b g(x)\,h(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\right. \\
&\quad +\left(\frac{1}{b-a}\int_a^b h(x)\,f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right) \\
&\quad \left.\left.+\left(\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b h(x)\,dx\right)\right]\right| \\
&\leq \frac{1}{3(b-a)}\int_a^b \left[|g(x)|\,|h(x)|\left\|f'\right\|_\infty + |h(x)|\,|f(x)|\left\|g'\right\|_\infty\right. \\
&\quad \left.+|f(x)|\,|g(x)|\left\|h'\right\|_\infty\right] A(x)\,dx,
\end{aligned} \tag{2.4}$$

where $A(x)$ is given by (2.2).

Remark 2. In the special case, when $h(x) = 1$ and hence $h'(x) = 0$, it is easy to observe that the inequality (2.4) obtained in Theorem 2 reduces to

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right)\right| \\
&\quad \leq \frac{1}{2(b-a)}\int_a^b \left[|g(x)|\left\|f'\right\|_\infty + |f(x)|\left\|g'\right\|_\infty\right] A(x)\,dx.
\end{aligned} \tag{2.5}$$

We note that the inequality obtained in (2.5) is a variant of the well known Grüss inequality in (1.2). For similar results, see [1,4,7].

The discrete versions of Theorems 1 and 2 are embodied in the following theorems.

Theorem 3. *Let $f(t), g(t), h(t)$ be real valued functions defined on $N_{a,b}$ for which $\Delta f(t), \Delta g(t), \Delta h(t)$ exist and $|\Delta f(t)| \leq M_1, |\Delta g(t)| \leq$*

$M_2, |\Delta h(t)| \leq M_3$ on $N_{a,b}$, where M_1, M_2, M_3 are nonnegative constants. Then

$$\begin{aligned} &\left| f(t) g(t) h(t) - \frac{1}{3(b-a)} \left[g(t) h(t) \sum_{s=a}^{b-1} f(s) \right.\right. \\ &\qquad \left.\left. + h(t) f(t) \sum_{s=a}^{b-1} g(s) + f(t) g(t) \sum_{s=a}^{b-1} h(s) \right] \right| \\ &\leq \frac{1}{3} \left[|g(t)| |h(t)| M_1 + |h(t)| |f(t)| M_2 + |f(t)| |g(t)| M_3 \right] B(t), \end{aligned} \tag{2.6}$$

for all $t \in N_{a,b}$, where

$$B(t) = \left[\frac{1}{2} + \left| t - \frac{a+b}{2} \right| \right]. \tag{2.7}$$

Remark 3. By taking $h(t) = 1$ and hence $\Delta h(t) = 0$ in Theorem 3 and by simple calculations, it is easy to see that the inequality (2.6) reduces to

$$\begin{aligned} &\left| f(t) g(t) - \frac{1}{2(b-a)} \left[g(t) \sum_{s=a}^{b-1} f(s) + f(t) \sum_{s=a}^{b-1} g(s) \right] \right| \\ &\leq \frac{1}{2} \left[|g(t)| M_1 + |f(t)| M_2 \right] B(t), \end{aligned} \tag{2.8}$$

for all $t \in N_{a,b}$. Further by taking $g(t) = 1$ and hence $\Delta g(t) = 0$ in (2.8), we get by simple computation

$$\left| f(t) - \frac{1}{b-a} \sum_{s=a}^{b-1} f(s) \right| \leq M_1 B(t), \tag{2.9}$$

for all $t \in N_{a,b}$. For inequalities similar to (2.8), (2.9), see [9].

