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TWO EXAMPLES FOR A HILLE–YOSIDA TYPE THEOREM IN ORDERED CONVEX CONES

BY

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1. Abstract. Resolvent families of kernels on an ordered convex cone were studies by Cornea and Licea [6]. The corresponding notion of a semi–group, named (generalized) semi-dynamical system (on an ordered convex cone) was considered in [11]. In [17], the connection between resolvents and semi–groups was established, the main result being a Hille–Yosida type theorem, giving the framework under which a semi–dynamical system may be associated with a resolvent. The present paper is a continuation of [17]. Its purpose is to provide full proofs of the assertion about two particular cases: standard H–cones [2] and vector lattices [19], in which the above mentioned result holds.

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2. Preliminaries. Throughout this paper X is an ordered convex cone which is a lattice, with the property: for any increasing and dominated sequence (x_n) there exists the lowest upper bound $\bigvee_n x_n \in X$. This situation will be denoted by $x_n \nearrow \bigvee_n x_n$. A map $V : X \to X$ is called a kernel if it is additive, increasing and continuous from below ($x_n \nearrow x \Longrightarrow Vx_n \nearrow Vx$).

Duality. We say that two ordered convex cones X and Y are in duality through: $[\cdot,\cdot] : X \times Y \longrightarrow [0,+\infty]$ if the following properties hold:

1. $[x+x',y] = [x,y] + [x',y]$; $[x,y+y'] = [x,y] + [x,y']$
2. $x \leq x' \Longrightarrow [x,y] \leq [x',y]$; $y \leq y' \Longrightarrow [x,y] \leq [x,y']$
3. $x_n \nearrow x \Longrightarrow [x_n,y] \nearrow [x,y]$; $y_n \nearrow y \Longrightarrow [x,y_n] \nearrow [x,y]$

The duality is called *separated* (resp. *strongly separated*) if $[x,y]=[x',y]$ $\forall y\in Y\Rightarrow x=x'$ (resp. $[x,y]\leq[x',y]$ $\forall y\in Y\Rightarrow x\leq x'$).

An element $x\in X$ is called *semi–finite* if

$\forall y\in Y$ $\exists y_n\nearrow y$ such that $[x,y_n]<+\infty$, $\forall n\in\mathbb{N}$.

Semi–dynamical systems. A semi-dynamical system on algebraic structure (s.d.s.a.s. for short) is a couple (X,Φ), where X is an ordered convex cone of positive elements and $\Phi:(0,+\infty)\times X\to X$ possesses the properties:

1. $\Phi(s,\Phi(t,x))=\Phi(s+t,x)$
2. $\Phi(t,x+x')=\Phi(t,x)+\Phi(t,x')$
3. $\Phi(t,\lambda.x)=\lambda.\Phi(t,x)$
4. $x\leq x'\Longrightarrow\Phi(t,x)\leq\Phi(t,x')$
5. $x_n\nearrow x\Longrightarrow\Phi(t,x_n)\nearrow\Phi(t,x),\ \forall t>0$

Let us recall [12] that we have defined the supermedian and excessive elements associated with a s.d.s.a.s. (X,Φ):

$$X_s:=\{x\in X|\Phi(t,x)\leq x,\forall t>0\}$$

$$X_e:=\{x\in X_s|x\text{ is semi–finite and }\bigvee_{t>0}\Phi(t,x)=x\}$$

Resolvents. As usual [6], a resolvent is a family $\mathcal{V}=(V_\alpha)_{\alpha>0}$ of kernels on X, such that:

$$V_\alpha\circ V_\beta=V_\beta\circ V_\alpha,\forall\alpha,\beta>0$$

$$V_\alpha=V_\beta+(\beta-\alpha)V_\alpha\circ V_\beta,\forall 0<\alpha<\beta$$

Analogously, we associate with the resolvent $\mathcal{V}=(V_\alpha)_{\alpha>0}$ the $\mathcal{V}$ - supermedian and $\mathcal{V}$–excessive elements as:

$$X_s^{\mathcal{V}}:=\{x\in X|\alpha V_\alpha x\leq x,\forall\alpha>0\}$$

$$X_e^{\mathcal{V}}:=\{x\in X_s^{\mathcal{V}}|x\text{ is semi–finite and }\bigvee_{\alpha>0}\alpha V_\alpha x=x\}$$

The s.d.s.a.s. Φ and the resolvent $\mathcal{V}$ on X are called *associated* if

$V_\alpha\Phi(t,x)=\Phi(t,V_\alpha x)$, $\forall x\in X$, $\forall t>0$, $\forall\alpha>0$

and (under a suitable measurability assumption on the map $(t,x,y)\mapsto[\Phi(t,x),y]$):

$$[V_\alpha x,y]=\int_0^\infty e^{-\alpha t}[\Phi(t,x),y]dt,\forall x\in X,\forall y\in Y,\forall\alpha>0$$

3. Standard H–cone. All the definitions and results about H-cones are as in [2]. X will be the dual of a standard H–cone S, considered in duality with $Y = S^{**}$ (in fact, X could be a standard H-cone, equal with its bi–dual). Hence X and Y are ordered convex cones and lattices in which $\bigvee_i x_i$ exists for any increasing and dominated family. It is known [2] that X and Y are in a strongly separated duality.

