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FLOW INVARIANCE AND NONLINEAR OPTIMIZATION PROBLEMS VIA FIRST AND SECOND ORDER GENERALIZED TANGENTIAL CONES*

BY

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Abstract. One presents some results on optimization problems via the technique of second order tangential cones to a nonempty subset. Such a technique is also used in the study of the flow invariance of a set with respect to a second order differential equation. Some classical results are included here (see the proof of Theorem 2.2 and Remark 2).

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1. Introduction. Let D be a nonempty subset of a Banach space X and let $F : X \to R$ be a functional on X.

This paper provides necessary (and some sufficient) conditions for a point $x \in D$ to be a local minimum or a local maximum of F on D.

We say that $F(x)$ is a local minimum of F on D if there is $\epsilon > 0$ such that:

$$Min\{F(z), z \in D, \|z - x\| < \epsilon\} = F(x), x \in D. \tag{1}$$

We also say that $F(x)$= Local Minimum $F(z)$: subject to $z \in D$ and that x above is a local minimizer (or a point of local minimum) of F on D.

Let $G : X \to R^n$ be a function. Set

$$D_G = \{x \in X;\ G(x) = 0\}. \tag{2}$$

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Denote by $\dot{F}(x)$ the Frechet derivative of F at x. Recall the definition of the tangential cone(or contingent cone) T_xD to D at $x \in D$, in the sense of BOULIGAND [4].

$$T_xD = \{v \in X;\ \lim_{t \downarrow 0} t^{-1}\text{dist}(x + tv; D) = 0\}. \tag{3}$$

We will also need the larger tangential cone $T_x^lD \supseteq T_xD$:

$$T_x^lD = \{v \in X;\ \lim_{t_p \downarrow 0} t_p^{-1}\text{dist}(x + t_pv; D) = 0\} \tag{4}$$

for some $t_p \downarrow 0$ as $p \to \infty$.

The second order tangential cone T_x^2D to D at $x \in D$, in the sense of PAVEL-URSESCU from 1982 (see [17]), plays a crucial role in Flight Mechanics [13, 15-19] and optimization problems.

$$T_x^2D = \{w \in X; \text{there is } v \in X;\ \lim_{t \downarrow 0} t^{-2}\text{dist}(x + tv + \frac{1}{2}t^2w; D) = 0\}. \tag{5}$$

It was suggested by a formula of Pavel from 1975 (see [15]) with $f(x)$ in place of w. Note that

$$\lim_{t \downarrow 0} t^{-1}\text{dist}(x + tv; D) = 0 \tag{6}$$

is equivalent to

$$x + tv + tr(t)) \in D,\ t > 0,\ \text{for some } r(t) \to 0, \text{as t} \downarrow 0 \tag{7}$$

and

$$\lim_{t \downarrow 0} t^{-2}\text{dist}(x + tv + \frac{1}{2!}t^2w; D) = 0\} \tag{8}$$

is equivalent to

$$x + tv + \frac{1}{2}t^2(w + r_1(t)) \in D, t > 0,\ \text{for some } r_1(t) \to 0, \text{as } t \downarrow 0. \tag{9}$$

Finally,

$$\lim_{t_p \downarrow 0} t_p^{-1}\text{dist}(x + t_pv; D) = 0\} \tag{10}$$

is equivalent to:

$$x + t_p v + t_p r_p \in D,\ t_p > 0,\ \text{for some } r_p \to 0, \text{as } t_p \downarrow 0. \tag{11}$$

For a given $w \in T_x^2 D$, a vector v in (5) is obviously an element of $T_x D$ and it is said to be an "associate" of w, or "associated" with w. Clearly, $T_x^2 D$ is a cone, as if $w \in T_x^2 D$ and v is an associate of w, then $\lambda^{1/2} v$ is an associate of λw so $\lambda w \in T_x^2 D$.

The following set introduced in 1982 by Pavel and Ursescu [17]

$$M(x, v, w) = \{v, w \in X;\ \lim_{t \downarrow 0} t^{-2} \text{dist}(x + tv + \frac{1}{2!} t^2 w; D) = 0\} \tag{12}$$

arises necessarily in flight Mechanics. Indeed, let us consider the problem

$$u''(t) = f(x(t)),\ u(0) = x,\ u'(0) = v,\ t \geq 0. \tag{13}$$

Let $w = f(x) = u''(0)$. Then a necessary condition for the trajectory $u(t)$ to remain in D, i.e. $u(t) \in D$, is $(v, w) \in M(x, v, w)$. This follows by Taylor formula and: $\text{dist}(x + tv + \frac{1}{2} t^2 w; D) \leq \|x + tv + \frac{1}{2!} t^2 w - u(t)\|$.

