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ON (k, μ)-CONTACT METRIC MANIFOLDS

BY

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Abstract. This paper presents a study of (k, μ)-contact metric manifold satisfying certain conditions.

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1. Introduction. In 1995 Blair, Koufogiorgos and Papantoniou [4] introduced the notion of contact metric manifolds with characteristic vector field ξ belonging to the (k, μ)-nullity distribution and such type of manifolds are called (k, μ)-contact metric manifolds. They obtained several results and examples of such a manifold and a full classification of this manifold has been given by Boeckx [5].

The object of this paper is to study (k, μ)-contact metric manifolds. A projectively flat Riemannian manifold is a space of constant curvature, but after preliminaries, in section 3, it is shown that a projectively flat (k, μ)-contact metric manifold is either (i) locally isometric to the Riemannian product $E^{n+1}(0) \times S^n(4)$ including the 3-dimensional case, or (ii) locally isometric to a Sasakian manifold of constant curvature 1, or (iii) locally isometric to a left invariant metric on the Lie group SU(2) or SL(2, R). Section 4 is devoted to the (k, μ)-contact metric manifolds satisfying the condition $R(X, \xi) \cdot P = 0$, where $R(X, Y)$ is considered as a derivation of the tensor algebra at each point of the manifold for tangent vectors X, Y and P is the projective curvature tensor. The last section deals with a projectively recurrent (k, μ)-contact metric manifolds and also such a manifold satisfying the condition div$P = 0$ has been studied, where 'div' denotes the divergence.

2. Preliminaries. A contact manifold is a C^∞-(2n+1) manifold M^{2n+1} equipped with a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M^{2n+1}. Given a contact form η it is well known that there exists a unique vector field ξ, called the characteristic vector field of η, such that $\eta(\xi)$=1 and $d\eta(X,\xi)$=0 for every vector field X on M^{2n+1}. A Riemannian metric is said to be associated metric if there exists a tensor field ϕ of type (1,1) such that

$$d\eta(X,Y) = g(X,\phi Y), \qquad \eta(X) = g(X,\xi), \tag{2.1}$$

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \phi^2 X = -X + \eta(X)\xi, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.3}$$

for all vector fields X, Y on M^{2n+1}. Then the structure (ϕ,ξ,η,g) on M^{2n+1} is called a contact metric structure and the manifold M^{2n+1} equipped with such a structure is called a contact metric manifold [1].

Given a contact metric manifold $M^{2n+1}(\phi,\ \xi,\ \eta,\ g)$ we define a $(1,1)$ tensor field h by h=$\frac{1}{2}\pounds_\xi\phi$, where $\pounds$ denotes Lie differentiation. Then h is symmetric and satisfies $h\phi = -\phi h$. Thus, if λ is an eigenvalue of h with eigenvector X, $-\lambda$ is also an eigenvalue with eigenvector ϕX. Also we have $Tr.h = Tr.\phi h = 0$ and $h\xi = 0$. Moreover, if ∇ denotes the Riemannian connection of g, then the following relation holds:

$$\nabla_X \xi = -\phi X - \phi h X. \tag{2.4}$$

The vector field ξ is a Killing vector with respect to g if and only if $h = 0$. A contact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$ for which ξ is a Killing vector is said to be a K-contact manifold. A Contact structure on M^{2n+1} gives rise to an almost complex structure on the product $M^{2n+1} \times R$. If this almost complex structure is integrable, the contact metric manifold is said to be Sasakian. Equivalently, a contact metric manifold is Sasakian if and only if the relation $R(X,Y)\xi = \eta(Y)X - \eta(X)Y$ holds for all X, Y where R denotes the curvature tensor of the manifold.