Theorem 4. *Let $f(t), g(t), h(t), \Delta f(t), \Delta g(t), \Delta h(t)$ be as in Theorem*

3. Then

(2.10)
$$\begin{aligned}
&\left|\frac{1}{b-a}\sum_{s=a}^{b-1} f(t)\,g(t)\,h(t) - \frac{1}{3}\left[\left(\frac{1}{b-a}\sum_{t=a}^{b-1} g(t)\,h(t)\right)\left(\frac{1}{b-a}\sum_{t=a}^{b-1} f(t)\right)\right.\right.\\
&\quad + \left(\frac{1}{b-a}\sum_{t=a}^{b-1} h(t)\,f(t)\right)\left(\frac{1}{b-a}\sum_{t=a}^{b-1} g(t)\right)\\
&\quad \left.\left. + \left(\frac{1}{b-a}\sum_{t=a}^{b-1} f(t)\,g(t)\right)\left(\frac{1}{b-a}\sum_{t=a}^{b-1} h(t)\right)\right]\right|\\
&\le \frac{1}{3(b-a)}\sum_{t=a}^{b-1}\left[|g(t)|\,|h(t)|\,M_1 + |h(t)|\,|f(t)|\,M_2 + |f(t)|\,|g(t)|\,M_3\right]B(t),
\end{aligned}$$

where $B(t)$ is given by (2.7).

Remark 4. If we take $h(t) = 1$ and hence $\Delta h(t) = 0$ in Theorem 4, then by simple calculations we get the following Grüss type discrete inequality

(2.11)
$$\begin{aligned}
&\left|\frac{1}{b-a}\sum_{t=a}^{b-1} f(t)\,g(t) - \left(\frac{1}{b-a}\sum_{t=a}^{b-1} f(t)\right)\left(\frac{1}{b-a}\sum_{t=a}^{b-1} g(t)\right)\right|\\
&\qquad \le \frac{1}{2(b-a)}\sum_{t=a}^{b-1}\left[|g(t)|\,M_1 + |f(t)|\,M_2\right]B(t).
\end{aligned}$$

For a slight variant of the inequality (2.11), see [9].

3. Proofs of theorems 1 and 2. From the hypotheses, for any $x, y \in [a, b]$ we have the following identities:

$$f(x) - f(y) = \int_y^x f'(t)\,dt, \tag{3.1}$$

$$g(x) - g(y) = \int_y^x g'(t)\,dt, \tag{3.2}$$

$$h(x) - h(y) = \int_y^x h'(t)\,dt. \tag{3.3}$$

Multiplying both sides of (3.1), (3.2) and (3.3) by $g(x)h(x)$, $h(x)f(x)$ and $f(x)g(x)$ respectively and adding the resulting identities we have

(3.4)
$$\begin{aligned}&3f(x)g(x)h(x)-[g(x)h(x)f(y)+h(x)f(x)g(y)+f(x)g(x)h(y)]\\&=g(x)h(x)\int_y^x f'(t)\,dt+h(x)f(x)\int_y^x g'(t)\,dt+f(x)g(x)\int_x^y h'(t)dt.\end{aligned}$$

Integrating both sides of (3.4) with respect to y over $[a,b]$ and rewriting we have

$$\begin{aligned}&f(x)g(x)h(x)-\frac{1}{3(b-a)}\left[g(x)h(x)\int_a^b f(y)\,dy\right.\\&\qquad\left.+h(x)f(x)\int_a^b g(y)\,dy+f(x)g(x)\int_a^b h(y)\,dy\right]\\&=\frac{1}{3(b-a)}\left[g(x)h(x)\int_a^b\left\{\int_y^x f'(t)\,dt\right\}dy\right.\\&\left.+h(x)f(x)\int_a^b\left\{\int_y^x g'(t)\,dt\right\}dy+f(x)g(x)\int_a^b\left\{\int_y^x h'(t)\,dt\right\}dy\right].\end{aligned}\tag{3.5}$$

From (3.5) and using the properties of modulus we have

$$\begin{aligned}&\left|f(x)g(x)h(x)-\frac{1}{3(b-a)}\left[g(x)h(x)\int_a^b f(y)\,dy\right.\right.\\&\qquad\left.\left.+h(x)f(x)\int_a^b g(y)\,dy+f(x)g(x)\int_a^b h(y)\,dy\right]\right|\\&\le\frac{1}{3(b-a)}\left[|g(x)||h(x)|\left\|f'\right\|_\infty+|h(x)||f(x)|\left\|g'\right\|_\infty\right.\\&\left.+|f(x)||g(x)|\left\|h'\right\|_\infty\right]\int_a^b|x-y|\,dy.\end{aligned}\tag{3.6}$$

It is easy to observe that

$$\int_a^b|x-y|\,dy=\frac{(x-a)^2+(b-x)^2}{2},\tag{3.7}$$

and

$$\frac{(x-a)^2+(b-x)^2}{2(b-a)}=\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)=A(x).\tag{3.8}$$

By making use of (3.7) and (3.8) in (3.6) we get the desired inequality in (2.1). The proof of Theorem 1 is complete.