There are several reasons to consider the dual of a standard H–cone: it allows to obtain (at least one–sided) strong separation ; it poses some kind of completness, for example $\mathcal{E}_{\mathcal{V}} = \overline{S} = S^{**}$; especially, for the (natural) compactness [2], recall also Alaoglu–Bourbaki's theorem [18].

When S is a H–cone, for separation it is enough to ask [2] [ex. 2.16, pg. 67]: $\forall s \in S, s \neq 0, \exists \mu \in S^*$ such that $\mu(s) \neq 0$.

Framework. We prove next that all the requirements from the framework in [17] are valid in this case. From the very definition of a standard H–cone, there exists a countable part $Y_0 \subseteq Y$, which separates X. The elements of Y_0 are universally continuous, Y_0 is increasingly dense and separates strongly X.

The finiteness assumption also holds [2] [prop. 4.2.3 b]: $[x, y] < +\infty$, $\forall x \in X$, $\forall y \in Y_0$.

We consider X endowed with the natural topology and $\mathcal{X}$ the Borel σ–algebra. By [2] [prop. 4. 2. 8.], the $\sigma(X, Y_0)$–topology coincides with the natural topology.

c) Since Y_0 is increasingly dense, each element $y \in Y$ is l.s.c. in the natural topology.

Since the dual of a standard H–cone is also a standard H–cone ([2] [cor. 4.4.10 a)]) the dual hypotheses also hold. Let us remark that each element from X is semi–finite, since the universally continuous elements form an increasingly dense part.

Remark. Once a weak unit u in the standard H–cone X is fixed, one represents X as a standard H–cone of functions (on a saturated) representation space. Moreover, choosing a nearly continuous generator $p \in X$, there exists a (absolutely continuous, submarkovian) resolvent of kernels on the representing space, such that $V_0 1 = p$ and $X = \mathcal{E}_{\mathcal{V}}$ (see [2] [th. 4.4.4]).

Proposition 1. *Let Φ and $\mathcal{V}$ be associated. Then: $X_c \cap X_s^{\mathcal{V}} \subseteq X_s \subseteq X_s^{\mathcal{V}}$ and $X_e = X_e^{\mathcal{V}}$.*

Proof. Since it is true that $\forall x \in X_e^{\mathcal{V}}\ \exists x_n \nearrow x$ with $x_n \in X_0$, we can use prop. 2 from [17]. □

Definition. *The system* Φ *is called* submarkovian *if:*

$\forall y \in Y\ \exists u \in X$, *which is a weak unit,* $[u, y] < +\infty$ *and* $\Phi(t, u) \leq u$, $\forall t > 0$.

When $X_s \equiv X$, *this property obviously holds. This condition is weaker than the usual submarkovian requirement, where the weak unit* u *is asked to be the same, for all* $y \in Y$.

The resolvent $\mathcal{V}$ *is called* submarkovian *if:*

$\forall y \in Y\ \exists u \in X$, *which is a weak unit,* $[u, y] < +\infty$ *and* $\alpha V_\alpha u \leq u$, $\forall \alpha > 0$.

Proposition 2. *Let* Φ *be a submarkovian s.d.s.a.s. on* X, *such that*

C) $t \mapsto [\Phi(t, x), y]$ *is right continuous,* $\forall x \in X$, $\forall y \in Y_0$.

There exists then a submarkovian resolvent $\mathcal{V}$ *on* X, *associated with* Φ. *Suppose moreover that* Ψ *is also submarkovian and*

C)* $t \mapsto [x, \Psi(t, y)]$ *is right continuous,* $\forall x \in X_0$, $\forall y \in Y$.

Then the dual resolvent (which always exists), is associated with Ψ *and is also submarkovian.*

Proof. Since Φ is submarkovian, we have:

$$\sup_{t>0}[u, \Psi(t, y)] = \sup_{t>0}[\Phi(t, u), y] \leq [u, y] < +\infty$$

This inequality extends to the convex combinations of $\Psi(t, y)$; as u is $\sigma(X, Y_0)$–l.s.c., the inequality extends to the closure. Using [2] [prop. 4.2.7], we obtain that $\overline{co}\{\Phi(t, x) | t \in (0, +\infty)\}$ is naturally compact, $\forall x \in X$.

Since we considered the full dual, it follows that $\exists \Psi$ s.d.s.a.s. on Y such that $[\Phi(t, x), y] = [x, \Psi(t, y)]$, $\forall t > 0$, $\forall x \in X$, $\forall y \in Y$. Indeed, let V be a kernel on X; then $V^* y$ is semifinite: let u a weak unit such that $[u, y] < +\infty$ and $\alpha V_\alpha u \leq u$. For each $x \in X$, let us denote $x_n := x \wedge nu$. Then $x_n \nearrow x$ while: $\alpha V_\alpha^* y(x_n) = [\alpha V_\alpha x_n, y] \leq \alpha V_\alpha(nu), y] \leq n[u, y] < +\infty$.

