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STABILITY PROBLEMS FOR DIFFERENCE EQUATIONS

BY

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Abstract. In this paper are investigated certain properties of some perturbed Volterra difference equations in connection with the h-stability of a linear Volterra difference equation by combining the method of the resolvent matrix solution of the linear equation and the theory of difference inequalities.

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1. Introduction. In recent years, considerable attention has been paid to the development of qualitative theory of difference equations, particularly, to the stability problems.

MEDINA and PINTO [8]-[11] extended the notion of exponential stability to a variety of reasonable systems called h-systems. The new concept, called h-stability, permits to obtain an uniform treatment for the concept of stability to difference equations. They have obtained results about stability which were, at least, weaker that those given by exponential stability and uniform Lipschitz stability under some perturbations.

In this paper, we shall investigate certain stability properties of some perturbed Volterra difference equations in connection with the h-stability of a linear Volterra difference equation by combining the method of the resolvent matrix solution of the linear equation and the theory of difference equations. The method of the matrix resolvent has been used in [2], [6], [7], [9], interesting results being obtained.

2. Preliminaries. We adopt the following notations in this paper: $N(n_0) = \{n_0, n_0+1, \dots\}$, where $n_0 \in \mathbb{N}$, $N_m(n_0) = \{n_0, n_0+1, \dots, n_0+m\}$,

$m \in \mathbb{N}$; $\mathbb{R}^q$ the q-dimensional real Euclidean space with norm $|x| = \sum_1^q |x_i|$, $x = (x_1, x_2, \ldots, x_q) \in \mathbb{R}^q$. For a $q \times q$ matrix $A = (a_{ij})$, define its norm $|A|$ by $|A| = \max_j \sum_{j=1}^{n} |a_{ij}|$.

Let $A(n)$ and $K(n,s)$ be $q \times q$ nonsingular matrices whose elements are real functions defined on $N(n_0)$, respectively, $N(n_0) \times N(n_0)$ and $\Delta u(n) = u(n+1) - u(n)$.

In the sequel we are interested in different relations between the solutions of the unperturbed linear Volterra difference equation

$$\Delta x(n) = A(n)\, x(n) + \sum_{s=n_0}^{n-1} K(n,s)\, x(s), \tag{2.1}$$

and the solutions of its perturbed Volterra type equations

$$\Delta y(n) = A(n)\, y(n) + \sum_{s=n_0}^{n-1} K(n,s)\, y(s) + \sum_{s=n_0}^{n-1} g(n,s,y(s)), \tag{2.2}$$

or

$$\begin{aligned} \Delta y(n) = A(n)\, y(n) &+ \sum_{s=n_0}^{n-1} K(n,s)\, y(s) \\ &+ \sum_{s=n_0}^{n-1} g(n,s,y(s)) + f(n,y(n)), \end{aligned} \tag{2.3}$$

where $x(n), y(n) \in \mathbb{R}^q$, $f(n,y)$ and $g(n,s,y)$ are real q-vector functions defined on $N(n_0) \times D$, respectively, for $n \geq s \geq n_0$ and $y \in D$, where $D = \{y \in \mathbb{R}^q : |y| < H \leq \infty\}$. Let $x(n) = x(n, n_0, x_0)$ be a solution of Eq. (2.1) with $x(n_0, n_0, x_0) = x_0$ and, respectively, $y(n) = y(n, n_0, y_0)$ a solution of Eqs. (2.2) or (2.3) with $y(n_0, n_0, y_0) = y_0$. We will assume that the above solutions are unique and, in addition, that $f(n,0) = g(n,s,0) = 0$, $n \geq s \geq n_0$.

We now recall some definitions of stability of the zero solution which will be used in the sequel.

