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ON THE $\mathcal{F}$-FLAT STRUCTURES AND ASSOCIATED CONNECTIONS

BY

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Abstract. The results of this text are devoted to differential geometry of vector bundles over foliated manifolds. The purpose of the paper is to study $\mathcal{F}$-flat connections in a vector bundle, and integrable $\mathcal{F}$-flat structures on a differentiable manifold (in the tangent bundle). The notion of a $\mathcal{F}$-flat structure is central in this work.

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1. Introduction. In this paper the word "foliation" means a foliated atlas and a decomposition of a manifold M into connected submanifolds of dimension p (leaves) [4]. Our idea is to replace the tangential foliation of a manifold M (defined by the connected components of M) by an arbitrary foliation $\mathcal{F}$ of M.

In the first section we develop a study of the $\mathcal{F}$-flat connections in a vector bundle over a foliated manifold. We consider the vector bundles η over the manifold M foliated by a foliation $\mathcal{F}$, for which the transition functions are constant along the leaves of $\mathcal{F}$. It is interesting to note that this condition can be expressed in terms of the curvature $\Omega(\nabla)$ of a certain connection ∇ in the vector bundle $\eta : \Omega(\nabla)$ vanishes along the leaves of $\mathcal{F}$ ([3]). In this work, it is presented a characterization of an $\mathcal{F}$-flat connection (Theorem 2.2).

*Á la mémoire des mes grands-parents Pachiţa et Teodor Lăcătuşu

We determine the coordinate transformations under which $\mathcal{F}$ defines an integrable $\mathcal{F}$-flat structure on the manifold M. The integrable $\mathcal{F}$-flat structures are particular foliations on M.

In the second section, it is convenient to think an $\mathcal{F}$-flat connection as an $\mathcal{F}$-flat structure. We obtain the geometric structure of the tangent vector bundle TM endowed with an integrable $\mathcal{F}$-flat structure (Theorem 3.2). The method of proof uses some existence results for the solutions of Partial Differential Equations which depend on parameters (or the theorem of Frobenius) and the parallelism defined by a connection along the leaves of $\mathcal{F}$ ([6]). This parallelism connects the properties of an $\mathcal{F}$-flat structure and $\mathcal{F}$-flat connections. We give some general results for such connections and structures. Remarkable situations are presented.

Some properties of special connections in a tangent vector bundle, associated to integrable $\mathcal{F}$-flat structures are given in the next section.

2. $\mathcal{F}$-flat connections. The geometrical objects considered in this paper are of class C^∞. The summation convention is used. The bases of the vector bundles are manifolds equipped with foliated structures. Let us fix some notations. Let M be a differentiable manifold of dimension m and $\mathcal{F} =\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$ be a foliation of dimension p on M. Suppose that $\mathcal{F}$ is maximal. Denote by TM the tangent bundle of M, $T\mathcal{F}$ the tangent bundle to $\mathcal{F}$, and $\tau\mathcal{F} =TM/T\mathcal{F}$ the transverse bundle of $\mathcal{F}$. Let $\eta = (E, \pi, M)$ be a vector bundle over M with the total space E, the fiber $\mathbb{R}^n$, projection π. Denote by s_α, s_β two frame fields of η on U_α and U_β, respectively, where $U_\alpha \cap U_\beta \neq \phi$. On $U_\alpha \cap U_\beta$ we have $s_\alpha = s_\beta A_{\beta\alpha}$, where $\{A_{\alpha\beta}\}_{\alpha,\beta\in I}$ are the transition functions of η. Now we can formulate the definition of the $\mathcal{F}$-flat structures in η as follows ([3]).

Definition 2.1. A set $\Lambda = \{(s_\alpha, A_{\alpha\beta}) \,/\, s_\alpha = s_\beta A_{\beta\alpha}, U_\alpha \cap U_\beta \neq \phi\}_{\alpha,\beta\in I}$ defines *an $\mathcal{F}$-flat structure in the vector bundle η* if the transition functions $A_{\alpha\beta}$ are constant matrices along the leaves of $\mathcal{F}$ (a. l. $\mathcal{F}$.). The vector bundle η endowed with an $\mathcal{F}$-flat structure is called *an $\mathcal{F}$-flat vector bundle.*

