ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII “AL.I. CUZA” DIN IAŞI (S.N.)
MATEMATICĂ, Tomul LIII, 2007, f.2

DIFFERENTIABLE M-CONVEX FUNCTIONS AND M-MONOTONE MAPPINGS

BY

MARIUS APETRII

Abstract. In an earlier paper, starting from Crouzeix (1977) characterisation of convex functions by quasiconvexity property of all ”dual perturbations”, we introduced a new class of functions, considering only the ”dual perturbations” in a given subset $M \subset X^*$. In this paper we are mainly concerned to characterize this special type of convex functions using its subdifferential and to study of a new type of monotone mappings related to a subset of the dual.

Mathematics Subject Classification 2000: 52A41, 47H05, 46B10, 26A51.
Key words: M-convexity, M-monocity, convex function, monotone map, quasimonotone map.

1. Introductions. In all the sequel, X denotes a real linear normed space and X^* its topological dual. The symbol $(\cdot,\cdot)$ will be used for the usual pairing between X and X^*, while $\langle\ \cdot,\cdot\ \rangle$ will be used for the associated bilinear functional, i.e. $\langle x, x^*\rangle = x^*(x)$, for all $x \in X$, $x^* \in X^*$.

If M is a nonempty subset of X^*, a function $f : X \to \mathbb{R} \cup \{+\infty\}$ is M-convex if for each $x^* \in M$ the level sets

$$L(f, \alpha, x^*) = \{x \in X / f(x) \le \alpha + \langle x^*, x\rangle\} \tag{1}$$

are convex for every $\alpha \in \mathbb{R}$.

We denote

$$\mathcal{C}(M) - \text{the class of all } M- \text{ convex functions.}$$

It is easily to show that if $M = X^*$ then $\mathcal{C}(M)$ is just the class of convex functions (see [12], Proposition 9, Chapter I), and if $0 \in M$ then every $M-$convex function is quasiconvex. Also, it is obviously that $\mathcal{C}(M)$ contains the class of all convex functions for every $M \subset X^*$.

Now, we remind some properties of this functions (see [2], [3], [4]).

Firstly, we characterized this functions using only its values on line segments, establishing the following characteristic inequality:

$$f(\lambda x + (1-\lambda)y) \leq f(y) + \lambda \inf_{x^* \in M_{x,y}} \langle x^*, x-y \rangle \,, \tag{2}$$

for every $x, y \in X$, and $\lambda \in [0,1]$, where

$$M_{x,y} = \{x^* \in M / \langle x^*, x-y \rangle \geq f(x) - f(y)\} \,. \tag{3}$$

Remark 1. $f : X \to \mathbb{R} \cup \{+\infty\}$ is a M- convex function if and only if for every $x, v \in X$, the associated function F, defined by

$$F(t) = f(x+tv), \; t \in \mathbb{R}, \tag{4}$$

is M_v-convex, where

$$M_v = \{\langle x^*, v \rangle / x^* \in M\} \,. \tag{5}$$

An important property of the convex functions is that every local minimum is a global one. This property, however, holds for more general families of functions (for instance, the family of semistrict quasiconvex functions, see [8]). In this line we consider the sets $M \subseteq X^*$ which satisfy the following property

(P) for every $x \in X \backslash \{0\}$, exists $\{x_n^*\}_{n \in \mathbb{N}} \subseteq M$ such that $\langle x_n^*, x \rangle \searrow 0$.

Remark 2. It is easily to prove that if $0 \in int(M)$, then the set M has the property (P). In [3] we proved that if $M \subset X^*$ has the property (P), then every M- convex function is a semistrictly quasiconvex function, thus every locally extreme point from *dom* f is a minimum global point. Moreover, the set of points at which f attains its global minimum is a convex set.

Concerning the continuity of this type of functions we remind the definition of the radial upper convex function and a continuity theorem, proved in [3].

Definition 3. *We say that the function* $f : X \to \mathbb{R} \cup \{+\infty\}$ *is radial upper convex if for every* $x \in X$*, there exists* $\varepsilon_x > 0$ *such that for every* $v \in X$ *with* $\|v\| = 1$*, the function* $F(t) = f(x + tv)$ *is* $(-\infty, -\varepsilon_x) \cup (\varepsilon_x, +\infty)-$*convex.*

Theorem 4 ([3]). *Let* $f : X \to \mathbb{R} \cup \{+\infty\}$ *be a radial upper convex function. If* f *is bounded from above in a neighborhood of one point* $x_0 \in$ *int(dom* f*), then* f *is continuous on int(dom* f*).*

2. Differentiable M-convex functions. Firstly, let us recall the definitions of the following subdifferentials:

Definition 5. *Given the proper function* $f : X \to \mathbb{R} \cup \{+\infty\}$*, the Fenchel subdifferential of such a function is the (generally multivalued) mapping* $\partial f : X \rightrightarrows X^*$ *defined by*

$$\partial^{Fen} f(x) = \{x^* \in X^* / \langle x^*, v\rangle \le f(x + v) - f(x), \forall v \in X\}, x \in X. \tag{6}$$

Definition 6. *The upper Dini subdifferential of* $f : X \to \mathbb{R} \cup \{+\infty\}$ *at* $x \in X$ *in the direction* $v \in X$ *is defined by*

$$\partial^{D+} f(x, v) = \{x^* \in X^* / \langle x^*, v\rangle \le \limsup_{t \searrow 0} \frac{f(x + tv) - f(x)}{t}\}, \tag{7}$$

and the upper Dini subdifferential of f *at* $x \in X$ *is defined by*

$$\partial^{D+} f(x) = \bigcap_{v \in X} \partial^{D+} f(x, v). \tag{8}$$

Definition 7. *We say that a function* $f : X \to \mathbb{R} \cup \{+\infty\}$ *is Gateaux differentiable in* $x \in X$ *if there exists* $x^* \in X^*$ *such that*

$$\langle x^*, v\rangle = \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t}, \text{ for every } v \in X. \tag{9}$$

Usualy, the functional x^**, called the gradient of* f *at* x*, is denoted by* $\nabla f(x)$*.*

Definition 8. *The Clarke-Rockafellar subdifferential of* $f : X \to \mathbb{R} \cup \{+\infty\}$ *at* $x \in X$ *is defined by*

$$\partial^{CR} f(x) = \{x^* \in X^* \mid \langle x^*, v\rangle \le f^{\dagger}(x, v), \forall v \in X\},$$

where

$$f^{\dagger}(x, v) = \sup_{\varepsilon > 0} \limsup_{\substack{x' \to x \\ f(x') \to f(x) \\ t \searrow 0}} \inf_{v' \in S(v;\varepsilon)} \frac{f(x' + tv') - f(x')}{t}.$$

The definitions and related properties of this subdifferentials can be found, for instance, in [5], [7], [9], [17], [18] and their references.