Remark 1. $T_x^2 D$ was extended to Banach manifolds by Motreanu and Pavel [12].

The p-order $T_x^p D$ tangential cones, $p > 2$ (see Pavel et al. 19), are an obvious extension of $T_x^2 D$, i.e.

(14)

$$T_x^p D = \{v_p \in X; \text{there are } v_i \in X;\ \lim_{t \downarrow 0} t^{-p} dist(x + tv_1 + \cdots \frac{1}{p!} t^p v_p; D) = 0\},$$

$i = 1, ..., p - 1$.

Remark 2. A formula similar to (14), with t^p in place of $\frac{1}{p!} t^p, p = 2, 3, ...$ was introduced in in 1998 by Ledzewicz and Schaettler ([10] as p-order approximating sequence to the set D. There are some minor overlaps with [10], whose authors are not aware of (they do not mention any of) the higher order cones since 1975, discussed above. However only the second order tangential cones above occur necessarily in Flight Mechanics and Optimization problems (as they are based on Taylor formula). More precisely, if the subset D is flow invariant with respect to the second order equation (13), then necessarily $f(x)$ belongs to the second order tangential cone of D at x (with $f(x)$ in place of w as in (5).

The following two Lemmas will be used:

Lemma 1. *Let $x \in D$ and $v_p \in X$ be such that $x + t_p v_p \in D$ for some $t_p \downarrow 0$. If $v_p \to v$ as $p \to \infty$, then $v \in T_x^l D$.*

Proof. If $v_p - v = r_p \to 0$ as $p \to \infty$ then $x + t_p v_p = x + t_p(v + r_p) \in D$, so according to (10), $v \in T_x^l D$. □

Recall also the following simple but useful result

Lemma 2. *If D is convex, then for any $x \in D$,*

$$\{v = z - x, z \in D\} \subseteq T_x D \subseteq T_x^l D.$$

Proof. If x, z are elements of D, then $x + t(z - x) \in D$ for all $t \in [0, 1]$, so in view of (6), $v = z - x \in T_x D$.

Some results in this paper have recently been extended in terms of higher order tangential cone. See [6,11].

2. Main results. For a given $w \in T_x^2 D$, set

$$S_w = \{v \in T_x D; \lim_{t \downarrow 0} t^{-2} dist(x + tv + \frac{1}{2!} t^2 w; D) = 0\}. \tag{15}$$

In words (see (5)), S_w is the set of all "associates" v of w. Note that in general S_w is strictly included in $T_x D$. For example, if D is the unit sphere of a real Hilbert space H of inner product $< \cdot, \cdot >$ and norm $|| \cdot ||$, i.e. $D = B_1 = \{x \in H; ||x|| = 1\}$ then [13, 17]

$$T_x B_1 = \{v \in H; < x, v >= 0\}, \tag{16}$$

$$T_x^2 B_1 = \{w \in H; \text{there exists } v \in H; < x, v >= 0, ||v||^2 + < x, w >= 0\} \tag{17}$$

$$S_w = \{v \in H; < x, v >= 0, ||v||^2 + < x, w >= 0\} \subset T_x B_1. \tag{18}$$

Our main results are given by:

Theorem 2.1. (*Necessary Conditions*). *Let D be a nonempty subset of the Banach space X, $F : X \to R$ a functional of class C^2 and let $F(x)$ be a local minimum of F on D. Then*

$$\dot{F}(x)(v) \geq 0, \text{ for all } v \in T_x(D). \tag{19}$$

Let $w \in T_x^2 D$. If $\dot{F}(x)(v) = 0$ for all $v \in S_w$, then necessarily

$$\ddot{F}(x)(v)(v) + \dot{F}(x)(w) \geq 0. \tag{20}$$

Clearly, Theorem 2.1 yields:

Corollary 1. *Suppose the hypotheses of Theorem 2.1 hold and $D = D_G$.*
(I). *If $\dot{F}(x)(v) = 0$ for all $v \in T_x(D_G)$, then*

$$\ddot{F}(x)(v)(v) + \dot{F}(x)(w) \geq 0, \tag{21}$$

for all v and w satisfying

$$G(x + tv + \frac{1}{2!}t^2(w + r(t))) = 0, \text{ for some } r(t) \to 0, as\ t \to 0. \tag{22}$$

(II). *Suppose in addition that the Fréchet derivative $\dot{G}(x)$ of G at x is onto from X into R^n, and G is of class C^2. Then necessarily*

$$\dot{F}(x)(v) = 0,\ \ddot{F}(x)(v)(v) + \dot{F}(x)(w) \geq 0, \tag{23}$$

for all (v, w) satisfying:

$$\dot{G}(x)(v) = 0,\ \ddot{G}(x)(v)(v) + \dot{G}(x)(w) = 0. \tag{24}$$

Theorem 2.2. (*Sufficient Conditions*). *Suppose the Fréchet derivative $\dot{G}(x)$ of G at x is onto from R^m into R^n, and G is of class C^2, too. Let $x \in D$ satisfy:*

$$\dot{F}(x)(v) = 0,\ \ddot{F}(x)(v)(v) + \dot{F}(x)(w) > 0, \tag{25}$$

for all (v, w) different from zero, satisfying:

$$\dot{G}(x)(v) = 0,\ \ddot{G}(x)(v)(v) + \dot{G}(x)(w) = 0. \tag{26}$$

Then $F(x)$ is a local minimum of F on D_G).

Note that the set of all v satisfying $\dot{G}(x)(v) = 0$ is the tangential (cone) space to D_G at x, while the set of all w in (22) is "the second order tangential cone to D_G at x". In the theory of motion on a given orbit on D_G in a force field f on D_G in Flight Mechanics, the vectors v and w above are the initial velocity at x and the acceleration at x, respectively. The proof of the

Corollary 1 (Part (II)) is based on the fact that Condition (21) is equivalent to:

$$x + tv + \frac{1}{2}t^2(w + r(t)) \in D_G, \tag{27}$$

$t > 0$, for some $r(t) \to 0, ast \downarrow 0$.

This is proved by Brower fixed point theorem by, PAVEL-URSESCU [17]. Significant examples are given.

Proof of Theorem 2.1. By definition, F is said to be twice Fréchet differentiable at x if

$$F(x+z) - F(x) = \dot{F}(x)(z) + \frac{1}{2}\ddot{F}(x)(z)(z) + ||z||^2 r(z) \tag{28}$$

with some $r(z) \to 0$ as $z \to 0$.

Replacing $z = tv + \frac{1}{2}t^2(w + r(t))$ into (28), we get:

$$\begin{aligned} 0 \leq F(x + tv + \frac{1}{2}t^2(w + r(t))) - F(x) = \dot{F}(x)(tv + \frac{1}{2}t^2(w + r(t)) \\ + \frac{1}{2}\ddot{F}(x)(tv + \frac{1}{2}t^2(w + r(t)))(tv + \frac{1}{2}t^2(w + r(t))) + t^2\alpha(t), \end{aligned} \tag{29}$$

with some $\alpha(t) \to 0$, as $t \to 0$.

Dividing by $t > 0$ and then letting $t \to 0$, one obtains $\dot{F}(x)(v) \geq 0$. If $\dot{F}(x)(v) = 0$, then dividing by t^2 and then letting $t \to 0$ one obtains (20), which completes the proof. □

Proof of Theorem 2.2. First of all it is easy to see that $\dot{F}(x)(v) = 0$, for all v in the null space of $\dot{G}$, i.e., $\dot{G}(x)(v) = 0$, is equivalent to

$$\dot{F}(x)(w) =< \lambda, \dot{G}(x)(w) > \tag{30}$$

for all $w \in X$, where $\lambda \in R^n$ and $< \cdot, \cdot >$ is the inner product of R^n.

This implies (by the elimination of the "acceleration w"), that Conditions (25) and (26) are equivalent to:

$$\ddot{F}(x)(v)(v) - \lambda\ddot{G}(x)(v)(v) > 0, \tag{31}$$

for all $v \neq 0$, with $\dot{G}(x)(v) = 0$, i.e. the Hessian $\ddot{F}(x) - \lambda\ddot{G}(x)$ is strictly positive definite on the tangential cone of D_G at x.