Definition 2.1. *The solution $x = 0$ of Eq, (2.1) is said to be:*

a) *stable (S) if, for every* $\varepsilon > 0$ *and* $n_0 \in \mathbb{N} \equiv N(1)$, *there exists* $\delta(\varepsilon, n_0) > 0$ *such that,* $|x_0| < \delta$ *and* $n \in N(n_0)$ *imply* $|x(n, n_0, x_0)| < \varepsilon$;

b) *uniformly stable (US) if it is stable and the above* δ *is independent of* n_0;

c) *asymptotically stable (AS) if it is stable and, for any* $n_0 \in \mathbb{N}$, *there exists* $\delta_0(n_0) > 0$ *such that* $|x_0| < \delta$ *implies* $|x(n, n_0, x_0)| \to 0$ *as* $n \to \infty$ *(i. e., it is attractive (Att));*

d) *exponentially stable (ES) if there exists* $0 < p < 1$ *and for every* $\alpha > 0$ *and* $n_0 \in \mathbb{N}$ *there exists* $\delta(\alpha, n_0) > 0$ *such that* $n \in N(n_0)$ *and* $|x_0| < \delta$ *imply* $|x(n, n_0, x_0)| \leq \alpha |x_0| p^{n-n_0}$;

e) *uniformly Lipschitz stable (ULS) if there exist* $M \geq 1$ *and* $\delta > 0$ *such that* $|x(n, n_0, x_0)| \leq M |x_0|$ *whenever* $|x_0| \leq \delta$ *and* $n \in N(n_0)$.

Definition 2.2. *The solution* $x = 0$ *if Eq. (2.1) is said to be:*

a) *h-stable (hS) if* $c \geq 1$, $\delta > 0$ *exist as well as a positive and bounded function* $h : N(n_0) \to \mathbb{R}$ *such that* $|x(n, n_0, x_0)| \leq c h(n) h^{-1}(n_0) |x_0|$, *for* $n \in N(n_0)$ *and* $|x_0| < \delta$;

b) *globally h-stable (GhS) if* $|x(n, n_0, x_0)| \leq c h(n) h^{-1}(n_0) |x_0|$ *holds for* $n \in N(n_0)$ *and every* $x_0 \in D$, *where* $D \subseteq \mathbb{R}^q$ *is a region which includes the origin* ($h^{-1}(n) = 1/h(n)$).

The notions of stability given in Definition 2.2 include several types of known stabilities as (US), (ES) and (ULS).

The resolvent matrix $R(n, s)$ of (2.1) ([9, Theorem 2.1]) is defined as the unique solution of the matrix difference equation

$$\Delta_l R(n, l) = -R(n, l+1) A(l) - \sum_{s=l+1}^{n-1} R(n, s+1) K(s, l), \quad n > l, \tag{2.4}$$

with $R(n, n) = I$, the identity matrix.

Using the resolvent matrix $R(n, l)$ the following variation of constant formula can be given.

Lemma 2.1. ([2; Lemma 2.2]) *The unique solution* $y(n, n_0, y_0)$ of Eq. (2.2) *is given by*

$$y(n, n_0, y_0) = R(n, n_0)\, y_0 + \sum_{l=n_0}^{n-1} R(n, l+1) \sum_{s=n_0}^{l-1} g(l, s, y(s)). \tag{2.5}$$

In view to develop relationships between the (hS) of $x = 0$ to Eq. (2.1) and different kind of stability to $y = 0$ of Eq. (2.2) or (2.3), we need of a discrete inegality of Volterra type which is a generalization of the well-known Bellman-Bihari inequality.

Lemma 2.2. ([3; Lemma 2.1]) *Let there exist functions* $u(n)$, $v(n)$, $h(n, s)$ *and* $\omega(p)$ *such that:*

a) $u(n)$, $v(n)$, $h(n, s)$ *are nonnegative for* $n \geq s \geq n_0$*;*

b) $\omega(\rho)$ *is positive, continuous and nondecreasing for* $\rho > 0$*;*

c) for any $n \in N(n_0)$ *the inequality*

$$u(n) \leq c + \sum_{l=n_0}^{n-1} \left(v(l)\, \omega(u(l)) + \sum_{s=n_0}^{l} h(l, s)\, \omega(u(s)) \right),$$

holds, where $c \geq u(n_0)$ *is a positive constant. Then, for any* $n \in N(n_0)$ *one obtains*

$$\Omega(u(n)) \leq \Omega(c) + \sum_{l=n_0}^{n-1} \left(v(l) + \sum_{s=n_0}^{l} h(l, s) \right), \tag{2.6}$$

where $\Omega(z) = \int \frac{dz}{\omega(z)}$, $z > 0$.