In the following, ∂ stands for the following abstract notion of subdifferential:

Definition 9. *We call subdifferential, denoted by ∂, any operator which associates a subset $\partial f(x)$ of X^* to any lower semicontinuous $f : X \to \mathbb{R} \cup \{+\infty\}$, and any $x \in X$, and satisfies the following properties:*

(***D1***) $\partial f(x) = \{x^* \in X^* \mid \langle x^*, y - x\rangle \leq f(y) - f(x), \forall y \in X\}$, *whenever f is convex.*

(***D2***) $0 \in \partial f(x)$, *whenever $x \in domf$ is a local minimum of f.*

(***D3***) $\partial(f + g)(x) \subset \partial f(x) + \partial g(x)$, *whenever g is a real-valued convex continuous function which is $\partial-$differentiable at x.*

Definition 10. *The function g is called $\partial-$differentiable at x if $\partial g(x)$ and $\partial(-g)(x)$ are nonempty. When $\partial g(x) \neq \varnothing$ we say that the function g is $\partial-$subdifferentiable at x.*

This abstract subdifferential allows one to recover, by a unique definition, a large class of subdifferentials. Among this class we can mention the Clarke-Rockafellar subdifferential ∂^{CR}, the lower and upper Dini subdifferentials ∂^{D-} and ∂^{D+}, the lower Hadamard subdifferential ∂^{H-}, the Fréchet subdifferential ∂^{F} the Lipschitz smooth subdifferential ∂^{LS}.

In this paper we will consider the class of subdifferentials which satisfy (D1)-(D3) and one of the following inclusions:

$$\partial \subset \partial^{D+} \text{ or } \partial \subset \partial^{CR}.$$

This assumption is not really restrictive, since the Clarke-Rockafellar and the upper Dini subdifferentials are the biggest (in the sense of inclusion) among the classical subdifferentials. In particular, we have

$$\partial^{LS} \subset \partial^{F} \subset \partial^{H-} \subset \partial^{CR},$$
$$\partial^{H-} \subset \partial^{D-} \subset \partial^{D+}.$$

Definition 11 ([7]). *A norm $\|\cdot\|$ on X is said to be $\partial-$smooth if all the (real-valued, convex, continuous) functions of the following form are ∂-subdifferentiable:*

(*i*) $d^2_{[a,b]}(x) = \min_{c \in [a,b]} \|x - c\|^2$, *where $[a, b]$ is a closed segment in X,*

(*ii*) $\Delta_2(x) = \sum_n \mu_n \|x - v_n\|^p$, *where $\sum_n \mu_n = 1, \mu_n \geq 0$ and v_n converges in X.*

We say that a Banach space admits a ∂-smooth renorm if it admits an equivalent norm which is ∂-smooth.

In [5], Aussel proved that the property

$$(Q_s) \text{ exists } x^* \in \partial f(x) \text{ such thatt } \langle x^*, y-x\rangle > 0 \text{ imply} \\ f(z) \le f(y), \text{ for every } z \in [x,y]\,,$$

is necessary and sufficient for characterization of lower semicontinuous quasiconvex functions on Banach spaces which admits a ∂-smooth renorm. Moreover, if the function is radially continuous then property (Q_s) can be replaced with

$$(Q) \quad \text{exists } x^* \in \partial f(x) \text{ such thatt } \langle x^*, y-x\rangle > 0 \text{ imply } f(x) \le f(y).$$

Remark 12. For quasiconvexity, the characterization with property (Q) is more useful because it is more easy to verify.

Now we present some result concerning to characterization of $M-$convex functions with their subdifferentials.

Theorem 13. *Let X be a Banach space with a ∂-smooth renorm, and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous $M-$convex function. Then*

$$\langle x^*, y-x\rangle \le \inf_{x_0^* \in int(M_{y,x})} \langle x_0^*, y-x\rangle\,, \tag{10}$$

for every $x, y \in X$, and $x^* \in \partial f(x)$.

Proof. If we suppose by a contradiction that (10) do not hold, then we can find $x, y \in X$, $x^* \in \partial f(x)$ and $x_0^* \in int(M_{y,x})$ such that $f - x_0^*$ is quasiconvex,

$$\langle x^*, y-x\rangle > \langle x_0^*, y-x\rangle \tag{11}$$

and

$$\langle x_0^*, y-x\rangle > f(y) - f(x). \tag{12}$$

Since ∂ satisfy property (D_3), then $x^* - x_0^* \in \partial(f - x_0^*)(x)$ and considering that $f - x_0^*$ is quasiconvex, from (Q_s) with $z = y$ and (11), we obtain that $(f - x_0^*)(x) \le (f - x_0^*)(y)$, which is not possible. $\square$

Remark 14. Using the characterization of a quasiconvex function with the property (Q), we can prov that under the additional assumption that f is radially continuous, the reverse implication of above theorem is also true.

Now we consider the case when the set M is open.

