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ON SOME EQUIVARIANT HOMOTOPY-BIMORPHISMS

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Abstract. Using the property of the homotopy category of pointed path-connected CW-spaces to be balanced, established in 1992 by DYER and ROITBERG [2], and a condition for equivariant homotopy equivalence by JAMES and SEGAL [3], in this paper we prove a sufficient condition for an equivariant homotopy bimorphism between $G-ANRs$ to be an equivalence (isomorphism).

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1. Introduction. A morphism of a category which is simultaneously an epimorphism and a monomorphism is called a *bimorphism*. The category is *balanced* if every bimorphism is an isomorphism.

In 1992 DYER and ROITBERG proved in [2] that the homotopy category $\mathcal{HCW}^*$of pointed path-connected CW-spaces is balanced. This result was also proved by DYDAK [1]. The two proofs use variants of the well-known theorem of J. H. C. Whitehead on homotopy equivalence of CW-complexes.

In this note we use this result for the study of some equivariant morphisms. We refer to some morphisms of the homotopy category of pointed $G-ANR$ spaces in the sence of JAMES and SEGAL [3]. The parallelism with the above is given by the fact that the category $\mathcal{HCW}^*$ can be identified with the homotopy category of pointed ANR spaces [5, App. 1, §2.3, Th. 13].

Let G be a topological group. By a G-ANR we mean a separable and metrizable $G-$space X with the following property: whenever B is a normal $G-$space and A is a invariant closed subspace of B, any $G-$map $A \to X$

can be extended over an invariant neighborhood of A in B.If X is a pointed space and its base point is fixed by all G, then we say that X is a pointed $G-ANR$ or a $G-ANR^*$.

The fixed point set of a $G-$space X with respect to a subgroup H of G is denoted by X^H. If X is a pointed $G-$spaces X^H is considered as a pointed space (with the same point base as X). For any (pointed) space A the (pointed) maps $A \to X^H$correspond precisely to the (pointed) $G-$maps $G/H \times A \to X$. It follows, in particular, that if X is a $G-ANR^*$ then X^H is an object of $\mathcal{HCW}^*$ for all $H \subset G$. When X and Y are (pointed) $G-$spaces a (pointed) $G-$map $f : X \to Y$ determines a (pointed) map $f^H : X^H \to Y^H$ of the fixed point sets for all subgroups $H \subset G$. If f is a (pointed) $G-$homotopy equivalence then f^H is a (pointed) homotopy equivalence for all H. In [3] a converse is proved in the case when G is a compact Lie group.

Theorem 1.1. ([3]) *Let G be a compact Lie group, and X, Y be $G-ANR's$. Then a $G-$map $f : X \to Y$ is a $G-$homotopy equivalence if the map $f^H : X^H \to Y^H$ is a homotopy equivalence for all closed subgroups H of G.*

2. Equivariant pointed homotopy. In this paragraph we emphasize the importance of the base point in the equivariant homotopy and the difference between the equivariant and non-equivariant situations. But also its relationships in the case of $G-ANR$ spaces.

Remark 2.1. Two $G-$spaces may be $G-$homotopy equivalent without being of the same based $G-$homotopy type.

Let X be the set of all points in $\mathbb{R}^2$ on straight-line segments joining $(0,1)$ to $(x_1,0)$, where x_1 runs through all points $1/n$, for each non-zero integer n, together with 0. This becomes a $\mathbb{Z}_2-$space if we consider the symmetry wit respect to $x=0$.

Then X is equivariant contractible by the following $\mathbb{Z}_2-$homotopy: $F\left(\left(\frac{t}{n},1-t\right),t'\right) = \left(\frac{t(1-t')}{n},1+tt'-t\right)$ and $F((0,1-t),t') = (0,1+tt'-t)$. But we can consider the pointed $\mathbb{Z}_2-$spaces (X,x_0) and $(\{x_0\},x_0)$ with the same base point $x_0=(0,0)$, and we can easily verify that these spaces are not of the same equivariant based homotopy type.

We have that X is an object of the category $\mathcal{H}-\mathbb{Z}_2-\mathcal{ANR}$ but (X,x_0) is not an object of $\mathcal{H}-\mathbb{Z}_2-\mathcal{ANR}^*$.

Lemma 2.2. *Let $f : (X, *) \to (Y, *)$ be a pointed morphism between pointed path-connected $G - ANR's$. If the unpointed map $f : X \to Y$ is a $G-$homotopy equivalence it is also a pointed $G-$homotopy equivalence.*

Proof. First we have that f is a week homotopy equivalence. Then by an equivariant version of Corollary 7.5.3 of [4], we have that there is a $G-$pointed map $g : (Y, *) \to (X, *)$ which is a right homotopy inverse for pointed f. Similarly there is a right $G-$ pointed homotopy inverse of g. □

Corollary 2.3. *Let G be a compact Lie group, and X, Y be pointed $G - ANR's$. Then a pointed $G-$map $f : X \to Y$ is a pointed $G-$homotopy equivalence if the map $f^H : X^H \to Y^H$ is a pointed homotopy equivalence for all closed subgroups H of G.*

3. Bimorphisms. Denote by $\mathcal{H}-G - \mathcal{ANR}^*$ the $G-$homotopy category of pointed $G-ANR$ spaces with the fixed pointed sets path-connected, for a compact Lie group G.