Hence we can apply [17] [prop. 3.2].

$\mathcal{V}$ is also submarkovian:

$$V_\alpha u = \int_0^\infty e^{-\alpha t} \Phi(t, y) dt \leq \int_0^\infty e^{-\alpha t} u dt = \frac{1}{\alpha} u,\ \forall \alpha > 0$$

□

Remark. The condition C*) was unfortunately omitted in [17]. The condition C) is a Ray–type condition. As $Y_0 \subsetneq Y$, it does not imply the weak continuity of the map, as can be seen from the case $Y =$ the cone of positive measures, with $Y_0 = \{\varepsilon_{x_n}\}$. Moreover, the system associated with a resolvent (see the next theorem) do posses such a property.

Corollary 3. *Let us suppose moreover that $X \equiv X_e$. Then there exists a (submarkovian) resolvent $\mathcal{V}$ on X, associated with Φ. We have $X_e^{\mathcal{V}} \equiv X$.*

Remark. The corollary is not meaningless, even if there always exists a resolvent such that $X = \mathcal{E}_{\mathcal{V}}$; it says that there exists a resolvent which is moreover associated with the s.d.s.a.s. (so maybe V_0 does not exists as a proper kernel).

Theorem 4. *Let $\mathcal{V}$ be a resolvent such that $X_e^{\mathcal{V}} \equiv X$. There exists then a s.d.s.a.s. Φ on X, associated with $\mathcal{V}$, for which $X_e \equiv X$.*

Moreover $\exists \Psi$ s.d.s.a.s. on Y such that $[\Phi(t,x), y] = [x, \Psi(t,y)]$, $\forall t > 0$, $\forall x \in X$, $\forall y \in Y$ for which $Y_e \equiv Y$.

Proof. The properties K') and A') (resp. K'*) and A'*)) from [17] [th. 4] follow from the submarkovian condition. In this case, any contraction has an adjoint, which is also a contraction. In the general case, when Y is not the full dual X^*, the existence postulated in A) is a compatibility requirement (between the kernel V and the cone Y).

As for B'), it is enough to apply the result to the subcone of the elements dominated by an element from X_s. As X_s contains at least a weak unit (from the submarkovian assumption), this subcone is increasingly dense in X. All we have to do is to extend the s.d.s.a.s. Φ to X, through

$$\tilde{V}x := \bigvee_n Vx_n$$

(for any increasing sequence $x_n \nearrow x$, from the subcone). B) follows from $X \equiv X_s^{\mathcal{V}}$; D') follows from $Y \equiv Y_s^{\mathcal{W}}$. □

Comments. The weak compactness required in K) results also under any of the assumptions:

B') $\forall x \in X$ $\exists u \in X_s^{\mathcal{V}}$ such that $x \leq u$ or:

D') $Y_0 \subset Y_s^{\mathcal{W}}$ (i.e. $\alpha W_\alpha y \leq y$, $\forall \alpha > 0$, $\forall y \in Y_0$)

(instead of the submarkovianity). Indeed, from B') we obtain:

$$\sup_{n,k}[(nV_n)^k x] \leq [u, y]$$

while from D') it follows that, for any $x \in X$ and $y \in Y_0$:

$$\sup_{n,k}[(nV_n)^k x, y] = \sup_{n,k}[x, (nW_n)^k y] \leq [x, y]$$

Corollary 5. (Inverse correspondences.) *The association establishes a bijection between the s.d.s.a.s. such that $X_e \equiv X$;*

C) $t \mapsto [\Phi(t, x), y]$ is right continuous, $\forall x \in X$, $\forall y \in Y_0$;

C) $t \mapsto [x, \Psi(t, y)]$ is right continuous, $\forall x \in X_0$, $\forall y \in Y$*

and the resolvents such that $X_e^{\mathcal{V}} \equiv X$.

The property $Y_e^{\mathcal{W}} \equiv Y$ holds: $[x, \alpha W_\alpha y] = [\alpha V_\alpha x, y] \leq [x, y]$, $\forall x \in X$ hence $\alpha W_\alpha y \leq y$, $\forall y \in Y$. Let us denote $y' := \bigvee_{\alpha>0} \alpha W_\alpha y$. Hence $[x, \alpha W_\alpha y] \nearrow [x, y']$, while $[x, \alpha W_\alpha y] = [\alpha V_\alpha x, y] \nearrow [x, y]$. Hence $y' = y$.

Finally, we discuss the possibility of extending the semi–group/resolvent to a semi–group/resolvent of true kernels (on some measurable space on which the cones X and Y are represented as cones of measurable functions/measures). If S is a cone of positive, measurable functions on a measurable space $(X, \mathcal{X})$ and the semi–dynamical system Φ comes from a true semi–group of kernels on $(X, \mathcal{X})$, then the associated resolvent $\mathcal{V}$ comes also from a true resolvent of kernels. Indeed, the weak integral

$$V_\alpha f(x) = \int_0^\infty e^{-\alpha t} P_t f(x) dt$$

has a meaning for any positive, measurable f.