Some interesting particular cases are given in [10; Remark 2.1]. The following results will be also used in the sequel.

Lemma 2.3. ([4; Lemma 2.2]) *Let* $h(n)$ *be a positive function defined for* $n \in N(n_0)$. *Assume there exists a positive constant* L *such that form* $n \in N(n_0)$, *the inequality*

$$\sum_{s=n_0}^{n} h(n)\, h^{-1}(s) \leq L \tag{2.7}$$

holds.

Then, $h(n) \to 0$ *as* $n \to \infty$.

Lemma 2.4. ([9; Theorem 7]) *The solution* $x = 0$ *of* Eq. (2.1) *is (hS) if and only if, there exists a constant* $c \geq 1$ *and a positive bounded function* $h : N(n_0) \to \mathbb{R}$ *such that* $|R(n, n_0)| \leq ch(n) h^{-1}(n_0), \quad n \in N(n_0)$, *where* $R(n, n_0)$ *is the resolvent matrix of* Eq. (2.1).

3. Main results. In this section we shall examine the relationships between the (hS) of $x = 0$ to Eq. (2.1) and those of (S) or (AS) of $y = 0$ to Eq. (2.2) or (2.3).

With respect to the function $g(n, s, y)$ we make the following hypothesis

(H1): $|g(n, s, y)| \leq b(n, s)\omega(|y|)$, $n \geq s \geq n_0$, $|y| < H$, where $b(n, s)$ is a positive function for $n \geq s \geq n_0$, and $\omega(\rho)$ is a continuous and nondecreasing function for $\rho > 0$ with $\Omega(0+) = -\infty$.

Theorem 3.1. *Assume that* $x = 0$ *of Eq. (2.1) is (hS) and hypothesis* (H1) *holds. If, in addition,*

$$h^{-1}(l+1) \sum_{s=n_0}^{l} b(l, s) \in l_1(n_0), \tag{3.1}$$

then, $y = 0$ *of* Eq. (2.2) *is stable.*

Proof. Firstly, we note that from the definition of (hS) there exists a positive constant T such that $h(n) \leq T$, $n \in N(n_0)$. Then, using (2.5), the hypothesis (H1), (3.1) and Lemma 2.4 one obtains

$$\begin{aligned} |y(n)| &\leq |R(n, n_0)|\,|y_0|\,| + \sum_{l=n_0}^{n-1} |R(n, l+1)| \sum_{s=n_0}^{l-1} |g(l, s, y(s))| \\ &\leq cTh^{-1}(n_0)\,|y_0| + cT \sum_{l=n_0}^{n-1} h^{-1}(l+1) \sum_{s=n_0}^{l} b(l, s)\,\omega(|y(s)|), \end{aligned}$$

for $n \in N(n_0)$. From here, applying Lemma 2.2 in particular case $v(n) \equiv 0$, we get $|y(n)| \leq \varphi(|y_0|)$ for $n \in N(n_0)$, where

$$\begin{aligned} \varphi(|y_0|) &= \Omega^{-1}\left(\Omega\left(cTh^{-1}(n_0)\,|y_0|\right) + cT \sum_{l=n_0}^{n-1} h^{-1}(l+1) \sum_{s=n_0}^{l} b(l, s)\right) \\ &\leq \Omega^{-1}\left(\Omega\left(cTh^{-1}(n_0)\,|y_0|\right) + c_1\right), \end{aligned}$$

with $c_1 = cT \sum\limits_{l=n_0}^{\infty} h^{-1}(l+1) \sum\limits_{s=n_0}^{l} b(l,s) < \infty$.

From $\Omega(0+) = -\infty$, we can choose $|y_0|$ small enough such that

$$\Omega\left(cTh^{-1}(n_0)|y_0|\right) + c_1 \in dom\,\Omega^{-1} \equiv (-\infty, \Omega(\infty)).$$

On the other hand, if we take $\delta(\varepsilon, n_0) = c^{-1}h(n_0)T^{-1}\Omega^{-1}(\Omega(\varepsilon) - c_1)$, it follows that $|y(n)| = |y(n, n_0, y_0)| < \varepsilon$ for $|y_0| < \delta(\varepsilon, n_0)$, i. e., the stability of $y = 0$ of Eq. (2.2). This completes the proof. □