Remark. If η is an $\mathcal{F}$-flat vector bundle, then any associated bundle to η is also $\mathcal{F}$-flat. The frames $\{s_\alpha\}$ are called *adapted to the $\mathcal{F}$-flat structure Λ of η.*

If $(U_\alpha, \varphi_\alpha) = \left(U_\alpha; x^i, x^{\widehat{i}}\right)$, $i = 1, 2, ..., p$, $\widehat{i} = p+1, ..., m$, is a chart of $\mathcal{F}$, $u = \varphi_\alpha^{-1}(x, \widehat{x}) \in U_\alpha \cap U_\beta$, then the matrices $A_{\alpha\beta}(u) = \left(A_a^b \circ \varphi_\alpha^{-1}(x, \widehat{x})\right)$,

$a, b = 1, 2, ..., n$, satisfy the following conditions: $\frac{\partial A_{\alpha\beta}}{\partial x^i} = 0$, $\forall \alpha, \beta \in I$. Here $x = \left(x^i\right)$ are the coordinates in a leaf of $\mathcal{F}$, $\widehat{x} = (\widehat{x^i})$ are secondary coordinates. We suppose that U_α, U_β, and $U_\alpha \cap U_\beta$ are connected, $\forall \alpha, \beta \in I$; then these Partial Differential Equations characterize the $\mathcal{F}$-flat structure Λ in η. If $\mathcal{F}$ is the tangent foliation of M, then an $\mathcal{F}$-flat structure is a flat structure in η ([8]). Moreover, if $\mathcal{F}'$ is a subfoliation of $\mathcal{F}$, then the $\mathcal{F}$-flat structure Λ is also an $\mathcal{F}'$-flat structure in η. We assume that the atlas $\mathcal{F}$ is maximal and denote $\mathcal{U} = \{U_\alpha\}_{\alpha \in I} = \{U, U', ...\}$.

Definition 2.2. A connection ∇ in the vector bundle η is *an $\mathcal{F}$-flat connection* if the curvature of ∇ is zero along the leaves of $\mathcal{F}$: $\Omega(\nabla)(X, Y) = 0$, $\forall X, Y \in \Gamma(T\mathcal{F})$.

Let i be the inclusion of a leaf $F \in \mathcal{F}$ in M. We will identify any tangent vector X to F with its image $di(X)$ through the differential di of i. We first study the properties of the $\mathcal{F}$-flat connections in η.

Remark. The matrices $A_U^{U'} = (A_a^{a'})$ are constant a.l. $\mathcal{F}$. iff $X(A_a^{a'}) = 0$, $X \in \Gamma(T\mathcal{F})$, $a, a' = 1, 2, ..., n$.

Lemma 2.1. *Let $\eta = (E, \pi, M)$ be a vector bundle over an $\mathcal{F}$-foliated manifold M, ∇ a connection in η, $s_U = (s_a)$, $s_{U'} = (s_{a'})$ be local frames of η, and $s_a = A_a^{a'} s_{a'}$ on $U \cap U' \neq \phi$. If two of the following conditions are true:*

1) s_U is parallel in connection ∇ a. l. .$\mathcal{F}$.;

2) $s_{U'}$ is parallel in connection ∇ a. l. .$\mathcal{F}$.;

3) $A_U^{U'} = (A_a^{a'})$ is a constant matrix a. l. .$\mathcal{F}$. (i.e. $A_a^{a'} = A_a^{a'}(\widehat{x})$),

then the third condition is also true ($a, a', b, b' = 1, ..., n$, $n = \dim E_u$, $u \in M$).

Proof. The conditions of this lemma are given, respectively, through the relations: $\nabla_X s_a = 0$, $\nabla_X s_{a'} = 0$, $X(A_a^{a'}) = 0$, $X \in \Gamma(T\mathcal{F})$. The equations of connection of ∇ are

$$\nabla_X s_a = \omega_a^b(X)\, s_b,\ \nabla_X s_{a'} = \omega_{a'}^{b'}(X)\, s_{b'},$$

where $\omega = (\omega_a^b)$, $\omega' = (\omega_{a'}^{b'})$ are the connection forms of ∇ relative to s_U and $s_{U'}$, respectively. On $U \cap U'$ we can write

$$\omega_a^b(X) = A_{b'}^b \omega_{c'}^{b'}(X)\, A_a^{c'} + A_{b'}^b dA_a^{b'}(X),\ X \in \Gamma(T\mathcal{F}).$$