Theorem 15. *Let X be a Banach space with a $\partial-$ smooth renorm, and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous $M-$convex function.*

If M is open, then the following assertions are equivalent:

(i) f is $M-$convex;

(ii) for every $x, y \in X$ and $x^* \in \partial f(x)$ we have

$$\langle x^*, y - x\rangle \leq \inf_{x_0^* \in M_{y,x}} \langle x_0^*, y - x\rangle . \tag{13}$$

Proof. Since M is an open set then for every $x, y \in X$,

$$\inf_{x_0^* \in M_{y,x}} \langle x_0^*, y - x\rangle = \inf_{x_0^* \in int(M_{y,x})} \langle x_0^*, y - x\rangle . \tag{14}$$

In this case the implication $(i) \Longrightarrow (ii)$ following from Theorem 13. For reverse implication we start with x_0^* from M and we want to prove that $f - x_0^*$ is quasiconvex. If we suppose that $f - x_0^*$ is not quasiconvex for some $x_0^* \in M$, then there exist $x, y \in X$ and $z \in (x, y)$ such that $(f - x_0^*)(z) > \max\{(f - x_0^*)(x), (f - x_0^*)(y)\}$. Without loose our generality, we can suppose that $x \neq y$, therefore we can find $\overline{x}^* \in X^*$ such that

$$\langle \overline{x}^*, y - x\rangle = -1. \tag{15}$$

Since M is an open set, then there exists $\varepsilon > 0$ such that $x_0^* + \varepsilon\overline{x}^* \in M$ and $(f - x_0^* - \varepsilon\overline{x}^*)(z) > \max\{(f - x_0^* - \varepsilon\overline{x}^*)(x), (f - x_0^* - \varepsilon\overline{x}^*)(y)\}$. Considering now the approximate mean value inequality from [7] for $f - x_0^* - \varepsilon\overline{x}^*$, we find $\bar{x} \in [x, z)$ and a sequence $x_n \to \bar{x}$, $x_n^* \in \partial(f - x_0^* - \varepsilon\overline{x}^*)(x_n)$ such that

$$\langle x_n^*, d - x_n\rangle > 0, \text{for every } n \in \mathbb{N} \text{ and for every } d \in [z, y] .$$

From (13) and (14) we obtain that

$$(f - x_0^*)(x_n) \leq (f - x_0^*)(d) + \varepsilon \langle \overline{x}^*, d - x_n\rangle , \text{for every } d \in [z, y] .$$

Since f is lower semicontinuous, from (15), tacking into account that $\bar{x} \in [x, z)$, passing to limit in above relation, we obtain that

$$(f - x_0^*)(\bar{x}) \leq \min_{d \in [z,y]} (f - x_0^*)(d) - \delta, \tag{16}$$

where $\delta = \varepsilon \frac{\|z-\bar{x}\|}{\|y-x\|} > 0$.

Similarly, we can find $\bar{y} \in (z, y]$ such that

$$(f - x_0^*)(\bar{y}) \le \min_{d\in[x,z]} (f - x_0^*)(d). \tag{17}$$

From (16) and (17) we obtain a contradiction, therefore $f - x_0^*$ is quasiconvex. for every x_0^* from M, and the proof is complete. □

Example 16. Let be $f : \mathbb{R} \to \mathbb{R}$, defined by

$$f(t) = \begin{cases} -x, & x < 0, \\ 1 + x, & x \ge 0. \end{cases}$$

This function is not radially continuous, but it is $(-1, 1)-$convex.

In the sequel we consider $\partial = \partial D^+$ and we take X a normed space without any restriction.

Theorem 17. *Let M be a subset of X^* and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a $M-$convex function.Then*

$$\langle x^*, y - x\rangle \le \inf_{x_0^*\in M_{y,x}} \langle x_0^*, y - x\rangle \tag{18}$$

for every $x, y \in X$, and $x^* \in \partial^{D+} f(x, y - x)$.

Proof. We take $x, y \in X$ and $x^* \in \partial^{D+} f(x, y - x)$. Thus we have :

$$\begin{aligned} \langle x^*, y - x\rangle &\le \limsup_{t\searrow 0} \frac{f(x + t(y - x)) - f(x)}{t} \\ &= \limsup_{t\searrow 0} \frac{f((1 - t)x + ty) - f(x)}{t}. \end{aligned}$$

Since f is $M-$convex we obtain that

$$\begin{aligned} \langle x^*, y - x\rangle &\le \limsup_{t\searrow 0} \frac{f(x) + t \inf_{x^*\in M_{y,x}} \langle x^*, y - x\rangle - f(x)}{t} \\ &= \inf_{x^*\in M_{y,x}} \langle x^*, y - x\rangle . \end{aligned}$$

i.e. (18) is fulfilled. □

Generally, the reverse implication is not true, but in the following theorem we shall show that this becomes true for lower semicontinuous functions.

Theorem 18. *Let M be a subset of X^* and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function.Then f is $M-$convex if and only if*

$$\langle x^*, y - x \rangle \leq \inf_{x_0^* \in M_{y,x}} \langle x_0^*, y - x \rangle \tag{19}$$

for every $x, y \in X$ and $x^* \in \partial^{D+} f(x, y - x)$.

Proof. The "if" part was proved in Theorem 17. For the reverse implication we must show that $f - x_0^*$ is quasiconvex for every $x_0^* \in M$.

We suppose, by contradiction, that there exists $x_0^* \in M$ such that $f - x_0^*$ is not quasiconvex, namely exists $x, y \in X$ and $z \in (x, y)$ such that $(f - x_0^*)(z) >\max\{(f - x_0^*)(x), (f - x_0^*)(y)\}$. Since f is lower semicontinuous, without losing our generality, we can suppose that y is a minimum point for the function $f - x_0^*$ on $[x, y]$.

If we take the function $F : [0, 1] \to \mathbb{R} \cup \{+\infty\}$ defined by $F(t) = (f - x_0^*)(x + t(z - x))$, then F is also lower semicontinuous and $F(0) < F(1)$. Since F is lower semicontinuous then there exists $c \in [0, 1)$ such that

$$\limsup_{t \searrow 0} \frac{F(c + t) - F(c)}{t} \geq F(1) - F(0),$$

or equivalently, there exists $z_0 = x + c(z - x) \in [x, z)$ such that

$$\begin{aligned}&\limsup_{t \searrow 0} \frac{(f - x_0^*)(z_0 + t(z - x)) - (f - x_0^*)(z_0)}{t} \\ &\geq (f - x_0^*)(z) - (f - x_0^*)(x).\end{aligned}$$