If, in the proof of Theorem 3.1 we shall use the result of Fubini type [1; Lemma 2]

$$\sum_{l=n_0}^{n-1} a(n, l+1) \sum_{s=n_0}^{l-1} b(l,s) = \sum_{l=n_0}^{n-1} \sum_{s=l+1}^{n-1} a(n, s+1)\, b(s,l), \tag{3.2}$$

then we can give

Corollary 3.1. *Assume that* $x = 0$ *of* Eq. (2.1) *is* (hS) *and* (H1) *holds. If, in addition,*

$$\sum_{s=l}^{\infty} h^{-1}(s+1)\, b(s,l) \in l_1(n_0), \tag{3.3}$$

then, $y = 0$ *of* Eq. (2.2) *is stable.*

Proof. In this case one obtains

$$|y(n)| \le cTh^{-1}(n_0) + cT \sum_{l=n_0}^{n-1} \left(\sum_{s=l}^{\infty} h^{-1}(s+1)\, b(s,l) \right) \omega(|y(l)|),$$

and therefore, by Lemma 2.2,

$$|y(n)| \le cTh^{-1}(n_0)|y_0| \exp(cc_1T), \quad c_1 = \sum_{l=n_0}^{\infty} \sum_{s=l}^{\infty} h^{-1}(s+1)\, b(s,l).$$

From here, it follows the stability of $y = 0$ of Eq. (2.2). □

Although, in general, $y = 0$ of Eq. (2.2) is not (hS) or (GhS), it is when $\omega(\rho) = \rho^{\alpha}$, $\alpha > 0$. More precisely, we have

Corollary 3.2. *Assume that* $x = 0$ *of* Eq. (2.1) *is* (hS) *and hypothesis* (H1) *holds with* $\omega(\rho) = \rho^{\alpha}$, $\alpha > 1$. *If, in addition*

$$\sum_{l=n_0}^{\infty} h^{-1}(l+1) \sum_{s=n_0}^{l} h^{\alpha}(s)\, b(l,s) = c_1 < \infty, \tag{3.4}$$

then, $y = 0$ *of* Eq. (2.2) *is also* (hS).

Proof. Indeed, since $\alpha > 1$, we have

$$h^{-1}(n)\,|y(n)| \leq ch^{-1}(n_0)\,|y_0| + c\sum_{l=n_0}^{n-1} h^{-1}(l+1) \sum_{s=n_0}^{l} h^{\alpha}(s)\, b(l,s)\left[h^{-\alpha}(s)\,|y(s)|\right]$$

and, in this case, $\Omega : (0,\infty) \to (-\infty, 0)$, $\Omega(z) = z^{1-\alpha}/(1-\alpha)$ and $\Omega^{-1} : (-\infty, 0) \to (0,\infty)$, $\Omega^{-1}(w) = [(1-\alpha)w]^{-1/(\alpha-1)}$. Again, by Lemma 2.2, one obtains

$$|y(n)| \leq h(n)\,\varphi(|y_0|) \tag{3.5}$$

for $|y_0|$ small enough, where

$$\varphi(|y_0|) = ch^{-1}(n_0)\,|y_0| \left[1 - (\alpha-1)\,c\left(ch^{-1}(n_0)\,|y_0|\right)^{\alpha-1} \cdot \sum_{l=n_0}^{n-1} h^{-1}(l+1) \sum_{s=n_0}^{l} h^{\alpha}(s)\, b(l,s)\right]^{-1/(\alpha-1)},$$

for $n \in N(n_0)$. If we denote $u(n) = ch^{-1}(n_0)\,|y_0|$, $\gamma = \alpha - 1 > 0$, $\beta(n) = c\sum\limits_{l=n_0}^{n-1} h^{-1}(s+1) \sum\limits_{s=n_0}^{l} h^{\alpha}(s)\, b(l,s)$, then $\varphi(|y_0|) = u\left[1 - \gamma u^{\gamma}\beta(n)\right]^{-1/\gamma} \leq u\left(1 - \gamma u^{\gamma} cc_1\right)^{-1/\gamma}$ with $\gamma cc_1 u^{\gamma} < 1$ for $|y_0|$ small enough. For a constant $0 < M \leq 1$, from $1 - cc_1 u^{\gamma} \geq M$ one obtains $|y_0| \leq c^{-1} h(n_0)\left(\frac{1-M}{cc_1}\right)^{1/\gamma} = \delta$. Thus, by (3.5), one get $|y(n)| \leq dh(n)\,h^{-1}(n_0)\,|y_0|$, $n \in N(n_0)$, $d = cM^{-1} \geq 1$, i. e., the (hS) of $y = 0$. □