The (k,μ)-nullity distribution of a contact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$ is a distribution [4]

$$\begin{aligned} N(k,\mu) : p \to N_p(k,\mu) = [Z \in T_pM : R(X,Y)Z \\ = k\{g(Y,Z)X - g(X,Z)Y] + \mu[g(Y,Z)hX - g(X,Z)hY]\} \end{aligned}$$

for any $X, Y \in T_pM$. Hence if the characteristic vector field ξ belongs to the (k,μ)-nullity distribution, then we have

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]. \tag{2.5}$$

Thus a contact metric manifold satisfying the relation (2.5) is called a (k,μ)-contact metric manifold. In particular, if $\mu = 0$, then the notion of (k,μ)-nullity distribution reduces to the notion of k-nullity distribution, introduced by Tanno [7]. A (k,μ)-contact metric manifold is Sasakian if $k = 1$. In a (k,μ)-contact metric manifold the following relations hold [4]:

$$h^2 = (k-1)\phi^2, \quad k \le 1, \tag{2.6}$$

$$(\nabla_X \phi)(Y) = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{2.7}$$

$$\begin{aligned} (\nabla_X h)(Y) &= \{(1-k)g(X,\phi Y) + g(X, h\phi Y)\}\xi \\ &+\eta(Y)[h(\phi X + \phi hX)] - \mu\eta(X)\phi hY, \end{aligned} \tag{2.8}$$

$$R(\xi, X)Y = k[g(X,Y)\xi - \eta(Y)X] + \mu[g(hX,Y)\xi - \eta(Y)hX], \tag{2.9}$$

$$\begin{aligned} \eta(R(X,Y)Z) &= k[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] \\ &+\mu[g(hY,Z)\eta(X) - g(hX,Z)\eta(Y)], \end{aligned} \tag{2.10}$$

$$S(X,\xi) = 2nk\eta(X), \tag{2.11}$$

$$Q\phi - \phi Q = 2[2(n-1) + \mu]h\phi, \tag{2.12}$$

$$\begin{aligned} S(X,Y) &= [2(n-1) - n\mu]g(X,Y) + [2(n-1) + \mu]g(hX,Y) \\ &+[2(1-n) + n(2k+\mu)]\eta(X)\eta(Y), \quad n \ge 1, \end{aligned} \tag{2.13}$$

$$r = 2n(2n - 2 + k - n\mu), \tag{2.14}$$

where S is the Ricci tensor of type $(0,2)$, Q is the Ricci-operator, i.e., $g(QX,Y) = S(X,Y)$ and r is the scalar curvature of the manifold. From (2.4), it follows that

$$(\nabla_X \eta)(Y) = g(X + hX, \phi Y). \tag{2.15}$$

We now state some results which will be used later on.

Lemma 2.1. [2]. *A contact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$ with $R(X,Y)\xi$=0 for all vector fields X, Y is locally isometric to the Riemannian product of a flat (n+1)-dimensional manifold and an n-dimensional manifold of positive curvature 4, i.e., $E^{n+1}(0) \times S^n(4)$.*

Lemma 2.2. [4]. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a contact metric manifold with ξ belonging to the (k,μ)-nullity distribution. Then $k \leq 1$. If $k = 1$, then $h = 0$ and M^{2n+1} is Sasakian. If $k < 1$, then M^{2n+1} admits three mutually orthogonal and integrable distributions $D(0)$, $D(\lambda)$ and $D(-\lambda)$ determined by the eigenspaces of h where $\lambda = \sqrt{1-k}$.*

Lemma 2.3. [4]. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a contact metric manifold with ξ belonging to the (k,μ)-nullity distribution. If $k < 1$, then for any X orthogonal to ξ, the ξ-sectional curvature $K(X,\xi)$ is given by*

$$\begin{aligned} K(X,\xi) = k + \mu g(hX,X) &= k + \lambda\mu \quad \text{if } X \in D(\lambda) \\ &= k - \lambda\mu \quad \text{if } X \in D(-\lambda). \end{aligned}$$

Lemma 2.4. [3]. *Let M^3 be a contact metric manifold on which $Q\phi = \phi Q$. Then M^3 is either Sasakian, flat or locally isometric to a left invariant metric on the Lie group $SU(2)$ or $SL(2,R)$.*