By a simple calculation, considering $t := t\frac{y - z_0}{z - x}$we obtain

$$\begin{aligned}&\limsup_{t \searrow 0} \frac{f(z_0 + t(y - z_0)) - f(z_0) - \langle x_0^*, t(y - z_0) \rangle}{t} \\ &\geq (f - x_0^*)(z) - (f - x_0^*)(x),\end{aligned}$$

which yields

$$\limsup_{t \searrow 0} \frac{f(z_0 + t(y - z_0)) - f(z_0) - \langle x_0^*, t(y - z_0) \rangle}{t} > \langle x_0^*, y - z_0 \rangle .$$

On the other hand, by virtute of the definition of upper Dini subdifferential, there exists $\overline{x}^* \in \partial^{D+} f(z_0)(y - z_0)$ such that $\langle \overline{x}^*, y - z_0 \rangle > \langle x_0^*, y - z_0 \rangle$. Taking into account the inequality (19) we obtain that $x_0^* \notin M_{y,z_0}$, namely $(f - x_0^*)(y) > (f - x_0^*)(z_0)$, which contradicts our assumption. $\square$

The problem which appears is to see certain conditions under which (19) is fulfilled only for elements from $\partial^{D+} f(x)$. This requires additional conditions either for the function f or for the set M. Starting with an earlier result of Aussel [5], established for quasiconvex functions, we can prove that if a function is radial continuous then in the Theorem 18 we can consider only elements from $\partial^{D+} f(x)$. By Theorem 4, the same things is true if the function f is radial upper convex.

Proposition 19. *Let M be a nonempty subset of X^* and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous $M-$convex function. Then for every $x^* \in \partial^{D+} f(x) \cap int(M)$ we have:*

$$f(y) - f(x) \geq \langle x^*, y - x \rangle\,, \text{ for every } y \in X \tag{20}$$

Proof. Since $x^* \in int(M)$, if we suppose that $f(y) - f(x) < \langle x^*, y - x \rangle$, for some $y \in X$, then $\inf_{x_0^* \in M_{y,x}} \langle x_0^*, y - x \rangle < \langle x^*, y - x \rangle$, which contradict Theorem 17. $\square$

Remark 20. We find again an important property of the Fenchel subdifferential from the convex case, but here this property are not fulfilled for every pair (x, x^*) of the subdifferential.

Corollary 21. *Let M be an open nonempty set and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous $M-$convex function. If $R(\partial^{D+} f) \subseteq M$, then f is convex.*

Proof. As is well known, to prove that f is convex is enough to show that $\partial^{D+} f$ is monotone [9].

Let (x, x^*),$(y, y^*) \in Gr(\partial^{D+} f)$. From above proposition, we obtain that

$$\langle x^*, y - x \rangle \leq f(y) - f(x)$$

and

$$\langle y^*, x - y \rangle \leq f(x) - f(y),$$

therefore

$$\langle x^* - y^*, y - x \rangle \geq 0,$$

which prove that $\partial^{D+} f$ is monotone. $\square$

It is well known that for differentiable functions, if in a point its differential is 0, then the point is not necessarily a locally extreme point as in the case of convex functions.

Now we are interested to find sets $M \subseteq X^*$ such that every $M-$convex function has the property that its extreme points are the same as the points for which 0 belongs to the subdifferential in these points.

The original definition of pseudoconvexity due to MANGASARIAN [14] for differentiable functions was extended by many authors (see [15]) in the following way.

A function f is said to be pseudoconvex if for any $x \in X$ there exist $x^* \in \partial^{D+} f(x)$ such that

$$\langle x^*, y - x \rangle \geq 0 \text{ implies } f(x) \leq f(y). \tag{21}$$

As in the differentiable case, every pseudoconvex function f satisfies the following fundamental properties [5]:

i) every local minimizer of f is a global minimizer;

ii) if $0 \in \partial^{D+} f(x)$ then x is a globally minimum point for f.

Theorem 22. *If $M \subseteq X^*$ has the property (P), then every lower semicontinuous $M-$convex function is pseudoconvex.*

Proof. Let $f \in \mathcal{C}(M)$ be a lower semicontinuous function and $x, y \in X$ such that $f(x) > f(y)$. Since M has the property (P), then

$$\inf_{x_0^* \in M_{y,x}} \langle x_0^*, y - x \rangle < 0.$$

Thus, taking into account Theorem 18, we have

$$\langle x^*, y - x \rangle \leq \inf_{x_0^* \in M_{y,x}} \langle x_0^*, y - x \rangle < 0, \text{ for every } x^* \in \partial^{D+} f(x)$$

which proves that f is a pseudoconvex function. $\square$

Definition 23. *We say that a function f is stable with respect to property (S) related to the set $M \subset X^*$ if*

$$x \to f(x) + \langle x^*, x \rangle \text{ has the property } (S) \text{ for every } x^* \in M.$$

We consider now the following properties:

$(C1)$ all lower level sets are convex;

$(C2)$ each local minimum is a global minimum;

$(C3)$ each stationary point is a global minimizer.

Theorem 24. *Let $M \subset X^*$ be a convex set with property (P) and $f : X \to \mathbb{R} \cup \{+\infty\}$. If f is $M-$convex then f is stable with respect to properties $(C1), (C2)$ and $(C3)$ related to the set $\frac{1}{2}M$.*

Proof. Since the set $\frac{1}{2}M$ has the property (P), from Remark 2, to prove the theorem is enough to show that for every $x_0^* \in \frac{1}{2}M$, the function $f - x_0^*$ is $\frac{1}{2}M-$convex. If $x^* \in \frac{1}{2}M$ then $x_0^* + x^* \in M$ because M is convex, therefore $(f - x_0^*) - x^*$ is quasiconvex. $\square$

If in above theorem we take $M = S^*(0; \varepsilon)$, for $\varepsilon > 0$, we obtain (on the other way) the stability property from $s-$quasiconvex function established by Phu and An in [16].

We specify that a function $f : D \subset X \to \mathbb{R} \cup \{+\infty\}$ is called *s-quasiconvex if there exists* $\sigma > 0$ such that

$$\frac{f(x_0) - f(x_1)}{\|x_0 - x_1\|} \leq \delta \text{ implies } \frac{f(x_\lambda) - f(x_1)}{\|x_\lambda - x_1\|} \leq \delta$$

for $|\delta| < \sigma$, $x_0, x_1 \in D$, $x_\lambda = (1-\lambda)x_0 + \lambda x_1$ and $\lambda \in [0, 1]$.