Corollary 3.3. *Assume that* $x = 0$ *of* Eq. (2.1) *is* (hS), *the hypothesis* (H1) *occurs with* $\omega(\rho) = \rho^{\alpha}$, $0 < \alpha \leq 1$, *and condition* (3.4) *is satisfied. Then,* $y = 0$ *of* Eq. (2.2) *is* (GhS).

Proof. We note that if $0 < \alpha < 1$ then $\Omega : [0, \infty) \to [0, \infty)$, $\Omega(z) = z^{1-\alpha}/(1-\alpha)$ and $\Omega^{-1} : [0, \infty) \to [0, \infty)$, $\Omega^{-1}(\omega) = [(1-\alpha) w]^{1/(1-\alpha)}$. By Lemma 2.2 we obtain

$$\begin{aligned} h^{-1}(n) |y(n)| \leq & ch^{-1}(n_0) |y_0| \left[1 + (1-\alpha) \left(ch^{-1}(n_0) |y_0|\right)^{\alpha-1} cc_1\right]^{1/(1-\alpha)} \\ & \leq ch^{-1}(n_0) |y_0| \left[1 + cc_1 (1-\alpha)\right]^{1/(1-\alpha)} \leq ch^{-1}(n_0) |y_0| \exp(cc_1), \end{aligned}$$

for $n \in N(n_0)$, if $|y_0| \leq c^{-1} h(n_0)$. Thus, we have

$$|y(n)| < dh(n) h^{-1}(n_0) |y_0|, \quad n \in N(n_0), \quad d = c \exp(cc_1) > 1,$$

for every $y_0 \in D \equiv \left\{y \in \mathbb{R}^q; |y| < c^{-1} c_2\right\}$, where $0 < c_2 < h(n_0)$. When $\alpha = 1$ then $\Omega : (0, \infty) \to (-\infty, \infty)$, $\Omega(z) = \ln z$ and $\Omega^{-1} : (-\infty, \infty) \to (0, \infty)$, $\Omega^{-1}(w) = e^w$ and therefore

$$\varphi(|y_0|) = \exp\left[\ln\left(ch^{-1}(n_0) |y_0|\right) + cc_1\right] = dh^{-1}(n_0) |y_0|, \quad d = c \exp(cc_1) > 1.$$

From (3.5) one obtains again $|y(n)| < dh(n) h^{-1}(n_0) |y_0|$, $d > 1$, for every y_0, $|y_0| < H$.

Hence $y = 0$ of Eq. (2.2) is (GhS). □

Remark 3.1. By similar arguments previously made it follows that, in particular case when $x = 0$ of Eq. (2.1) is (ULS), (H1) holds and $\sum_{s=n_0}^{l} b(l, s) \in l_1(n_0)$, the zero solution of (2.2) is (US).

We are now going to establish a connection between the (hS) of $x = 0$ to Eq. (2.1) and the (AS) of $y = 0$ to Eq. (2.2).

Theorem 3.2. *Assume that $x = 0$ of* Eq. (2.1) *is* (hS)*, the hypothesis* (H1) *holds and the function $h(n)$ from the definition of* (hS) *is such that $h(n) \to 0$ as $n \to \infty$ and satisfies the boundedness condition*

$$h(n) h^{-1}(s) \leq M < \infty, \quad n_0 \leq s \leq n. \tag{3.5}$$

Moreover, we assume that

$$\sum_{l=n_0}^{\infty} \sum_{s=n_0}^{l} b(l, s) \leq K < (cM)^{-1}, \tag{3.6}$$

and

$$\lim_{m\to\infty}\sum_{s=n_0}^{n} b(m,s)=0 \quad \textit{for all} \quad n\in N(n_0). \tag{3.7}$$

occur. Then, $y=0$ *of* Eq. (2.2) *is* (AS).