From the hypotheses of Theorem 2, the identity (3.4) holds for any $x, y \in [a,b]$. Integrating both sides of (3.4) with respect to x over $[a,b]$ and rewriting we have

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(x)\,g(x)\,h(x)\,dx \\
&\quad -\frac{1}{3}\left[\left(\frac{1}{b-a}\int_a^b g(x)\,h(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b f(y)\,dy\right)\right. \\
&\quad +\left(\frac{1}{b-a}\int_a^b h(x)\,f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(y)\,dy\right) \\
&\quad \left.+\left(\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b h(y)\,dy\right)\right] \\
&\quad =\frac{1}{3(b-a)^2}\int_a^b\left[g(x)\,h(x)\int_a^b\left\{\int_y^x f'(t)\,dt\right\}dy\right. \\
&\quad +h(x)\,f(x)\int_a^b\left\{\int_y^x g'(t)\,dt\right\}dy \\
&\quad \left.+\,f(x)\,g(x)\int_a^b\left\{\int_y^x h'(t)\,dt\right\}dy\right]dx.
\end{aligned} \tag{3.9}$$

From (3.9) and using the properties of modulus we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,h(x)\,dx\right. \\
&\quad -\frac{1}{3}\left[\left(\frac{1}{b-a}\int_a^b g(x)\,h(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\right. \\
&\quad +\left(\frac{1}{b-a}\int_a^b h(x)\,f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right) \\
&\quad \left.\left.+\left(\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b h(x)\,dx\right)\right]\right| \\
&\quad \le\frac{1}{3(b-a)^2}\int_a^b\left[\left\{|g(x)|\,|h(x)|\,\|f'\|_\infty\right.\right. \\
&\quad \left.\left.+|h(x)|\,|f(x)|\,\|g'\|_\infty+|f(x)|\,|g(x)|\,\|h'\|_\infty\right\}\int_a^b|x-y|\,dy\right]dx.
\end{aligned} \tag{3.10}$$

By making use of (3.7) and (3.8) in (3.10) we get the required inequality in (2.4) and the proof of Theorem 2 is complete.

4. Proofs of theorems 3 and 4. For any $t, s \in N_{a,b}$, it is easy to observe that the following identities hold:

$$f(t) - f(s) = \sum_{m=s}^{t-1} \Delta f(m), \tag{4.1}$$

$$g(t) - g(s) = \sum_{m=s}^{t-1} \Delta g(m), \tag{4.2}$$

$$h(t) - h(s) = \sum_{m=s}^{t-1} \Delta h(m). \tag{4.3}$$

Multiplying both sides of (4.1), (4.2) and (4.3) by $g(t)h(t)$, $h(t)f(t)$ and $f(t)g(t)$ respectively and adding the resulting identities we get

$$\begin{aligned} &3f(t)g(t)h(t) - [g(t)h(t)f(s) + h(t)f(t)g(s) + f(t)g(t)h(s)] \\ &= g(t)h(t)\sum_{m=s}^{t-1} \Delta f(m) + h(t)f(t)\sum_{m=s}^{t-1} \Delta g(m) + f(t)g(t)\sum_{m=s}^{t-1} \Delta h(m). \end{aligned} \tag{4.4}$$