By [2] [prop. 4.3.9. pg. 119], the associated resolvent of true kernels is unique.

Lemma 6. *Let S be a standard H–cone of functions on the saturated space X. Let $\mathcal{V}$ be a resolvent of (true) kernels on X, such that $S = \mathcal{E}_{\mathcal{V}}$.*

There exists a semi–group of (true) kernels on X, which is associated with $\mathcal{V}$.

Proof. Of course, $\mathcal{V}$ defines, by restriction, a resolvent on S. Hence, we obtain a s.d.s.a.s. Φ on S, associated with $\mathcal{V}$. Now, each $V := \Phi(t, \cdot)$,

$t > 0$ can be extended to a true kernel on X. Indeed, for each $x \in X$ we have $\varepsilon_x \in S^*$. From hypothesis: $V^*\varepsilon_x \in S^*$. Indeed, for each $s \in S$ let us consider $s_n := s \wedge n$. Then $s_n \nearrow s$ and $Vs_n \leq nV1 < +\infty$. In fact, since $V^*\varepsilon_x(1) < +\infty$, from [2] [prop. 4.2.12 d)] it follows that $V^*\varepsilon_x$ is a H–measure on the saturated space X. Hence V is extended to a true kernel through:

$$\tilde{V}f(x) := \int_X f(y)d(V^*\varepsilon_x)(y)$$

By [2] [prop. 4.3.9.], this extension is unique. The measurability of $\tilde{V}f$ is obtained from [13] [Lemma 1] (see also [10] [I. T20]). □

Remarks. In order to obtain V represented as a true kernel on a semi–saturated space, one has to know that each H–integral $V^*\varepsilon_x$ is (naturally) dominated either by an element from S_0^* [2]; or by another H–measure. In order to obtain V represented as a true kernel on a nearly saturated space, one has to know that each $V^*\varepsilon_x$ is a sum of universally continuous elements [2] [pg. 175]. Since the resolvent associated with a standard H–cone is absolutely continuous, the kernels may be considered on any nearly saturated space. The same would be true for the semi–group, if the associated semi–group is also absolutely continuous.

These extension results allow to obtain as particular cases analogous theorems of: [5] [th. 10. 2. 1.], [2] [th. 6.1.2].

4. Vector lattices. Let us recall some useful definitions and results about vector lattices from [19]. Let E be a vector lattice. We suppose that: **(i) for any increasing and dominated sequence** (x_n) **from** E, $\bigvee_n x_n$ **exists in** E**.**

We recall that E^+ will denote the space of all functionals on E, which are differences of linear, positive functionals; a functional f on E is called positive if $x \geq 0 \Longrightarrow f(x) \geq 0$. An order interval is a set of the form $[x, y] := \{z \in E | x \leq z \leq y\}$; a linear functional is called order bounded if it is bounded on each order interval. E^b denotes the (ordered vector) space of order bounded functionals on E.

$\sigma(E, E^+)$ is the coarsest topology on E, such that each functional from E^+ is continuous.

For each $x \in E$ we define the linear functional $\tilde{x} : E^+ \longrightarrow \mathbb{R}$ by $\tilde{x}(f) := f(x)$. On E^+ we have also the weak*-topology, denoted $\sigma(E^+, E)$, defined

as the coarsest topology on E^+, such that each functional $\tilde{x}$ defined by elements from E, is continuous. Clearly $\sigma(E^+, E) \leq \sigma(E^+, E^{++})$.

τ_0 (see [19] [§V.6, pg. 230]) is the finest locally convex topology, in which each order interval is bounded.

The following facts hold:

$(E, \tau_0)' = E^b$ (where E' means the topological dual); any linear and positive functional on E is τ_0–continuous.

(ii) There exists a locally convex, Hausdorff separated topology τ on E, such that:

a) $\sigma(E, E^+) \leq \tau \leq \tau_0$**;**

b) τ **(or, in the weakest form,** $\sigma(E, E^+)$**) is order continuous, meaning that:**

$x_n \nearrow x \Longrightarrow x_n \longrightarrow x$ **in** τ**.**

If τ is order continuous, then for any $f \in E^+$ and $x_n \nearrow x$ it follows that $f(x_n) \nearrow f(x)$.

For the order continuity in dual, we have the following result:

Proposition 7. *Let us suppose that (E, τ) is a Hausdorff separated locally convex lattice in which the topology is given by a family (p) of semi–norms satisfying: $p(|x|) = p(x)$, $\forall x \in E$ (this holds if we suppose that the topology is locally solid* [19] [pg. 235]*).*

Then for any increasing and dominated family (f_i) of positive, linear continuous functionals on E there exists $f = \bigvee_i f_i$ and $f_i \longrightarrow f$ (the filter section) in $\sigma(E^+, E)$.