Lemma 2.5. [6] [8]. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a contact metric manifold with harmonic curvature tensor and ξ belonging to the (k,μ)-nullity distribution. Then M is either (i) an Einstein Sasakian manifold, or (ii) an η-Einstein manifold, or (iii) locally isometric to the Riemannian product $E^{n+1}(0) \times S^n(4)$ including a flat contact metric structure for $n = 1$.*

3. Projectively flat (k,μ)-contact metric manifolds. In this section we consider a (k,μ)-contact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$ which is projectively flat. The projective curvature tensor is given by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y]. \tag{3.1}$$

Since the manifold under consideration is projectively flat, (3.1) yields us

$$R(X,Y)Z = \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y]. \tag{3.2}$$

Substituting Z by ξ in (3.2) and using (2.5) and (2.11) we obtain

$$\mu[\eta(Y)hX - \eta(X)hY] = 0. \tag{3.3}$$

Putting $Y = \xi$ in (3.3) we get either $\mu = 0$ or $h = 0$. If $h = 0$, then (2.6) implies that $k = 1$ and hence the manifold is Sasakian. Again a projectively flat Sasakian manifold is of constant curvature 1. We now consider the case for $\mu = 0$ and $k \neq 1$. Since in a (k,μ)-contact metric manifold $k \leq 1$ (by Lemma 2.2), it follows that $k < 1$. Hence we classify the case for $\mu = 0$ and $k < 1$. Then putting $X = \xi$ in (3.2) and using (2.9) and (2.11) we obtain

$$k[g(Y,Z)\xi - \eta(Z)Y] = \frac{1}{2n}[S(Y,Z)\xi - 2nk\eta(Z)Y].$$

Taking the inner product on both sides by ξ of the above relation we get

$$S(Y,Z) = 2nkg(Y,Z). \tag{3.4}$$

Again for $\mu = 0$, (2.13) takes the form

$$\begin{aligned} S(Y,Z) &= 2(n-1)g(Y,Z) + 2(n-1)g(hY,Z) \\ &\quad + [2(1-n) + 2nk]\eta(Y)\eta(Z). \end{aligned} \tag{3.5}$$

From (3.4) and (3.5) it follows that

$$(2n-2-2nk)g(Y,Z) + (2n-2)g(hY,Z) = (2n-2-2nk)\eta(Y)\eta(Z). \tag{3.6}$$

Substituting Z by ξ in (3.6) we obtain $(2n-2)g(hY,Z) = 0$ for all Y, Z. This gives either $n = 1$ or $h = 0$. But if $h = 0$, then $k = 1$, contradicts to the assumption that $k < 1$.Hence we must have $n = 1$. Thus for $\mu = 0$ and $k < 1$, the (k,μ)-contact metric manifold reduces to a 3-dimensional manifold M^3. Again, since $\mu = 0$, the condition $\xi \in N(k,\mu)$ reduces to

$\xi \in N(k)$ which is equivalent to the condition $Q\phi = \phi Q$ for 3-dimensional manifold. Consequently using Lemma 2.4 we obtain that(for $\mu = 0$, $k < 1$ and $n = 1$) either the manifold is flat (for $k = 0 = \mu$) or locally isometric to a left invariant metric on the Lie group SU(2) or SL(2, R) [SU(2) for $0 < k < 1$ and SL(2, R) for $k < 0$].