Summing both sides of (4.4) over s from a to $b-1$ and rewriting we have

$$\begin{aligned} &f(t)g(t)h(t) - \frac{1}{3(b-a)}\left[g(t)h(t)\sum_{s=a}^{b-1} f(s)\right. \\ &\left. + h(t)f(t)\sum_{s=a}^{b-1} g(s) + f(t)g(t)\sum_{s=a}^{b-1} h(s)\right] \\ &= \frac{1}{3(b-a)}\sum_{s=a}^{b-1}\left[g(t)h(t)\sum_{m=s}^{t-1} \Delta f(m)\right. \\ &\left. + h(t)f(t)\sum_{m=s}^{t-1} \Delta g(m) + f(t)g(t)\sum_{m=s}^{t-1} \Delta h(m)\right]. \end{aligned} \tag{4.5}$$

From (4.5) and using the properties of modulus we have

$$\begin{aligned}
&\left| f(t)\, g(t)\, h(t) - \frac{1}{3(b-a)} \left[g(t)\, h(t) \sum_{s=a}^{b-1} f(s) \right.\right. \\
&\qquad \left.\left. + h(t)\, f(t) \sum_{s=a}^{b-1} g(s) + f(t)\, g(t) \sum_{s=a}^{b-1} h(s) \right] \right| \\
&\qquad \leq \frac{1}{3(b-a)} \left[|g(t)|\, |h(t)|\, M_1 + |h(t)|\, |f(t)|\, M_2 \right. \\
&\qquad \left. + |f(t)|\, |g(t)|\, M_3 \right] \times \left| \sum_{s=a}^{b-1} (t-s) \right|.
\end{aligned} \tag{4.6}$$

By using the summation formula for Arithmetic Progression, it is easy to observe that

$$\begin{aligned}
\left| \sum_{s=a}^{b-1} (t-s) \right| &= \left| t(b-a) - \frac{b-a}{2} [2a+b-a-1] \right| \\
&= \left[\frac{1}{2} + \left| t - \frac{a+b}{2} \right| \right] (b-a) = B(t)\,(b-a).
\end{aligned} \tag{4.7}$$

Using (4.7) in (4.6) we get (2.6). The proof of Theorem 3 is complete.

The proof of Theorem 4 can be completed by following the proof of Theorem 3 and closely looking at the proof of Theorem 2 given above. Here we omit the details.

REFERENCES

1. Dragomir, S.S. – *Some integral inequalities of Grüss type,* Indian J.Pure and Appl. Math.,, 31(2000), 379-415.
2. Dragomir, S.S.; Rassias, T.M.(Eds.) – *Ostrowski type Inequalities and Applications in Numerical Integration,* Kluwer Academic Publishers, Dordrecht, 2002.
3. Grüss, G. – *Über das maximum des absoluten Betrages von* $\frac{1}{b-a}\int_a^b f(x)\, g(x)\, dx - \frac{1}{(b-a)^2}\int_a^b f(x)\, dx \int_a^b g(x)\, dx$, Math.Z., 39(1935), 215-226.
4. Mitrinović, D.S.; Pečarić, J.E.; Fink, A.M. – *Classical and New Inequalities in Analysis,* Kluwer Academic Publishers, Dordrecht, 1993.

5. Mitrinović, D.S.; Pečarić, J.E.; Fink, A.M. – *Inequalities for Functions and Their Integrals and Derivatives*, Kluwer Academic Publishers, Dordrecht, 1994.

6. Ostrowski, A.M. – *Über die Absolutabweichung einer differentiebaren Funktion von ihrem Integralmitelwert,* Comment. Math. Helv., 10(1938), 226-227.

7. Pachpatte, B.G. – *On Trapezoid and Grüss like integral inequalities,* Tamkang J.Math., 34(2003), 365-369.

8. Pachpatte, B.G. – *A note on Ostrowski and Grüss type discrete inequalities,* Tamkang J.Math., 35(2004), 61-65.

9. Pachpatte, B.G. – *On a new generalization of Ostrowski's inequality,* J. Inequal. Pure and Appl. Math. 5(2), Art.36, 2004.

10. Pachpatte, B.G. – *On Grüss type discrete inequalities,* Math. Ineq. & Appl., 7(2004), 13-17.

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