Proof. Let $g \in E'$ be such that $f_i(x) \leq g(x)$, $\forall x \geq 0$, $\forall i$. There exists a semi–norm p and a constant $C > 0$ such that $|g(x)| \leq Cp(x)$, $\forall x \in E$. We define $f(x) := \sup_i f_i(x)$, Now f is a well defined linear functional on E and: $|f_i(x)| \leq |f_i(x^+) - f_i(x^-)| \leq f_i(|x|) \leq g(|x|) \leq Cp(|x|) = Cp(x)$ proving thus the continuity of f. Now $f_i \longrightarrow f$ in $\sigma(E^+, E)$. □

In a general topology, we have the following result [19] [(4.3) cor. 3, pg. 224]:

Let E be an ordered, Hausdorff separated locally convex semi–reflexive space, in which the positive cone is normal. Then for any increasing family (x_i), exists $x_0 := \bigvee_i x_i$ and the section filter converges to x_0.

In the case of normed spaces, let us recall from [1] [pg. 473] that L^p for $1 \leq p < +\infty$, c_0 and the dual of a C^*–algebra are order continuous; while

L^∞ and $\mathcal{C}(K)$ are not, (unless finite dimensional).

There exists a stronger notion: a KB–space if satisfies one of the following equivalent conditions:

(i) any increasing sequence (x_n) with $\sup_n \|x_n\| < +\infty$ is convergent;

(ii) E is a band in E'';

(iii) no closed subspace of E is isomorphic with c_0.

Clearly, any KB–space is order continuous; c_0 is order continuous, but not KB.

Let us recall that a *lattice seminorm* is a seminorm p, with the additional property: $|x| \leq |y| \Longrightarrow p(x) \leq p(y)$.

For separation, it suffices to ask: for each $x \neq 0$ there exists a lattice semi–norm p such that $p(x) \neq 0$.

(iii) ("Schauder basis") there exists a sequence (e_n) in E and a sequence (f_n) in E^+, such that for each $x \in E$ we have

$$x = \sum_{n=1}^{\infty} f_n(x) e_n$$

(in $\sigma(E, E^+)$).

Under the above assumptions (i)–(iii), X will denote the positive cone in E^+ and Y will denote the positive cone in E. By hypothesis, Y is an ordered convex cone and a lattice, with the property: for any increasing and dominated sequence (y_n) from Y, $\bigvee_n y_n$ exists in Y.

We recall (see [19] [pg. 204] the following results:

Proposition 8. *If E is a vector lattice, then the Riesz' splitting property holds:*

$\forall x, y \geq 0$ and $0 \leq z \leq x+y$ $\exists x', y' \geq 0$ such that $z = x'+y'$, $x' \leq x$, $y' \leq y$

Proposition 9. *If the Riesz' splitting property holds in E, then the positive cone is generating (i.e. $E = C - C$); E^b is a conditionally complete vector lattice and $E^b = E^+$.*

Hence, X is an ordered convex cone and a lattice, with the property: for any increasing and dominated sequence (x_n) from X, $\bigvee_n x_n$ exists in X.

In fact, the above property holds even for increasing and dominated families.

The duality is naturally defined as $[x, y] := x(y)$. Clearly the properties
1. $[x + x', y] = [x, y] + [x', y]$; $[x, y + y'] = [x, y] + [x, y']$
2. $x \leq x' \Longrightarrow [x, y] \leq [x', y]$; $y \leq y' \Longrightarrow [x, y] \leq [x, y']$

hold, while $[x_n, y] \nearrow [x, y]$ holds by the definition of $\bigvee_n x_n$; also $[x, y] \leq [x', y]$, $\forall y \in Y \Longrightarrow x \leq x'$ by the definition of the order relation in X. Finally, the property $y_n \nearrow y \Longrightarrow [x, y_n] \nearrow [x, y]$ follows from the supposed order continuity.

For the other separation, let us remark that $[x, y] = [x, y']$, $\forall x \in X \Longrightarrow y = y'$ by the fact that τ is supposed Hausdorff separated. From [19] [cor. 2 4.1. pg.223], it follows moreover that the order is archimedian and the canonical embedding $E \hookrightarrow E^{++}$ is injective.

In order to prove the strong separation $[x, y] \leq [x, y'] \, \forall x \in X \Longrightarrow y \leq y'$, we need a lemma:

Lemma 10. *For each $x \in X$ and $y, y' \in Y$ $\exists x_1, x_1' \in X$ such that $x = x_1 + x_1'$ and $x(y \vee y') = x_1(y) + x_1'(y')$.*