Proof. To prove the stability of $y=0$ of Eq. (2.2) we follow the same way as in the proof of Theorem 3.1. Taking into account our hypothesis one obtains

$$|y(n)|\le cM|y_0|+cM\sum_{l=n_0}^{n-1}\sum_{s=n_0}^{l} b(n,s)\,\omega(|y(s)|), \quad n\in N(n_0),$$

from where, by Lemma 2.2 $|y(n)|\le\Omega^{-1}(\Omega(cM|y_0|)+cMK), n\in N(n_0)$. Reasoning as in the final of the proof of Theorem 3.1, it follows the proposed goal.

Further, taking $\varepsilon=1$ it follows the existence of $0<\delta_0=\delta(1,n_0)<1$ such that $n_0\in N$ and $|y_0|<\delta_0$ imply

$$|y(n)|=|y(n,n_0,y_0)|<\min(H,1). \tag{3.8}$$

In the sequel, we consider that solution with $|y_0|<\delta_0$ and assume there exists y_0 and $y(n)=y(n,n_0,y_0)$ such that

$$\limsup_{n\to\infty}|y(n)|=\nu=\omega^{-1}(\mu)>0. \tag{3.9}$$

Since $cMK<1$, there exists a constant θ such that $cMK<\theta<1$. By (3.9), there exists $n_1\in N(n_0)$ such that

$$|y(n)|<\omega^{-1}\left(\mu\theta^{-1}\right) \quad \text{for all} \quad n\in N(n_1). \tag{3.10}$$

From (3.7) there exists $m_1\in N(n_1)$ such that

$$\sum_{s=n_0}^{n_1} b(m,s)\le(\theta-cMK)\,\mu/2\theta\omega(1)\,cM \tag{3.11}$$

for all $m \in N(m_1)$. Then we have

$$|y(n)| \leq ch(n)\left[h^{-1}(n_0)\delta_0 + \sum_{l=n_0}^{m_1-1} h^{-1}(l+1) \sum_{s=n_0}^{l-1} b(l,s)\,\omega(|y(s)|)\right]$$
$$+ c\sum_{l=m_1}^{n-1} h(n)\,h^{-1}(l+1) \sum_{s=n_0}^{n_1-1} b(l,s)\,\omega(|y(s)|)$$
$$+ c\sum_{l=m_1}^{l-1} h(n)\,h^{-1}(l+1) \sum_{s=n_1}^{l-1} b(l,s)\,\omega(|y(s)|).$$

By (3.5) and (3.11) we obtain

$$c\sum_{l=m_1}^{n-1} h(n)\,h^{-1}(l+1) \sum_{s=n_0}^{n_1-1} b(l,s)\,\omega(|y(s)|)$$
$$\leq \frac{cM(\theta - cKM)\mu\omega(1)}{2\theta cM\omega(1)} = \frac{(\theta - cMK)\mu}{2\theta}$$

and by (3.6)

$$c\sum_{l=m_1}^{n-1} h(n)\,h^{-1}(l+1) \sum_{s=n_1}^{l-1} b(l,s)\,\omega(|y(s)|)$$
$$\leq cMK\omega\left(\omega^{-1}\left(\mu\theta^{-1}\right)\right) = \frac{cMK\mu}{\theta}.$$

These yield

$$|y(n)| \leq ch(n)\left[h^{-1}(n_0)\delta_0 + \sum_{l=n_0}^{m_1-1} h^{-1}(l+1) \sum_{s=n_0}^{l-1} b(l,s)\,\omega(|y(s)|)\right]$$
$$+ \frac{(\theta + cMK)\mu}{2\theta}.$$

Since $h(n) \to 0$ as $n \to \infty$, one obtains $\mu \leq \frac{(\theta+cMK)\mu}{2\theta} < \mu$ which is impossible. Consequently, $\lim\limits_{n\to\infty} \sup |y(n)| = 0$, i.e., $y = 0$ of Eq. (2.2) is (Att), hence (AS). The proof is now complete. □

Remark 3.2. Notice that there exist functions $h(n)$ possessing the asked properties. Thus, if we take $h(n) = \prod_{s=0}^{n-1} \alpha(s)$, and therefore $h^{-1}(n_0) = \prod_{s=0}^{n_0-1} \alpha^{-1}(s)$, where $\alpha(s)$ is a positive function for $s \in \mathbb{N}$, then the condition of (hS) becomes $|x(n, n_0, x_0)| \leq c|x_0| \prod_{s=n_0}^{n-1} \alpha(s)$. Such a type of behavior to solutions was widely used in the theory of stability because of it assures the existence of a Liapunov function possessing some properties [12].