We now classify the case for $\mu = 0$, $k < 1$ and $n > 1$, i.e., for greater than 3-dimensional manifold. Then putting $Y = hY$ in (3.6) and using (2.6) we obtain

$$(2n-2-2nk)g(hY,Z) + (1-k)(2n-2)[g(Y,Z) - \eta(Y)\eta(Z)] = 0. \tag{3.7}$$

Again from (3.6) we have (as $n > 1$)

$$g(hY,Z) = \frac{(2n-2-2nk)}{2n-2}[-g(Y,Z) + \eta(Y)\eta(Z)]. \tag{3.8}$$

Using (3.8) in (3.7) we get

$$\left[\frac{(2n-2-2nk)^2}{2n-2} + (k-1)(2n-2)\right]\{-g(Y,Z) + \eta(Y)\eta(Z)\} = 0.$$

This gives us $k[n^2k - (n-1)(n+1)] = 0$, which yields either $k = 0$, or $k = \frac{n^2-1}{n^2} < 1$. Hence for $\mu = 0$, $k < 1$ and $n > 1$ we have either $k = 0$, or $k = \frac{n^2-1}{n^2}$. Thus for greater than 3-dimensional (k,μ)-contact metric manifold we obtain either $k = 0 = \mu$, or $\mu = 0$ and $k = \frac{n^2-1}{n^2}(\neq 0)$. For $k = 0 = \mu$, using Lemma 2.1 we get that the manifold $M^{2n+1}(n > 1)$ is locally isometric to the Riemannian product $E^{n+1}(0) \times S^n(4)$.

Finally we consider the case for $\mu = 0$, $k = \frac{n^2-1}{n^2} < 1$ and $n > 1$. Let $\{e_i : i = 1, 2, ..., 2n+1\}$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $Y = Z = e_i$ in (3.6) and taking summation over $1 \leq i \leq 2n+1$ we get

$$k = \frac{n-1}{n} < 1.$$

Hence we must have (for $n > 1$)

$$\frac{n^2-1}{n^2} = \frac{n-1}{n},$$

which implies $1 = 0$, absurd. Thus the case $k = \frac{n^2-1}{n^2}(\neq 0) < 1$ and $n > 1$ can not occur. Consequently we must have $k = 0 = \mu$ for $k < 1$ and $n > 1$. Hence considering all the cases we can state the following:

Theorem 1. *A projectively flat* (k,μ)*-contact metric manifold* $M^{2n+1}(\phi, \xi,\eta,g)$ *is either* (i) *locally isometric to the Riemannian product* $E^{n+1}(0)\times S^n(4)$ *including the 3-dimensional case, or* (ii) *locally isometric to a Sasakian manifold of constant curvature 1, or* (iii) *locally isometric to a left invariant metric on the Lie group SU(2) or SL(2, R).*

4. (k,μ)**- contact metric manifolds satisfying the condition** $R(X,\xi)\cdot P=0$**.** This section deals with a (k,μ)- contact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$ satisfying the condition

$$R(X,\xi)\cdot P=0. \tag{4.1}$$

From (4.1) we have

$$\begin{aligned} &g(R(\xi,Y)P(U,V)W,\xi)-g(P(R(\xi,Y)U,V)W,\xi)\\ &\quad -g(P(U,R(\xi,Y)V)W,\xi)-g(P(U,V)R(\xi,Y)W,\xi)=0. \end{aligned} \tag{4.2}$$

In view of (2.10) and (2.11) we get from (3.1) that

$$\begin{aligned} &\eta(P\ \ (X,Y)Z)=k[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)]+\mu[g(hY,Z)\eta(X)\\ &\quad -g(hX,Z)\eta(Y)]-\frac{1}{2n}[S(Y,Z)\eta(X)-S(X,Z)\eta(Y)]. \end{aligned} \tag{4.3}$$