Proof. We consider $Y \times Y$ as an ordered convex cone and define $p : Y \times Y \longrightarrow [0, +\infty)$ as $p(u, u') := x(u \vee u')$. Clearly p is subadditive, positively homogeneous and increasing. If we denote $C := \{(u+\alpha y, u+\alpha y') | u \in Y, \alpha \geq 0\}$, then $p|_C$ is additive, as the property $(u + \alpha y) \vee (u + \alpha y') = u + \alpha y \vee y'$ holds [19] [pg. 206]. From [8] [th.1.3.1. pg. 11], there exists a functional ν on $Y \times Y$, which is additive, positively homogeneous, and $\nu = p$ on C; $\nu \leq p$ on $Y \times Y$. We define: $x_1(u) := \nu(u, 0)$; $x_1'(u) := \nu(0, u)$. Hence x_1, x_1' are additive, positive functionals on Y. Moreover, since $u \leq u'$ means $u' - u \in Y$, we obtain that x_1, x_1' are also monotone, hence elements of X. Since $(u, u) \in C$, $\forall u \in Y$, we get:

$$(x_1 + x_1')(u) = \nu(u, 0) + \nu(0, u) = \nu(u, u) = p(u, u) = x(u)$$

Finally, since $(y, y') \in C$ we obtain:

$$x_1(y) + x_1'(y') = \nu(y, 0) + \nu(0, y') = \nu(y, y') = p(y, y') = x(y \vee y')$$

□

From this lemma, we obtain the strong separation. Indeed, $[x, y] \leq [x, y'] \, \forall x \in X$ means that $\tilde{y} \leq \tilde{y'}$, hence $\tilde{y} \vee \tilde{y'} = \tilde{y'}$. Now:

$$\widetilde{y \vee y'}(x) = x(y \vee y') = x_1(y) + x_1'(y') \leq (\tilde{y} \vee \tilde{y'})(x)$$

(the last inequality by [19] [1.5, pg. 212]). The other inequality being obvious, we have proved that $\widetilde{y \vee y'} = \tilde{y} \vee \tilde{y}' = \tilde{y}'$. From the separation, we get $y \vee y' = y'$, which means exactly $y \leq y'$.

Using the hypotheses (iii) (on "Schauder basis"), we define $X_0 := \{f_n^+, f_n^-\}$ and $Y_0 := \{e_n^+, e_n^-\}$: these are countable parts from X, resp. Y, for which the finiteness condition obviously holds. Clearly, each $x \in X$ is measurable. Moreover, since for each $y \in Y$ the relation

$$y = \sum_{n=1}^{\infty} \tilde{e_n}(y) f_n$$

holds, we obtain the measurability of y.

Hence, under the present assumptions (i)–(iii), all the properties from the framework [17] are fulfilled.

Remark. If (e_n) and (f_n) are positive, then X_0 and Y_0 separate strongly.

Proposition 11. *Let Φ and $\mathcal{V}$ be associated. Then: $X_c \cap X_s^{\mathcal{V}} \subseteq X_s \subseteq X_s^{\mathcal{V}}$ and $X_e = X_e^{\mathcal{V}}$.*

Proof. The condition (ii) from [17] [prop. 2] holds. □

Definition. *The system Φ is called* submarkovian *if:*

$$\forall x \in X \exists M > 0 \exists p \text{ lattice seminorm on } Y, \text{ such that } p(y) < 1 \Longrightarrow$$

$$\exists y' \in Y_s \text{ for which } |y| \leq y' \text{ and } [x, y'] \leq M$$

Proposition 12. *If Φ is submarkovian, then K) holds.*

Proof. Indeed, for each $y \in Y$ with $p(y) < 1$ we have:

$$|[\Phi(t,x), y]| \leq [\Phi(t,x), y] \leq [\Phi(t,x), y'] = [x, \Psi(t,y')] \leq [x, y'] \leq M$$

Hence, $\Phi(t,x)$ lies in the polar of the neighborhood of the origin $\{y | p(y) < 1\}$. It results that the whole set $\overline{co}\{\Phi(t,x)\}$ is in the same polar. By the Alaoglu–Bourbaki theorem, this set is weak*–compact. □

Definition. *The resolvent $\mathcal{V}$ is called* submarkovian *if:*

$$\forall x \in X \exists M > 0 \exists p \text{ lattice seminorm on } Y, \text{ such that } p(y) < 1 \Longrightarrow$$

$$\exists y' \in Y_s^{\mathcal{W}} \text{ for which } |y| \leq y' \text{ and } [x, y'] \leq M$$

Proposition 13. *If $\mathcal{V}$ is submarkovian, then K') holds.*

Proof. Indeed, for each $y \in Y$ with $p(y) < 1$ we have:

$$\left|[(nV_n)^k x, y]\right| \leq [(nV_n)^k x, |y|] \leq [(nV_n)^k x, y'] = [x, (nW_n)^k y'] \leq [x, y'] \leq M$$

The conclusion follows as above. □

Remark. If Φ is submarkovian, then the associated resolvent si also submarkovian, since $Y_s \subseteq Y_s^{\mathcal{W}}$. If we consider, in the definition of the submarkovianty, the stronger requirement "$y' \in Y_e^{\mathcal{W}}$", then the converse also holds.

In the case of a dual of a normed space, the submarkovian requirement may be formulated: $\forall x \exists M_x$ such that $\|\Phi(t, x)\| \leq M_x, \forall t > 0$. For the resolvent, it is enough to ask:

$$\forall x \exists M_x \text{ such that } \|nV_n x\| \leq M_x, \forall n$$

Proposition 14. *Let Φ be a submarkovian s.d.s.a.s. on X, such that*

C) $t \mapsto [\Phi(t, x), y]$ is right continuous, $\forall x \in X$, $\forall y \in Y_0$.