The authors show that this functions can be characterized as follows a function f is s-quasiconvex if and only if there exists $\varepsilon > 0$ such that $f - x^*$ is quasiconvex, for every x^* with the property $\|x^*\| < \varepsilon$.

Remark 25. The relation (22) can be obtained (on the other way) from Theorem 17, taking M a ball with the center in the origin. In fact, as we see in the relation (22), the class of s-quasiconvex functions are the same with the class of M-convex functions, with $0 \in \int(M)$.

Remark 26. In finite dimensional spaces if $M \subset X^*$is a convex set with property (P) then every $M-$convex function is $s-$quasiconvex. This is not true if the spaces is infinite dimensional, as we see if we take $X = l^1$, $M = \{\alpha_n e_n \mid \text{where } \alpha_n \in (-\frac{1}{n}, \frac{1}{n}),\ n \in \mathbb{N},\ \{e_i\}_{i \in \mathbb{N}^*} - \text{canonical bases in } l^1\}$ and the functions defined by

$$f(x) = \begin{cases} \sup\limits_{x^* \in M} \langle x^*, \quad x\rangle, x \neq 0 \\ -1, \qquad\qquad x = 0. \end{cases}$$

3. $M-$**monotone mappings.** Having as starting point the subdifferentials of M-convex function, in [4] we introduce a new type of monotonicity, associated to a subset of the dual.

For an operator $A : X \rightrightarrows X^*$, we denote

$$D(A) = \{x \in X / Ax \neq \emptyset\}, \text{ the domain of } A,$$

$$R(A) = \bigcup_{x \in X} Ax, \text{ the range of } A,$$

$$Gr(A) = \{(x, x^*) \in X \times X^* / x \in D(A), x^* \in Ax\}, \text{ the graph of } A,$$

$$A^{-1}x^* = \{x \in X / x^* \in Ax\}, \; x^* \in X^*.$$

Recall that the operator A is called *monotone* if the following condition is fulfilled:

$$\langle x^* - y^*, x - y \rangle \geq 0 \text{ for every two couples } (x, x^*), (y, y^*) \in Gr(A). \tag{22}$$

When

$$\begin{aligned} &\langle y^*, y - x \rangle < 0 \text{ imply that } \langle x^*, y - x \rangle \leq 0 \\ &\text{for every two couples } (x, x^*), (y, y^*) \in Gr(A), \end{aligned} \tag{23}$$

the operator A is called *quasimonotone*, and when $\langle y^*, y - x \rangle < 0$ imply that $\langle x^*, y - x \rangle < 0$ for every two couples $(x, x^*), (y, y^*) \in Gr(A)$, the operator A is called *pseudomonotone*.

Let M be a nonempty subset of X^* .

Definition 27. *We say that the operator $A : X \rightrightarrows X^*$ is $M-$monotone if*

$$\langle x^*, y - x \rangle \leq \inf_{x_0^* \in M_{y,x}(y^*)} \langle x_0^*, y - x \rangle \tag{24}$$

for every $(x, x^*), (y, y^*) \in Gr(A)$, where

$$M_{y,x}(y^*) = \{x_0^* \in M / \langle y^* - x_0^*, y - x \rangle < 0\}.$$

Throughout this paper, for a given nonempty subset $M \subseteq X^*$, we will denote by

$$\mathcal{M}(M), \text{the class of all } M - \text{monotone operators.}$$

Remark 28. *If* $\inf_{x_0^* \in M_{y,x}(y^*)} \langle x_0^*, y - x \rangle = \langle y^*, y - x \rangle$ for every $x, y \in X$ and $y^* \in X^*$ (for example if $M = X^*$), then (24) becomes to (22), thus every $M-$monotone operator is monotone. If $\inf_{x_0^* \in M_{y,x}(y^*)} \langle x_0^*, y - x \rangle \leq 0$ for every $x, y \in X$ and $y^* \in X^*$(for example if $0 \in M$), then (24) becomes to (24), thus every $M-$monotone operator is quasimonotone.

For the beginning, we recall some property of this operators proved in [4].

Proposition 29. *Let A be an operator between X and X^*.*
i) If $A \in \mathcal{M}(M)$, then for every $x^* \in X^*$, $A - x^* \in \mathcal{M}(M - x^*)$;
ii) $A \in \mathcal{M}(M)$, if and only if $A - x^*$ is quasimonotone, for every $x^* \in M$;
iii) If $M_1 \subset M_2$, then $\mathcal{M}(M_2) \subseteq \mathcal{M}(M_1)$;
iv) $\mathcal{M}(M) = \mathcal{M}(\overline{M}^{w^*})$, where $\overline{M}^{w^*}$ is the adherence of M with respect to w^*-topology;
v) If $x^* \in int\,(M)$, then $A - x^*$ is pseudomonotone;
vi) If $A \in \mathcal{M}(M)$, then for each $\lambda > 0$, $\lambda A \in \mathcal{M}(\lambda M)$.

Remark 30. From (*ii*) and Remark 28, tacking $M = X^*$, we obtain the characterization of monotone mappings gives by Aussel, Corvellec şi Lassonde [6].

Concerning to the relations between $\mathcal{M}(M_1)$ and $\mathcal{M}(M_2)$ for two nonempty sets $M_1, M_2 \subset X^*$, we have:

Proposition 31. *Let $M_1, M_2 \subset X^*$ be two nonempty sets. If*

$$M_{1v} \subset \overline{M_{2v}}, \text{ for every } v \in X, \tag{25}$$

then $\mathcal{M}(M_2) \subset \mathcal{M}(M_1)$.