Corollary 3.1. *Let $f : X \to Y$ be a morphism in $\mathcal{H}-G - \mathcal{ANR}^*$ with X^H path connected spaces for all closed subgroups H of G. Then if $f^H : X^H \to Y^H$ are bimorphisms in $\mathcal{HCW}^*$, f is an equivalence in $\mathcal{H}-G - \mathcal{ANR}^*$.*

Theorem 3.2. *Suppose that X is a $G - \mathcal{ANR}^*$ with the property that each inclusion $X^H \hookrightarrow X$ is a weak monomorphism in $\mathcal{HCW}^*$. Then a bimorphism $f : X \to Y$ in $\mathcal{H}-G-\mathcal{ANR}^*$ is an equivalence in this category.*

(We recall that $g : A \to B$ is a weak monomorphism in $\mathcal{HCW}^*$if it satisfies the weak left cancellation, $g \circ u = * \Rightarrow u = *$, whenever $u : C \to A$).

Lemma 3.3. *In the conditions of Th. 3.2, $\pi_k(f^H) : \pi_k(X^H) \to \pi_k(Y^H)$ is a monomorphism for all $k \geq 1$ and all subgroups H.*

Proof. Since f is a monomorphism it follows immediately that $\pi_k(f)$ is a monomorphism and since the inclusion $j : X^H \hookrightarrow X$ is assumed a weak monomorphism in $\mathcal{HCW}^*$we have also that $\pi_k(j)$ is a monomorphism. Then, from the commutative diagram

$$\begin{array}{ccc} \pi_k(X^H) & \xrightarrow{\pi_k(f^H)} & \pi_k(Y^H) \\ {\scriptstyle \pi_k(j)}\downarrow & & \downarrow{\scriptstyle \pi_k(j')} \\ \pi_k(X) & \xrightarrow{\pi_k(f)} & \pi_k(Y) \end{array}$$

we deduce that $\pi_k(f^H)$ is a monomorphism. □

Lemma 3.4. *If $f : X \to Y$ is an epimorphism in $\mathcal{H}-G-\mathcal{ANR}^*$ then for all closed subgroup H of G, the pointed maps $f^H : X^H \to Y^H$ are epimorphisms in $\mathcal{HANR}^*$.*

Proof. Consider the mapping cylinder M_f of f. This is the space obtained from Y and $X \times I/\{*\} \times I$ by identifying, for each $x \in X$, the points $(x, 1)$ and $f(x)$. It is easy to define an action of G on M_f by $g\widehat{b} = \widehat{gb}$ and $g\widehat{(x,t)} = \widehat{(gx,t)}$. With respect to this $G-$space structure the homotopy equivalence $h : M_f \to Y$ is a $G-$homotopy equivalence and the inclusion $j : X \hookrightarrow M_f$ (as $X \times 0$) is an inclusion of $G-$spaces. Thus we have that f is an epimorphism in $\mathcal{H}-G-\mathcal{ANR}^*$ iff j is an epimorphism in $\mathcal{H}-G-\mathcal{ANR}^*$. But $f = h \circ j$, which implies $f^H = h^H \circ j^H$ and h^H is an equivalence in $\mathcal{HANR}^*$ [3]. Thus it is sufficient to prove that $j^H : X^H \to (M_f)^H$ is an epimorphism in $\mathcal{HANR}^*$. We prove this in a more general form.

Let be Y, Y_0 two $G-ANR^*$'s with $Y_0 \subseteq Y$ and $j : Y_0 \to Y$ the morphism in $\mathcal{H}-G-\mathcal{ANR}^*$ defined by the inclusion. By analogy with the proof of Lemma 1 from [1] we can prove that j is an epimorphism in $\mathcal{H}-G-\mathcal{ANR}^*$ if and only if there is a $G-$map $\Gamma : Y \times [-1,1] \to (Y \times \{-1,1\} \cup Y_0 \times [-1,1])$ such that $\Gamma \mid Y \times \{-1,1\} \cup \{*\} \times [-1,1] : Y \times \{-1,1\} \cup \{*\} \times [-1,1] \to Y \times \{-1,1\} \cup Y_0 \times [-1,1]$ is the inclusion.

Then if we suppose that j is an epimorphism in $\mathcal{H}-G-\mathcal{ANR}^*$ and the $G-$map Γ is given, $\Gamma \mid Y^H \times [-1,1]$ has the image in $(Y^H \times \{-1,1\} \cup Y_0^H \times [-1,1])$ and $\Gamma \mid Y^H \times \{-1,1\} \cup \{*\} \times [-1,1] : Y^H \times \{-1,1\} \cup \{*\} \times [-1,1] \to Y^H \times \{-1,1\} \cup Y_0^H \times [-1,1]$ is the inclusion. So, by [1], $f^H : Y_0^H \to Y^H$ is an epimorphism in $\mathcal{HANR}^*$. □

3.5. Proof of Theorem. From 3.3 and 3.4 we have that each $f^H : X^H \to Y^H$ is an epimorphism in $\mathcal{HCW}^*$ and $\pi_k(f^H)$ is a monomorphism for all $k \geq 1$. Then by [1] we have that f^H is an equivalence in $\mathcal{HCW}^*$ and therefore Corollary 2.3 implies that f is a pointed $G-$homotopy equivalence.

Corollary 3.6. *Let (X, x_0) an object in $\mathcal{H}-G-\mathcal{ANR}^*$. The the following conditions are equivalent:*

a) *(X, x_0) is based equivariantly contractible,*

b) *the inclusion $(\{x_0\}, x_0) \hookrightarrow (X, x_0)$ is an epimorphism in $\mathcal{H}-G-\mathcal{ANR}^*$,*

c) *for all closed subgroup H of G, (X^H, x_0) is based contractible.*

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