If $0 < \alpha(s) = p < 1$, $s \in N(n_0)$, we have $h(n) = p^n \equiv e^{-\lambda n}$, $\lambda > 0$, i. e., $|x(n, n_0, x_0)| \leq c|x_0| \gamma^{n-n_0}$, what means the (ES) of $x = 0$. Obviously, (3.5) is satisfied and $h(n) \to 0$ as $n \to \infty$. More general, $h(n) = p^{a(n)}$ where $\alpha(n) > 0$ and $\alpha(n) \to \infty$ as $n \to \infty$. Then, $|x(n, n_0, x_0)| \leq c|x_0| p^{\alpha(n)-\alpha(n_0)}$, $0 < p < 1$. When $\alpha(s) = (s+1)^{-1}$ one obtains $h(n) = 1/n!$; when $\alpha(s) = 1$ for $s = 0, 1, \ldots, n-2$ and $\alpha(n-1) = 1/n$ we have $h(n) = 1/n$.

Further, let us suppose that $x = 0$ of Eq. (2.1) is (hS) with $h(n) = \prod_{s=0}^{n-1} \alpha(s)$. Then, by Lemma 2.4, we have $|R(n, n_0)| \leq c \prod_{s=n_0}^{n-1} \alpha(s)$, $c \geq 1$, for $n \in N(n_0)$. With respect to the positive function $\alpha(s)$, $s \in N(n_0)$, we make the assumption that for all $n \in N(n_0)$,

$$\sum_{l=n_0}^{n} \sum_{s=l}^{n-1} \alpha(s) \equiv \sum_{l=n_0}^{n} h(n) h^{-1}(l) \leq K. \tag{3.12}$$

According to Lemma 2.3, it follows that $\lim_{n\to\infty} \prod_{s=n_0}^{n} \alpha(s) = 0$. We are now able to formulate:

Corollary 3.4. *Assume that $x = 0$ of* Eq. (2.1) *is* (hS) *with $h(n)$ as above, the hypothesis* (H1) *holds with $\omega(\rho) \equiv \rho$ and $b(n, s)$ satisfies* (3.7) *and*

$$\sup \left\{ \sum_{s=n_0}^{n} b(n, s) : n \in N(n_0) \right\} \leq (cK)^{-1}. \tag{3.13}$$

Then, $y = 0$ of Eq. (2.2) *is* (AS).

Because of the proof follows the same way as that from [5; Theorem 4.1] it will be omitted. Notice that when $x = 0$ of Eq. (2.1) is (EAS), i. e., $h(n) = p^n$, $0 < p < 1$, condition (3.12) occurs with $K = (1-p)^{-1}$. More exactly, we have:

Corollary 3.5. *Assume that* $x = 0$ *of* Eq. (2.1) *is* (ES)*, the hypothesis* (H1) *holds with* $\omega(\rho) \equiv \rho$ *and* $b(n,s)$ *satisfies*

$$\sup\left\{\sum_{s=n_0}^{n} b(n,s) : n \in N(n_0)\right\} \leq \gamma \tag{3.14}$$

with $\gamma < -c^{-1}\ln p$. *Then,* $y = 0$ *of* Eq. (2.2) *is also* (ES).