Proof. If $M_{1v} \subset M_{2v}$ for every $v \in X$, then for each $x, y \in X$ and $y^* \in X^*$ we have $M''_{y,x}(y^*) \supset M''_{y,x}(y^*)$, and so by (24) we obtain that $\mathcal{M}(M_2) \subset \mathcal{M}(M_1)$. □

Is obvious that if

$$M_{1v} \subset \overline{M_{2v}} \text{ and } M_{2v} \subset \overline{M_{1v}} \text{ for every } v \in X,$$

then $\mathcal{M}(M_1) = \mathcal{M}(M_2)$. Obviously, a sufficient condition for (25) is $M_1 \subset \overline{M_2}^{w^*}$, but this is not necessary, as we see if we take in $\mathbb{R}^2$ the sets $M_1 = \{x \in \mathbb{R}^2 / \, \|x\| = 1\}$ and $M_2 = \{x \in \mathbb{R}^2 / \, \|x\| = 2\}$.

Theorem 32 ([4]). *Let $M_2 \subset X^*$ be an open nonempty set. If $\varnothing \neq M_1 \subset M_2$, then $\mathcal{M}(M_1) = \mathcal{M}(M_2)$ if and only if*

$$\begin{aligned} \text{for every } x^* &\in M_2, \text{ and } x \in X \setminus \{0\}, \text{ there exists a sequence} \\ (x_n^*)_{n\in\mathbb{N}} &\subseteq M_1 \text{ such that } \langle x_n^* - x^*, x\rangle \searrow 0. \end{aligned} \tag{26}$$

Corollary 33. *Every quasimonotone mapping is $M-$monotone if and only if $M = \{0\}$.*

$\mathcal{M}(M)$ is the class of monotone operators if and only if the set M satisfy the following property:

$$\begin{array}{l} \textit{for every } \alpha \in \mathbb{R}, \ \textit{and } x \in X \setminus \{0\}, \\ \textit{exists a sequence } (x_n^*)_{n\in\mathbb{N}} \subseteq M \ \ \textit{such that } \langle x_n^*, x\rangle \to \alpha. \end{array} \tag{27}$$

It is easily to prove that if $M \subseteq X^*$ has the property (P), then

$$\inf_{x_0^* \in M_{y,x}(y^*)} \langle x_0^*, y - x\rangle < 0$$

and by (24) we obtain that every $M-$monotone operator is pseudomonotone.

In [4] we proved that if the set M has the property

(D) for every two points x, y of M exists $z \in M$ between x and y,

then for every $M-$monotone operator $A : X \rightrightarrows X^*$ we have that $A^{-1}_{/M}$ is monotone. Considering $M = X^*$, we obtained the well known result that A^{-1} is monotone whenever:

$$A \text{ is monotone }.$$

Now we present the relationship between the $M-$convexity of a function and the $M-$monotonicity of its subdifferential.

Theorem 34. *Let $M \subseteq X^*$ be a nonempty subset of X^* and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. Then f is $M-$convex if and only if $\partial^{D+} f$ is $M-$monotone.*

Proof. To prove the "if" part we start with two pairs $(x^*, x), (y^*, y)$ from $\partial^{D+} f$. Since $x^* \in \partial^{D+} f(x)$, we have:

$$\langle x^*, y - x\rangle \le \inf_{x_0^* \in M_{y,x}} \langle x_0^*, y - x\rangle .$$

Considering (24), is enough to prove that

$$M_{y,x}(y^*) \subseteq M_{y,x}. \tag{28}$$

If we suppose that (28) is not true, then there exists $x_0^* \in M_{y,x}(y^*) \backslash M_{y,x}$. Therefore $x_0^* \in M$, but

$$\langle y^* - x_0^*, y - x \rangle < 0 \tag{29}$$

and

$$\langle x_0^*, x - y \rangle > f(x) - f(y). \tag{30}$$

From (30) it follows that $x_0^* \in M_{x,y}$, and tacking into account that $y^* \in \partial^{D+} f(y)$, we obtain

$$\langle y^*, x - y \rangle \leq \inf_{y_0^* \in M_{x,y}} \langle y_0^*, x - y \rangle \leq \langle x_0^*, x - y \rangle\,,. \tag{31}$$

which by 29 is not possible. Conversely, if we suppose that $\partial^{D+} f$ is M-monotone and f is not M-convex, then there exists $x_0^* \in M$ such that $\partial^{D+} f \; - x_0^*$ is quasimonotone and $f - x_0^*$ is not quasiconvex, which is also not possible. $\square$

In the following, we characterize the $M-$monotonicity with the aid of the generalized Jacobian.

Let $F : C \to X^*$, $C \subseteq X$.

We recall that the generalized Jacobian of F at $x \in S$ (in the sense of Clarke [11]) is the set :

$$\partial F(x) = conv \left\{ \lim_i \nabla F(x_i) / x_i \to x, F - \text{differentiable at } x_i \right\},$$

where "*conv*" denotes the convex hull and $DF(x)$ is the usual Jacobian of F at x. Elements of $\partial F(x)$ are called subgradients.

In preparation of the next result on nonmonotone maps, we recall the following notation from [13]:

$$\begin{aligned} D_+ F(x; u) &= \sup \left\{ \langle u, Au \rangle / A \in \partial F(x) \right\} \\ D_- F(x; u) &= \inf \left\{ \langle u, Au \rangle / A \in \partial F(x) \right\} \end{aligned}$$

where $x \in C$, $u \in X$.

Remark 35. If F is locally Lipschitz then $\partial F(x)$ is a nonempty and a compact set. Moreover, $x \to D_+ F(x; u)$ is upper semicontinuous.

In [13] Schaible and Dinh The Luc have given in the case of locally Lipschitz function, a characterization of quasimonotone operators, as we see in the next theorem.

Theorem 36 ([13]). *The map F is quasimonotone on S if and only if the following conditions hold for every $x \in S$, $u \in R^n$:*

(i) $\langle u, (x)\rangle = 0$ implies $D_+F(x; u) \geq 0$;

(ii) $\langle u, F(x)\rangle = 0$, $0 \in \{\langle u, Au\rangle : A \in \theta F(x)\}$ and $\left\langle u, F(x + \overline{t}u)\right\rangle > 0$ for some $\overline{t} < 0$ imply the existence of $\widetilde{t} > 0$ such that $\langle u, F(x + tu)\rangle \geq 0$ for all $t \in [0; \widetilde{t}]$.

We are interested in finding such sets M for witch $M-$convex functions can be characterized using i) and ii) rewritten in a simple way.