Indeed, in view of the hypothesis that $w \to \dot{G}(x)(w)$ is onto, for each v with $\dot{G}(x)(v) = 0$, there is w such that

$$\dot{G}(x)(w) = -\ddot{G}(x)(v)(v). \tag{32}$$

By a classical result (BERTSEKAS [3, p. 272]), this (i.e. (31)) implies that $F(x)$ is a local minimum of F on D_G. The proof is complete. □

3. Necessary Conditions in terms of T_xD

Theorem 3.1. *Let D be a nonempty convex subset of $R^n, x \in D$ and $F : D \to R$ of class C^2 such that:*

1. $\dot{F}(x)(v) \geq 0$, *for all* $v \in T_x(D)$.
2. $\ddot{F}(x)(v)(v) > 0$, *for all* $v \in T_x(D)$, *with* $v \neq 0$.

Then $F(x)$ is a strict local minimum of F on D. Precisely, there are two positive numbers a and b such that: $F(z) - F(x) \geq a\|z - x\|^2$, for all $z \in D, \|z - x\| < b$.

Proof. Suppose by contradiction that the conclusion of this theorem is not true, i.e. that there is a sequence $z_p \in D$ with $z_p \to x$ such that

$$F(z_p) - F(x) < \frac{1}{p}\|z_p - x\|^2, \text{ for all } p = 1, 2, \ldots \tag{33}$$

Set $d_p = \|z_p - x\|$ and $z_p - x = d_p v_p$ with $v_p = \frac{z_p - x}{\|z_p - x\|}$. Since D is convex, $z_p - x = d_p v_p \in T_x(D)$, so $v_p \in T_x(D)$.

As $\|v_p\| = 1$, we may assume that v_p is convergent to an element v_0, so $v_0 \in T_x(D)$ as $T_x(D)$ is closed. Clearly $\|v_0\| = 1$. In view of the Taylor formula, (33) yields

$$\begin{aligned} \frac{1}{p}\|z_p - x\|^2 > F(z_p) - F(x) &= d_p\dot{F}(x)(v_p) + \frac{1}{2}d_p^2\ddot{F}(x)(v_p)(v_p) + d_p^2 r_p \\ &\geq \frac{1}{2}d_p^2\ddot{F}(x)(v_p)(v_p) + d_p^2 r_p, \text{ with some } r_p \to 0, \text{ for } p \to \infty. \end{aligned} \tag{34}$$

Dividing by d_p^2 and then letting $r_p \to 0$ we get $0 \geq \ddot{F}(x)(v_0)(v_0)$ which is in conflict with the second hypothesis 2. This completes the proof. □

Note that by Theorem 2.1, the hypothesis 1 of Theorem 3.1 is also necessary for the conclusion of this theorem.

An example showing the applicability of Theorem 3.1:

Example 3.1. Let D be the triangle of vertices (0,0), (2,0), (2,1) in R^2 and $F(x_1, x_2) = \frac{1}{3}x_1^3 + x_1^2 + x_1 - x_2^2 + x_2$ with $x = (x_1, x_2)$. Clearly $\dot{F}(x) = (x_1^2 + 2x_1 + 1, -2x_2 + 1)$ so $\dot{F}(0,0) = (1,1)$ and the Hessian matrix $\ddot{F}(0,0)$ is

$$\ddot{F}(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}. \tag{35}$$

It is also easy to check that at $x = (0,0)$, $T_x(D) = \{v = (v_1, v_2), v_1 \geq 2v_2 \geq 0\}$, so $v \neq 0$ implies $v_1 \neq 0$.

Therefore $\dot{F}(0,0)(v) = v_1 + v_2 > 0$, for all $v \in T_x(D)$, with $v \neq 0$. Clearly $\ddot{F}(x)(v)(v) = 2(v_1^2 - v_2^2) > 0$ for all $v \in T_x(D)$, with $v \neq 0$. According to Theorem 3.1, the "critical" point $x = (0,0)$ is a point of local strict minimum for F on D, namely $F(0,0) = 0$.