Also from (3.1), it follows that

$$\eta(P(X,Y)\xi)=0 \text{ for all } X,Y. \tag{4.4}$$

Replacing X by ξ in (4.3) and then using (2.11) and $h\xi=0$ we get

$$\eta(P(\xi,Y)Z)=kg(Y,Z)+\mu g(hY,Z)-\frac{1}{2n}S(Y,Z). \tag{4.5}$$

Now using (2.9), $h\xi=0$ and (4.4) we obtain from (4.2) by straightforward calculation

$$\begin{aligned} &k[\tilde{P}(U,V,W,Y)-\eta(Y)\eta(P(U,V)W)-g(Y,U)\eta(P(\xi,V)W)\\ &\quad +\eta(U)\eta(P(Y,V)W)-g(Y,V)\eta(P(U,\xi)W)\\ &\quad +\eta(V)\eta(P(U,Y)W)+\eta(W)\eta(P(U,V)Y)]\\ &\quad +\mu[\tilde{P}(U,V,W,hY)-g(hY,U)\eta(P(\xi,V)W)\\ &\quad +\eta(U)\eta(P(hY,V)W)-g(hY,V)\eta(P(U,\xi)W)\\ &\quad +\eta(V)\eta(P(U,hY)W)+\eta(W)\eta(P(U,V)hY)]=0, \end{aligned} \tag{4.6}$$

where $\tilde{P}(U,V,W,Y) = g(P(U,V)W,Y)$.
We now consider the following cases :
case 1. $k = 0 = \mu$,
case 2. $k = 0, \mu \neq 0$,
case 3. $k \neq 0, \mu = 0$,
case 4. $k \neq 0, \mu \neq 0$.
For case 1, we have from(2.5) that $R(X,Y)\xi = 0$ for all X, Y and hence by Lemma 2.1, the manifold is locally the Riemannian product $E^{n+1}(0) \times S^n(4)$. For case 2, it follows from (4.6) that

$$\begin{aligned} &\tilde{P}(U,V,W,hY) - g(hY,U)\eta(P(\xi,V)W) + \eta(U)\eta(P(hY,V)W) \\ &\quad -g(hY,V)\eta(P(U,\xi)W) + \eta(V)\eta(P(U,hY)W) \\ &\quad +\eta(W)\eta(P(U,V)hY) = 0. \end{aligned} \tag{4.7}$$

Since $k = 0$, (2.6) yields us

$$h^2X = X - \eta(X)\xi \text{ for all } X. \tag{4.8}$$

Putting $Y = hY$ in (4.7) and then using (4.8) we obtain by virtue of (4.4) that

$$\begin{aligned} \tilde{P}(U,V,W,Y) &-\eta(Y)\eta(P(U,V)W) - g(Y,U)\eta(P(\xi,V)W) \\ &+\eta(U)\eta(P(Y,V)W) - g(Y,V)\eta(P(U,\xi)W) \\ &+\eta(V)\eta(P(U,Y)W) + \eta(W)\eta(P(U,V)Y) = 0. \end{aligned} \tag{4.9}$$

Putting $Y = U = e_i$ in (4.9) and then taking summation for $1 \leq i \leq 2n+1$, we get

$$-2n\eta(P(\xi,V)W) + \eta(W)\eta(P(e_i,V)e_i) = 0. \tag{4.10}$$

From (3.1), it follows that (in this case)

$$\eta(P(e_i,V)e_i) = 0. \tag{4.11}$$

Using (4.11) and (4.5) in (4.10) we obtain

$$S(V,W) = 2n\mu g(hV,W). \tag{4.12}$$

By virtue of (4.12) we obtain from (4.3) that (as $k = 0$)

$$\eta(P(X,Y)Z) = 0 \quad \text{for all } X, Y, Z. \tag{4.13}$$

In view of (4.13), we get from (4.9) $\tilde{P}(U,V,W,Y)=0$ for all U,V,W,Y, i.e., $P(U,V)W=0$ for all U,V,W. Thus the manifold is projectively flat.

Next for case 3, (4.6) gives us the relation (4.9) and hence proceeding in the same way as in the case 2, we can show that the manifold is projectively flat in this case also.