Proof. If we put $x(n) = \sup\{|y(s)| : s = n_0, n_0+1, \ldots, n\}$, then we have

$$p^{-n}|y(n)| \leq p^{-n}z(n) \leq c|x_0|p^{-n_0} + c\sum_{l=n_0}^{n-1}\left[\sum_{s=n_0}^{l} b(l,s)\right]p^{-l}z(l)\,,\ n \in N(n_0)\,.$$

From here, one obtains

$$|y(n)| \leq c|x_0|p^{n-n_0}e^{c\gamma(n-n_0)} = c|x_0|e^{(\ln p + c\gamma)(n-n_0)} = c|x_0|q^{n-n_0},\ n \in N(n_0)\,,$$

where $q = e^{\ln p + c\gamma} < 1$. □

If in Theorem 3.2 one replace the two conditions regarding $h(n)$ by more restrictive (2.7), then we can formulate the following result:

Theorem 3.3. *Assume that* $x = 0$ *of* Eq. (2.1) *is* (hS) *and that* (H1) *and* (2.7) *occur. In addition, assume that*

$$h^{-1}(l+1)\sum_{s=n_0}^{l} b(l,s) \in l_1(n_0)\,. \tag{3.14}$$

Then, $y = 0$ *of* Eq. (2.2) *is* (AS).

Proof. From the definition of (hS), and from Lemma 2.3, there exists $T > 0$ such that $h(n) \leq T$, $n \in N(n_0)$. By (3.1) there exists a constant

$K>0$ such that $\sum\limits_{l=n_0}^{\infty} h^{-1}(l+1) \sum\limits_{s=n_0}^{l} b(l,s) \leq K$. Reasoning as in the proofs of previous theorems one obtains

$$|y(n)| \leq cTh^{-1}(n_0)\,|y_0| + cT\sum_{l=n_0}^{n-1} h^{-1}(l+1)\sum_{s=n_0}^{l} b(l,s)\,\omega(|y(s)|)\,,\ n\in N(n_0)\,,$$

from where, by Lemma 2.2,

$$|y(n)| \leq \Omega^{-1}\left(\Omega\left(cTh^{-1}(n_0)\,|y_0|\right) + KT\right), \quad n \in N(n_0)\,. \tag{3.15}$$

From (3.15), by the same argument from the proof of Theorem 3.1, it follows the stability of $y=0$ of Eq. (2.2). From its stability, taking $\varepsilon=1$, there exists a $\delta_0=\delta(1,n_0)>0$ such that $|y_0|<\delta_0$ implies $|y(n)|<1$, $n\in N(n_0)$. Thus we get

$$\begin{aligned} |y(n)| \leq ch(n) &\left[h^{-1}(n_0)\,\delta_0 + \sum_{l=n_0}^{n_1-1} h^{-1}(l+1)\sum_{s=n_0}^{l-1} b(l,s)\,\omega(1)\right] \\ &+ ch(n)\sum_{l=n_1}^{n-1} h^{-1}(l+1)\sum_{s=n_0}^{l-1} b(l,s)\,\omega(1)\,, \end{aligned} \tag{3.16}$$

where $n_1\in N(n_0)$, $n_0<n_1<n$, is large enough such that

$$\sum_{l=n_1}^{\infty} h^{-1}(l+1)\sum_{s=n_0}^{l} b(l,s) < \varepsilon T^{-1}.$$

Thus, using (3.16), we get $\lim\limits_{n\to\infty}|y(n)|=0$ if $|y_0|<\delta_0$, because of $h(n)\to 0$ as $n\to\infty$ and $\varepsilon>0$ is arbitrary. This completes the proof. □

Remark 3.3. Notice that for $h(n)=p^n$, $0<p<1$, both conditions (2.7) and (3.5) occur, but for $h(n)=\dfrac{1}{n}$, only (3.5) is satisfied. Finally, when $h(n)=1$ condition (3.5) occurs but $h(n)\not\to 0$ as $n\to\infty$.

Remark 3.4. The above results can be extended to the Eq. (2.3) if as regard to the functions $f(n,y)$ and $g(n,s,y)$ one among the following hypotheses occurs:

(H2): there exist positive functions $a(n)$ and $b(n,s)$ defined for $n \geq s \geq n_0$ such that for $|y| < H$:

a) $|f(n,y)| \leq a(n)\,\omega(|y|)$, $|g(n,s,y)| \leq b(n,s)\,\omega(|y|)$;

b) $|f(n,y)| \leq a(n)\,|y|^{\alpha}$, $|g(n,s,y)| \leq b(n,s)\,|y|^{\alpha}$, $\alpha > 0$;

c) $|f(n,y)| \leq a(n)$, $|g(n,s,y)| \leq b(n,s)\,\omega(|y|)$.

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