Remark 3.1. Obviously, $\ddot{F}(0,0)(v)(v) = 2(v_1^2 - v_2^2)$ is not positive definite on R^2 so the classical criterion can only say that (0,0) is a saddle point of F on R^2 (as $\mathrm{Det}\ddot{F}(0,0) = -4 < 0$). No other results can lead to the conclusion that the "critical" point $x = (0,0)$ is a point of local strict minimum for F on D. Without convexity of D we have

Theorem 3.2. *Let D be a nonempty subset of $R^n, x \in D$ and $F : D \to R$ of class C^2 such that:*

1. $\dot{F}(x) = 0$,
2. $\ddot{F}(x)(v)(v) > 0$, *for all* $v \in T_x^l(D)$, *with* $v \neq 0$.

Then $F(x)$ is a strict local minimum of F on D. Precisely, there are two positive numbers a and b such that:

$$F(z) - F(x) \geq a\|z - x\|^2, \ forall\ z \in D, \|z - x\| < b.$$

Proof. The proof is similar to the proof of Theorem 3.1. In this case v_p may not belong to $T_x^l(D)$ (as D is not supposed to be convex) but in view of Lemma 1, its limit $v_0 \in T_x^l(D)$. Finally, in this case the inequality (34) becomes:

$$\begin{aligned} \frac{1}{p}\|z_p - x\|^2 > F(z_p) - F(x) &= d_p\dot{F}(x)(v_p) + \frac{1}{2}d_p^2\ddot{F}(x)(v_p)(v_p) + d_p^2 r_p \\ &= \frac{1}{2}d_p^2\ddot{F}(x)(v_p)(v_p) + d_p^2 r_p, \text{ with some } r_p \to 0, \text{for } p \to \infty. \end{aligned} \tag{36}$$

Dividing by d_p^2 and then letting $r_p \to 0$ we get $0 \geq \ddot{F}(x)(v_0)(v_0)$ which is in conflict with the second hypothesis 2. This completes the proof. □

Example 3.2. Let $D = D_G$ be the surface in R^3 of equation $G(x_1, x_2) = x_1^9 - 3x_1^2x_2^3 + x_2^5 = 0$, with $x = (x_1, x_2)$ and $F(x_1, x_2) = x_1^7 + x_1^2 + x_1^2x_2 - x_2^5$. Clearly $\dot{F}(x) = (7x_1^6 + 2x_1 + 2x_1x_2, x_1^2 - 5x_2^4)$ so $\dot{F}(0,0) = (0,0) = \dot{G}(0,0)$ and the Hessian matrix $\ddot{F}(0,0)$ is given by

$$\ddot{F}(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}. \tag{37}$$

It can be easily verified that $T_x^l(D) \subset \{v = (v_1, v_2), -3v_1^2v_2^3 + v_2^5 = 0\}$. Therefore $v \in T_x^l(D)$ with $v \neq 0$, implies $v_1 \neq 0$.

Obviously, $\ddot{F}(0,0)(v)(v) = 2v_1^2 > 0$ for $v \in T_x^l D, v \neq 0$, so by Theorem 3.2, the "critical" point $x = (0,0)$ is a point of local strict minimum for F on D_G.

As the Hessian matrix $\ddot{F}(0,0)$ is singular and $\dot{F}(0,0) = (0,0) = \dot{G}(0,0)$, the classical criteria are not applicable to this example.

4. Necessary conditions in terms of fourth order derivatives. We now extend the necessary conditions above in terms of fourth order derivatives:

Denote by $F^{(k)}(x)(v^k)$, the Fréchet derivative of order k of F at x and in the direction of v and $(v^k) = (v)(v)(v)(v)$.

Theorem 4.1. *Let D be a nonempty convex subset of $R^n, x \in D$ and $F : D \to R$ of class C^4 such that:*

1. *$F^{(k)}(x)(v^k) \geq 0$, for all $v \in T_xD$ and $k = 1, 2, 3$ and*
2. *$F^{(4)}(x)(v^4) > 0$, for all $v \in T_xD$, with $v \neq 0$.*