There exists then a submarkovian resolvent $\mathcal{V}$ on X, associated with Φ. Suppose moreover that the dual Ψ is also submarkovian and

C) $t \mapsto [x, \Psi(t, y)]$ is right continuous, $\forall x \in X_0$, $\forall y \in Y$.*

holds. Then the dual resolvent $\mathcal{W}$ on Y is submarkovian and associated with Ψ.

Proof. Using prop. 12, we can apply [17] [prop. 3.2.]. □

Remark. Due to the choice of X and Y, the existence of the adjoint of a kernel (condition A)) is clearly fulfilled. In the case of normed spaces, the normality of the positive cone (i.e. $E - E^+ - E^+$) is an additional assumption.

Theorem 15. *Let $\mathcal{V} = (V_\alpha)_{\alpha\geq 0}$ be a submarkovian resolvent with the properties:*

B') $\forall x \in X\ \exists u \in X_s^{\mathcal{V}}$ such that $x \leq u$.

D') $Y_0 \subset Y_s^{\mathcal{W}}$ (i.e. $\alpha W_\alpha y \leq y$, $\forall \alpha > 0$, $\forall y \in Y_0$).

There exists then a submarkovian s.d.s.a.s. Φ on X, associated with $\mathcal{V}$, with the property:

C) $t \mapsto [\Phi(t,x), y]$ is right continuous, $\forall x \in X$, $\forall y \in Y_0$.

If moreover the dual resolvent $\mathcal{W}$ is also submarkovian and:

B) $\forall y \in Y\ \exists v \in Y_s^{\mathcal{W}}$ such that $y \leq v$.*

D) $X_0 \subset X_s^{\mathcal{V}}$ (i.e. $\alpha V_\alpha x \leq x$, $\forall \alpha > 0$, $\forall x \in X_0$).*

hold, then the dual system Ψ on Y is also submarkovian, is associated with $\mathcal{W}$ and

C) $t \mapsto [x, \Psi(t,y)] = [\Phi(t,x), y]$ is right continuous, $\forall x \in X_0$, $\forall y \in Y$.*

Proof. We can apply [17] [th. 3.3.] □

Theorem 16. (inverse correspondences.) *The association establishes a bijection between the submarkovian s.d.s.a.s. Φ, with the properties:*

C) $t \mapsto [\Phi(t,x), y]$ is right continuous, $\forall x \in X$, $\forall y \in Y_0$.

C) $t \mapsto [x, \Psi(t,y)]$ is right continuous, $\forall x \in X_0$, $\forall y \in Y$.*

B) $\forall x \in X\ \exists u \in X_s$ such that $x \leq u$.

B) $\forall y \in Y\ \exists v \in Y_s$ such that $y \leq v$.*

D) $Y_0 \subset Y_s$ (i.e. $\Psi(t,y) \leq y$, $\forall t > 0$, $\forall y \in Y_0$).

D) $X_0 \subset X_s$ (i.e. $\Phi(t,x) \leq x$, $\forall t > 0$, $\forall x \in X_0$).*

and the submarkovian resolvents with the properties:

B') $\forall x \in X\ \exists u \in X_s^{\mathcal{V}}$ such that $x \leq u$.

D') $Y_0 \subset Y_s^{\mathcal{W}}$ (i.e. $\alpha W_\alpha y \leq y$, $\forall \alpha > 0$, $\forall y \in Y_0$).

B) $\forall y \in Y\ \exists v \in Y_s^{\mathcal{W}}$ such that $y \leq v$.*

D) $X_0 \subset X_s^{\mathcal{V}}$ (i.e. $\alpha V_\alpha x \leq x$, $\forall \alpha > 0$, $\forall x \in X_0$).*

Proof. It is a direct consequence of prop. 14 and th. 15, using [17] [th. 3.4.]. □

The definition of V_0. It is important to have also the kernel V_0 corresponding to a resolvent, when discussing the condition D). Since for $0 < \alpha < \beta$ we have $V_\alpha x = V_\beta x + (\beta - \alpha) V_\alpha V_\beta x$, it follows that the family $(V_\alpha x)_{\alpha>0}$ is specifically decreasing. Hence, for those $x \in X$ for which the above family is dominated, there exists $V_0 x := \curlyvee_{\alpha>0} V_\alpha x = \bigvee\limits_{\alpha>0} V_\alpha x$ (in

the sense that the suprema exists for each decreasing sequence of strictly positive numbers). If we denote $D(V_0) := \{x \in X | (V_\alpha x)_{\alpha>0} \text{ is dominated}\}$, then we obtain a kernel $V_0 : D(V_0) \subseteq X \to X$. Clearly $D(V_0)$ is a naturally solid convex subcone. If (x_n) is an increasing sequence, dominated by an element from $D(V_0)$, then $\bigvee x_n \in D(V_0)$. V_0 is additive, positively homogeneous and increasing. Moreover, if $x_n \nearrow x$ in $D(V_0)$, then $V_0 x_n \nearrow V_0 x$. The resolvent $\mathcal{V} = (V_\alpha)_\alpha$ will be called proper if $D(V_0) \equiv X$.