Theorem 37. *Let M be an open subset of X^* and let $F : C \to X^*$ be a locally Lipschitz function on the open and convex set $C \subseteq X$. Then F is $M-$monotone on C if and only if for every $x \in C$ and $u \in X$ the following property holds:*

$$\langle u, F(x)\rangle \in \{\langle u, x^*\rangle \, /x^* \in M\} \text{ implies } D_-F(x; u) \geq 0. \tag{32}$$

Proof. For the "only part", we suppose that exists $x \in C$ and $u \in X$ such that $\langle u, F(x)\rangle \in \{\langle u, x^*\rangle \, /x^* \in M\}$ and $D_-F(x; u) < 0$. From this, we obtain that there exist $x^* \in M$ with $\langle u, F(x)\rangle = \langle u, x^*\rangle$ and a sequence $t_n \searrow 0$, such that $\langle u, F(x + t_n u) - x^*\rangle < 0$ for every $n \in \mathbb{N}$. Tacking $n \in \mathbb{N}$ with $x + t_n u \in C$, then since M is a open set, we find $x_0^* \in X^*$ such that $x^* + x_0^* \in M$ and

$$\langle u, F(x + t_n u) - x^*\rangle < \langle u, x_0^*\rangle < 0. \tag{33}$$

Now, tacking into account that F is $M-$monotone, then $F - x^* - x_0^*$ is quasimonotone. But

$$\langle x + t_n u - x, F(x) - x^* - x_0^*\rangle = t_n \langle u, F(x) - x^*\rangle - t_n \langle u, x_0^*\rangle > 0$$

and from (33)we obtain that

$$\begin{aligned} &\langle x + t_n u - x, F(x + t_n u) - x^* - x_0^*\rangle \\ &= t_n \left(\langle u, F(x + t_n u) - x^*\rangle - \langle u, x_0^*\rangle\right) < 0, \end{aligned}$$

which is not possible because is a contradiction with the quasimonotonicity property of $F - x^* - x_0^*$. This mean that if $x \in C$ and $u \in X$ such that

$\langle u, F(x)\rangle \in \{\langle u, x^*\rangle /x^* \in M\}$, then $D_-F(x;u) \geq 0$. In above calculations we considered that $u \neq 0$. (The case $u = 0$ is obviously.)

Conversely, we assume that (32) holds and we want to prove that F is $M-$ monotone. Let x^* be an arbitrary element of M. We want to show that for $F - x^*$ the property i) and ii) from Theorem 36 are fulfilled. If $\langle u, F(x) - x^*\rangle = 0$ then $\langle u, F(x)\rangle \in \{\langle u, x^*\rangle /x^* \in M\}$, from where $D_-F(x;u) \geq 0$. Therefore $D_+F(x;u) \geq 0$ i.e. $D_+(F - x^*)(x;u) \geq 0$, and i) is proved. To prove ii) we start with $x \in C$ and $u \in X$ such that $\langle u, F(x) - x^*\rangle = 0, 0 \in \{\langle u, Ax\rangle /A \in \partial(F - x^*)(x)\}$, $\left\langle u, F(x + \overline{t}u) - x^*\right\rangle >$ 0 for $\overline{t} < 0$. If we suppose by a contradiction that ii) not holds, then for every $t > 0$ there exists $\widetilde{t} \in (0,t)$ such that $\left\langle u, F(x + \widetilde{t}u) - x^*\right\rangle < 0$ which means that there exists a sequence $t_n \searrow 0$ with

$$\langle u, F(x + t_n u) - x^*\rangle < 0, \text{for every } n \in \mathbb{N}.$$

Using the mean theorem we find a sequence $\overline{t}_n \searrow 0$ such that $D_-(F)(x + \overline{t}_n u; u) < 0$, for every $n \in \mathbb{N}$.

Since the set M is open and F is locally Lipschitz, we can find a neighborhood V of x such that

$$\langle u, F(y)\rangle \in \{\langle u, x^*\rangle /x^* \in M\}, \text{ for every } y \in V.$$

Tacking now $n \in \mathbb{N}$ sufficiently large then $x + \overline{t}_n u \in V$, therefore $D_-(F)(x + \overline{t}_n u; u) \geq 0$ and we obtain a contradiction. Hence, the proof is completed.$\square$

Remark 38. When $M = X^*$ we refind a well known result concerning the monotone case: all the gradients are positive semidefinite.

In [1] An introduced the s-quasimonotone operators and study in the differentiable case the stability to the property:

$(P1)$ $v^T F(x) = 0$ implies $v^T J_F(x)v \geq 0$;

$(P2)$ $v^T F(x) = v^T J_F(x)v = 0$, $t' < 0$ such that $v^T F(x + t'v) > 0$, then there exists $\overline{t} > 0$ such that $v^T F(x + tv) \geq 0$ for every $t \in [0;\overline{t}]$;

$(P3)$ $v^T F(x) = v^T J_F(x)v = 0$, then there exists $\overline{t} > 0$ such that $v^T F(x + tv) \geq 0$ for every $t \in [0;\overline{t}]$;

$(P4)$ there exists $\delta > 0$ such that

$$u \in \mathbb{R}^n, \left|\frac{u^T}{\|u\|}F(x)\right| < \delta \text{ implies } v^T J_F(x)v \geq 0.$$

We denote by $J_F(x)$ the Jacobian matrix evaluated at x.

We remind the definition from [1] of s-quasimonotone operators:

The map $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$is said to be s-quasimonotone if there exists $\sigma_F > 0$ such that

$$\frac{(x_1 - x_0)^T}{\|x_1 - x_0\|} F(x_0) \geq \delta \Longrightarrow \frac{(x_1 - x_0)^T}{\|x_1 - x_0\|} F(x_1) \geq \delta$$

for $|\delta| < \sigma_F$, $x_0, x_1 \in D$.

For gradient maps, s-quasimonotonicity is related to s-quasiconvexity [16]. In the differentiable case, An [1] prove that s-quasimonotone mappings are stable with respect to property $(P1) - (P4)$ (i.e. this properties are maintained if the map is perturbed by an element with sufficiently small norm).