Then $F(x)$ is a strict local minimum of F on D. Precisely, there are two positive numbers a and b such that:

$$F(z) - F(x) \geq a\|z - x\|^4, for all z \in D, \|z - x\| < b.$$

Proof. As expected, the proof in very similar to the one of Theorem 3.1. It is by contradiction. Precisely, the analogous of (33) is:

$$F(z_p) - F(x) < \frac{1}{p}\|z_p - x\|^4, \text{ for all } p = 1, 2, \ldots \tag{38}$$

and the analogous of (34) is:

$$\frac{1}{p}\|z_p - x\|^4 > F(z_p) - F(x) = \sum_{k=1}^{3} \frac{1}{k!} d_p^k F^{(k)}(x)(v_p^k) + \frac{1}{4!} d_p^4 F^{(4)}(x)(v_p^4) + d_p^4 r_p \text{ with some } r_p \to 0, \text{ for } p \to \infty. \tag{39}$$

This, $v_p \in T_x D$ and the hypothesis 1 imply

$$\frac{1}{p}\|z_p - x\|^4 \geq \frac{1}{4!} d_p^4 F^{(4)}(x)(v_p^4) + d_p^4 r_p \text{ with some } r_p \to 0, \text{ for } p \to \infty. \tag{40}$$

Dividing by d_p^4 and then letting $r_p \to 0$ we get $0 \geq F^4(x)(v_0^4)$, with $v_0 \in T_x D, v_0 \neq 0$, which is in conflict with the second hypothesis 2. This completes the proof. □

Theorem 4.2. *Let D be a nonempty subset of $R^n, x \in D$ and $F : D \to R$ of class C^4 such that:*

1. *$F^{(k)}(x) = 0$, $k = 1, 2, 3$, $k = 1, 2, 3$ and*
2. *$F^{(4)}(x)(v^4) > 0$, for all $v \in T_x^l D$, with $v \neq 0$.*

Then $F(x)$ is a strict local minimum of F on D. Precisely, there are two positive numbers a and b such that:

$$F(z) - F(x) \geq a\|z - x\|^4, \; forall\, z \in D, \|z - x\| < b.$$

Proof. In this case the inequality (38) and Hypothesis 1 yield

$$\frac{1}{p}\|z_p - x\|^4 > \frac{1}{4!} d_p^4 F^{(4)}(x)(v_p^4) + d_p^4 r_p \text{ with some } r_p \to 0, \text{ for } p \to \infty. \tag{41}$$

Dividing by d_p^4 and then letting $r_p \to 0$ we get $0 \geq F^4(x)(v_0^4)$, with $v_0 \in T_x^l D, v_0 \neq 0$, which is in conflict with the second hypothesis 2. This completes the proof. □

Example showing the applicability of Theorems 4.1 and 4.2, respectively:

Example 4.1. Let D be the triangle of vertices (0,0), (2,0), (2,1) in R^2 and $F(x_1, x_2) = x_1^4 + x_1 - x_2^4 + x_2$ with $x = (x_1, x_2)$. Clearly $\dot{F}(0, 0) = (1, 1)$. We know that at $x = (0, 0)$, $T_x(D) = \{v = (v_1, v_2), v_1 > v_2 \geq 0$, for $v_1 \neq 0\} \cup (0, 0)$, so $v \neq 0$ implies $v_1 \neq 0$.

Therefore $\dot{F}(0,0)(v) = v_1 + v_2 > 0$, for all $v \in T_x(D)$, with $v \neq 0$. Clearly $F^{(4)}(x)(v^4) = 4!(v_1^4 - v_2^4) > 0$, for all $v \in T_x(D)$, with $v \neq 0$.

According to Theorem 4.1, the "critical" point $x = (0,0)$ is a point of local strict minimum for F on D, namely $F(0,0) = 0$.

Example 4.2. Let $D = D_G$ be the surface in R^3 of equation $G(x_1, x_2) = x_1^5 + x_1^3 x_2^2 - x_2^5 = 0$ and $F(x_1, x_2) = x_1^4 - x_2^3 x_1$ with $x = (x_1, x_2)$. It is easy to check that at $x = (0,0)$, $T_x(D) = \{v = (v_1, v_2); v_1^5 + v_1^3 v_2^2 - v_2^5 = 0\}$. Clearly $F^{(4)}(x)(v^4) = 4!v_1^4 > 0$, for all $v \in T_x(D)$, with $v \neq 0$.

According to Theorem 4.2, the "critical" point $x = (0,0)$ is a point of local strict minimum for F on D_G, namely $F(0,0) = 0$.

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