Finally we consider the case 4. In this case putting $Y=hY$ in (4.6) and then using (2.6) we get

(4.14)
$$\begin{aligned}
&k[\tilde{P}(U,V,W,hY)-g(hY,U)\eta(P(\xi,V)W)+\eta(U)\eta(P(hY,V)W)\\
&\quad -g(hY,V)\eta(P(U,\xi)W)+\eta(V)\eta(P(U,hY)W)+\eta(W)\eta(P(U,V)hY)]\\
&\quad -\mu(k-1)[\tilde{P}(U,V,W,Y)-\eta(Y)\eta(P(U,V)W)-g(Y,U)\eta(P(\xi,V)W)\\
&\quad +\eta(U)\eta(P(Y,V)W)-g(Y,V)\eta(P(U,\xi)W)+\eta(V)\eta(P(U,Y)W)\\
&\quad +\eta(W)\eta(P(U,V)Y)]=0.
\end{aligned}$$

From (4.6) and (4.14) (since $k\neq 0,\mu\neq 0$), it follows that

(4.15)
$$\begin{aligned}
&[k^2+\mu^2(k-1)][\tilde{P}(U,V,W,Y)-\eta(Y)\eta(P(U,V)W)\\
&\quad -g(Y,U)\eta(P(\xi,V)W)+\eta(U)\eta(P(Y,V)W)-g(Y,V)\eta(P(U,\xi)W)\\
&\quad +\eta(V)\eta(P(U,Y)W)+\eta(W)\eta(P(U,V)Y)]=0.
\end{aligned}$$

Now $k^2+\mu^2(k-1)\neq 0$ means $\{k+\mu\sqrt{1-k}\}\{k-\mu\sqrt{1-k}\}\neq 0$ which implies $k+\mu\sqrt{1-k}\neq 0$ and $k-\mu\sqrt{1-k}\neq 0$. Also

$$TM=[\xi]\oplus[D(\lambda)]\oplus[D(-\lambda)],$$

where $D(\lambda)$ (resp. $D(-\lambda)$) is the distribution defined by vector fields $hX=\lambda X$ (resp. $hX=-\lambda X$), $\lambda=\sqrt{1-k}$ (which follows from (2.6)). Hence the relation $k^2+\mu^2(k-1)\neq 0$ basically means that the sectional curvatures of plane sections containing ξ are non-vanishing i.e., $K(X,\xi)\neq 0$ for any vector field $X\perp\xi$. Again, from Lemma 2.3, it follows that $\mathrm{K}(X,\xi)=0$ if and only if

$$\begin{aligned}
k+\lambda\mu=0 &\quad \text{for } X\in D(\lambda)\\
k-\lambda\mu=0 &\quad \text{for } X\in D(-\lambda),
\end{aligned}$$

where $\lambda=\sqrt{1-k}$. Then we have $k+\mu\sqrt{1-k}=0$ and $k-\mu\sqrt{1-k}=0$. This two relations gives us $k=0=\mu$. But in this case (i.e., for case 4) we have assumed that $k\neq 0$ and $\mu\neq 0$. Consequently we must have $K(X,\xi)\neq$

0 for all $X \perp \xi$ in this case. Hence we must have $k^2 + \mu^2(k-1) \neq 0$. Then (4.15) implies that the relation (4.9) holds and hence (4.10) also holds. From (3.1) we have

$$\eta(P(e_i, V)e_i) = -2nk\eta(V). \tag{4.16}$$

In view of (4.16) and (4.5) we obtain from (4.10) that

$$S(V,W) = 2nkg(V,W) + 2n\mu g(hV,W) + 2nk\eta(V)\eta(W). \tag{4.17}$$

Using (4.17) in (4.3) we get (for $k \neq 0$, $\mu \neq 0$) the relation (4.13) and hence as before the manifold is projectively flat. Therefore considering all the cases we see that the manifold satisfying $R(X,\xi)\cdot P = 0$ is projectively flat. Hence by virtue of Theorem 1, we can state the following:

Theorem 2. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a (k,μ)- contact metric manifold satisfying the condition $R(X,\xi)\cdot P = 0$. Then the manifold is either* (i) *locally isometric to the Riemannian product $E^{n+1}(0)\times S^n(4)$ including the 3-dimensional case, or* (ii) *locally isometric to a Sasakian manifold of constant curvature 1, or* (iii) *locally isometric to a left invariant metric on the Lie group SU(2) or SL(2, R).*

5. Projectively recurrent (k,μ)- contact metric manifolds. A non-flat Riemannian manifold M is said to be projectively recurrent if the projective curvature tensor P satisfies the condition $\nabla P = \alpha \otimes P$, where α is an everywhere non-zero 1-form. We now define a function f on M by $f^2 = g(P,P)$, where the metric g is extended to the inner product between the tensor fields in the standard fashion. Then we know that $f(Yf) = f^2\alpha(Y)$. So from this we have $Yf = f\alpha(Y)$, because f≠0. This implies that

$$X(Yf) = \frac{1}{f}(Xf)(Yf) + (X\alpha(Y))f.$$

Hence $X(Yf) - Y(Xf) = \{X\alpha(Y) - Y\alpha(X)\}f$. Therefore we get

$$(\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})f = \{X\alpha(Y) - Y\alpha(X) - \alpha([X,Y])\}f.$$

Since the left hand side of the above equation is identically zero and f≠0 on M by our assumption, we obtain

$$d\alpha(X,Y) = 0, \text{ that is, the 1-form } \alpha \text{ is closed.} \tag{5.1}$$

Now from $(\nabla_X P)(U,V)Z = \alpha(X)P(U,V)Z$, we get

$$(\nabla_U \nabla_V P)(X,Y)Z = \{U\alpha(V) + \alpha(U)\alpha(V)\}P(X,Y)Z.$$

Hence using (5.1) we get

$$(R(X,Y)\cdot P)(U,V)Z = [2d\alpha(X,Y)]P(U,V)Z = 0.$$

Therefore, for a projectively recurrent manifold, we have

$$R(X,Y)\cdot P = 0 \ \ \forall X,Y,$$

which implies that $R(X,\xi)\cdot P = 0$. Thus by virtue of Theorem 2, we can state the following:

Theorem 3. *A projectively recurrent (k,μ)- contact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$ is either* (i) *locally isometric to the Riemannian product $E^{n+1}(0)\times S^n(4)$ including the 3-dimensional case, or* (ii) *locally isometric to a Sasakian manifold of constant curvature 1, or* (iii) *locally isometric to a left invariant metric on the Lie group SU(2) or SL(2, R).*

Since for a projectively symmetric contact metric manifold, we have $(\nabla_U P)(X,Y)Z = 0$, which implies $R(X,Y)\cdot P = 0$, we can state the following:

Corollary. *A projectively symmetric (k,μ)- contact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$ is either* (i) *locally isometric to the Riemannian product $E^{n+1}(0)\times S^n(4)$ including the 3-dimensional case, or* (ii) *locally isometric to a Sasakian manifold of constant curvature 1, or* (iii) *locally isometric to a left invariant metric on the Lie group SU(2) or SL(2, R).*

We now consider a (k,μ)- contact metric manifold satisfying the condition $divP = 0$. From (4.1) we get

$$(divP)(X,Y)Z = \frac{2n-1}{2n}[(\nabla_X S)(Y,Z) - ((\nabla_Y S)(X,Z)].$$

Then $divP = 0$ if and only if $(\nabla_X Q)Y = (\nabla_Y Q)X$, where $S(X,Y) = g(QX,Y)$. This means that the curvature tensor R of the manifold is harmonic. Consequently by virtue of Lemma (2.5), we can state the following:

Theorem 4. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a (k,μ)-contact metric manifold satisfying the condition $divP = 0$. Then M is either* (i) *an Einstein*

Sasakian manifold, or (ii) *an η-Einstein manifold, or* (iii) *locally isometric to the Riemannian product $E^{n+1}(0) \times S^n(4)$ including a flat contact metric structure for $n = 1$.*

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