The difficulty is illustrated by the case when the resolvent is associated with a s.s.d.a.s. Φ: while $V_\alpha x$ can be defined as an element of X, integrating with respect to a probability measure, this is no more the case for $V_0 x = \int_0^\infty \Phi(t, x)dt$. This weak integral, if it exists, is a priori an element from Y^*; by identification with supremum, it remains in fact in X.

The usual trick is to replace the original resolvent $\mathcal{V}$ by $\mathcal{V}^\varepsilon := (V_{\alpha+\varepsilon})_{\alpha>0}$ (associated with the s.d.s.a.s. $\Phi^\varepsilon(t, \cdot) := e^{-\varepsilon t}\Phi(t, \cdot)$). In this way, the original V_0 is replaced by V_ε which is defined on X.

Properties of $V_0 x$. For $x \in D(V_0)$ the resolvent equation shows that $\alpha V_\alpha(V_0 x) \preccurlyeq V_0 x$, hence $V_0 x$ is even specifically supermedian. Moreover, when $\mathcal{V}$ is associated with Φ, then $V_0 x$ is Φ–supermedian:

$$\Phi(t, V_0 x) = \Phi(t, \bigvee_{\alpha>0} V_\alpha x) = \bigvee_{\alpha>0} \int_0^\infty e^{-\alpha\tau}\Phi(\tau, \Phi(t, x))d\tau$$

$$= \bigvee_{\alpha>0} \int_0^\infty e^{-\alpha\tau}\Phi(\tau + t, x)d\tau = \bigvee_{\alpha>0} e^{\alpha t} \int_t^\infty e^{-\alpha s}\Phi(s, x)ds$$

$$= \int_t^\infty \Phi(s, x)ds \leq V_0 x$$

Moreover, we have $\Phi(t, V_0 x) \nearrow V_0 x$ for $t \searrow 0$; hence $V_0 x$ is even Φ–excessive, on condition to be semi–finite. Indeed:

$$0 \leq \int_0^t [\Phi(s, x), y]ds = \int_0^\infty \chi_{[0,t]}(s)[\Phi(s, x), y]ds$$

hence we need a domination: $s \mapsto [\Phi(s, x), y]$ is integrable on $[0, +\infty]$ for an increasingly dense part of $y \in Y$. The same computation shows that the map $t \mapsto [\Phi(t, V_0 x), y]$ is continuous, for each $y \in Y_0$.

Let us suppose that the resolvent $\mathcal{V}$ is proper and: $V_\alpha x = V_\alpha x', \forall \alpha > 0 \Rightarrow x = x'$. There exists then a countable part $Y_0' \subset Y_s$ which separates X. Indeed, we choose $Y_0' := W_0(Y_0)$.

Remarks. If $\mathcal{V}$ is associated with Φ, then we obtain further that $\Phi(t,x) = \Phi(t,x')$, $\forall t > 0$. The fact that this property implies $x = x'$ is an axiom for the semi–dynamical system in the sense of BUCUR-BEZZARGA [4].

In order to obtain the condition D), the change of the resolvent into $\mathcal{V}^\varepsilon$ does not seem to work. In that way, the weak topology $\sigma(X, W_0(Y_0))$ may be different from the original one. Moreover, there may appear some non–measurable elements with respect to the new borel σ–algebra.

Let X, Y be in a separated duality.

Let us suppose that

A') $\exists \mathcal{W}$ resolvent on Y such that $[V_\alpha x, y] = [x, W_\alpha y]$, $\forall \alpha > 0$, $\forall x \in X$, $\forall y \in Y$

holds and that $Y_e^{\mathcal{W}}$ separates. Then for each $x, x' \in D(V_0)$ such that $V_0 x$ is cancellable, from $V_0 x = V_0 x'$ it follows that $x = x'$.

Indeed, from the resolvent equation (and the fact that $V_0 x$ is cancellable) it follows that $V_\alpha x = V_\alpha x'$, $\forall \alpha > 0$. For $y \in Y_e^{\mathcal{W}}$ we have:

$$[\alpha V_\alpha x, y] = [\alpha V_\alpha x', y] \Longleftrightarrow [x, \alpha W_\alpha y] = [x', \alpha W_\alpha y]$$

hence $[x, y] = [x', y]$. □

Under the same hypotheses, $W_0(Y_0)$ separates the set of those $x \in D(V_0)$ such that $V_0 x$ is cancellable.

Indeed, for x, x' such that $V_0 x$ and $V_0 x'$ are cancellable, from $[x, W_0 y] = [x', W_0 y]$ we get $[V_0 x, y] = [V_0 x', y]$ (for $y \in Y_0$). Hence $V_0 x = V_0 x'$ and the conclusion follows from the preceding result. □

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