Theorem 39 ([1]). *A map F is s-quasimonotone if and only if there exists $\epsilon > 0$ such that $F + a$ is pseudomonotone for all $a \in \mathbb{R}^n$ satisfying $\|a\| < \epsilon$.*

In [4] we established the following theorem:

Theorem 40 ([4]). *$F : D \subset \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 2$ is s-quasimonotone if and only if there exists $r > 0$ such that F is $Fr(S^*(0; r))-$monotone.*

In nondifferentiable case, the properties $(P1) - (P4)$ becomes:

$(P^{'}1)$ $\langle u, F(x)\rangle = 0$ implies $D_{-}F(x; u) \geq 0$;

$(P^{'}2)$ $\langle u, F(x)\rangle = 0$, $0 \in \{\langle u, Au\rangle : A \in \partial F(x)\}$ and $\langle u, F(x + \bar{t}u)\rangle > 0$ for some $\bar{t} < 0$, then there exists $\widetilde{t} > 0$ such that $\langle u, F(x + tu)\rangle \geq 0$ for every $t \in [0; \widetilde{t}]$;

$(P^{'}3)$ $\langle u, F(x)\rangle = 0$, $0 \in \{\langle u, Au\rangle : A \in \partial F(x)\}$ then there exists $\widetilde{t} > 0$ such that $\langle u, F(x + tu)\rangle \geq 0$ for every $t \in [0; \widetilde{t}]$;

$(P^{'}4)$ there exists $M \subset X^*$ such that $\langle u, F(x)\rangle \in \{\langle u, x^*\rangle \mid x^* \in M\}$ imply $D_{-}F(x; u) \geq 0$.

Theorem 41. *Let M be a open convex set from X^* and let $F : C \to X^*$ be a locally Lipschitz map defined on the open set $C \subseteq X$. If F is $M-$monotone on C, then F is stable with respect to $(P^{'}1)$, $(P^{'}2)$, $(P^{'}3)$ and $(P^{'}4)$ related to $\frac{1}{2}M$.*

Proof. Is enough to prove that F is stable with respect to $(P^{'}4)$. For this, we start with x_0^* from $\frac{1}{2}M$. If $\langle u, (F - x_0^*)(x)\rangle \in \{\langle u, x^*\rangle \mid x^* \in \frac{1}{2}M\}$

then $\langle u, F(x)\rangle \in \{\langle u, x^* + x_0^*\rangle \mid x^* \in \frac{1}{2}M\}$. Since M is a convex set we obtain that $\langle u, F(x)\rangle \in \{\langle u, x^*\rangle \mid x^* \in M\}$ and considering Theorem 37 we obtain that $D_-F(x; u) \geq 0$, therefore $D_-(F - x_0^*)(x; u) \geq 0$. This ended the proof because x_0^* was arbitrary from $\frac{1}{2}M$. □

If in above theorem we take $M = S^*(0; \varepsilon)$, for $\varepsilon > 0$, we obtain the stability property for continuous differentiable $s-$quasimonotone operators defined on $\mathbb{R}^n$, earlier established by AN in [1].

REFERENCES

1. AN, P.T. – *Stability of generalized monotone maps with respect to their characterizations,* Optimization, 55(2006), 289-299.
2. APETRII, M. – *A new type of convexity defined by dual perturbations,* An. Univ. De Vest Timişoara, seria Matematică Informatică, 45(2007), 11-20.
3. APETRII, M. – *A dual generalization of convex functions, Revue d'Analyse Numerique et de Theorie de l'Approximation* (to appear).
4. APETRII, M. – *A dual generalization of monotone mappings, 6-th Congress of Romanian Mathematicians*, Bucureşti, 2007 (to appear).
5. AUSSEL, D. – *Subdifferential properties of quasiconvex and pseudoconvex functions: unified approach,* J. Optim. Theory Appl, 97(1998), 29-45.
6. AUSSEL, D.; CORVELLEC, J.N.; LASSONDE, M. – *Subdifferential characterization of quasiconvexity and convexity*, J. Convex Anal., 1(1994), 195-201.
7. AUSSEL, D.; CORVELLEC, J.N.; LASSONDE, M. – *Mean value property and subdifferential criteria for lower semicontinuous functions,* Trans. Amer. Math. Soc., 347, No.10(1995), 4147-4161.
8. AVRIEL, M.; DIEWERT, W.E.; SCHAIBLE, S.; ZANG, I. – *Generalized Concavity,* Plenum Press, New York and London, 1988.
9. BARBU, V.; PRECUPANU, T. – *Convexity and Optimization in Banach Spaces,* D. Reidel Publish. Co., Dordrecht, 1986.
10. BORWEIN, J.M.; PREISS, D. – *A smooth variational principle with applications to subdifferentiability and to differentiability of convex functions*, Trans. Amer. Math. Soc., 303(1987), 517-527.
11. CLARKE, F.H. – *Optimization and nonsmooth analysis*, Wiley, New York, 1983.
12. CROUZEIX, J.P. – *Contribution à l'étude des fonctions quasi-convexes,* Thèse de Docteur en Sciences, Univ. Clermont-Ferrand II, 1977.
13. LUC, D.T.; SCHAIBLE, S. – *Generalized monotone nonsmooth maps*, J. Convex Anal., 3(1996), 195-205.

14. MANGASARIAN, O.L. – *Pseudo-convex functions*, J. Soc. Indust. Appl. Math. Control, 3(1965), 281-290.

15. PENOT, J.P.; QUANG, H.P. – *Generalized convexity of functions and generalized monotonicity of set-valued maps*, J. Optim. Theory Appl., 92(1997), 343-356.

16. PHU, H.X.; AN, P.T. – *Stable generalization of convex function*, Optimization, 38(1996), 309-318.

17. PRECUPANU, T.; STAMATE, C. – *Approximative quasisubdifferentials*, Optimization, 56(2007), 339 - 354.

18. ROCKAFELLAR, R.T. – *Generalized directional derivatives and subgradients of non-convex functions*, Canad. J. Math., 32(1980), 257-280.

Received: 7.VI.2007

"Al.I. Cuza" University,
Faculty of Mathematics,
Bd. Carol I, no 11, 700506, Iaşi,
ROMÂNIA
mapetrii@uaic.ro