We prove that 1), 3) $\Rightarrow$2). From the previous relations, we obtain $\omega_{c'}^{b'}(X)=0$. Hence $\nabla_X s_{a'}=0$. This implies that $s_{U'}$ is parallel in connection ∇ a. l. .$\mathcal{F}$. The implications 1), 2) $\Rightarrow$ 3); 2), 3) $\Rightarrow$ 1) follow similarly. $\square$

Theorem 2.1. *Let η be a vector bundle over an $\mathcal{F}$-foliated manifold M and ∇ be a connection in η. We denote by $\{U\}$ an open cover $\mathcal{U}$ of M with local framefields (s_U) of η. If the frames s_U, $U\in\mathcal{U}$, are parallel in connection ∇ a.l. $\mathcal{F}$., then the set $\Lambda=\{(U,s_U),\ s_U=s_{U'}A_U^{U'}\}_{U,U'\in\mathcal{U}}$ defines an $\mathcal{F}$-flat structure in η and the connection ∇ is $\mathcal{F}$-flat (∇ defines the $\mathcal{F}$-flat structure Λ).*

Proof. From hypothesis, $\nabla_X s_U=\omega_U(X)\,s_U=0$, i.e. $\omega_U(X)=0$, $X\in\Gamma(T\mathcal{F})$, where ω_U is the connection form of ∇ with respect to the frame fields s_U. If we consider $\nabla_X s_{U'}=\omega_{U'}(X)\,s_{U'}=0$, then $\omega_{U'}(X)=0$. In other words, by Lemma 1.1, we can write $X(A_a^{a'})=0$, $X\in\Gamma(T\mathcal{F})$, where $s_a=A_a^{a'}s_{a'}$, on $U\cap U'\neq\phi$. Thus, $A_a^{a'}=A_a^{a'}(\widehat{x})$, so Λ defines an $\mathcal{F}$-flat structure in η. We now show that ∇ is $\mathcal{F}$-flat. Let $\omega_U=\left(\omega_a^b\right)$, where $\omega_a^b=\Gamma_{ak}^b dx^k+\Gamma_{a\widehat{k}}^b dx^{\widehat{k}}$ are the local expressions of ω_a^b. Then $\omega_a^b\left(\frac{\partial}{\partial x^k}\right)=\Gamma_{ak}^b=0$. Hence $\omega_a^b=\Gamma_{a\widehat{k}}^b dx^{\widehat{k}}$. The equations of structure of ∇, $\Omega_a^b=d\omega_a^b+\omega_c^b\Lambda\omega_a^c$, implies that $\Omega_a^b(X,Y)=0$, $X,Y\in\Gamma(T\mathcal{F})$. This proves that the connection ∇ is $\mathcal{F}$-flat and defines an $\mathcal{F}$-flat structure Λ ([3]). $\square$

Remarks. 1) If $\omega_U=\left(\omega_a^b\right)$ is the connection form of ∇ relative to s_U, then $\nabla_X s_U=0$, $X\in\Gamma(T\mathcal{F})$, is equivalent to $\omega_a^b=\Gamma_{a\widehat{k}}^b(x,\widehat{x})\,dx^{\widehat{k}}$. This can be used to prove that certain Pontrjagin classes of E vanishes ([3]).

2) Since the connection ∇ is $\mathcal{F}$-flat, the parallel transport in connection ∇ along a curve c in a leaf $F\in\mathcal{F}$ depends only on the homotopy class of c in F with fixed end points.

Lemma 2.2. *Let ∇ be a connection $\mathcal{F}$-flat in the vector bundle $\eta=(E,\pi,M)$. Then there exists an open cover $\mathcal{U}=\{U\}$ of M with local frame fields $\{s_U\}=(\underset{U}{s_a})$ of η, parallel in the connection ∇ along the leaves of $\mathcal{F}$, i.e. $\nabla_X\ \underset{U}{s_a}=0$, $\forall X\in\Gamma(T\mathcal{F})$, $a=1,2,...,n$, $U\in\mathcal{U}$.*

Proof. Let $\sigma=(\sigma_a)$ be an arbitrary local frame field of η over U, where U is a trivializing neighborhood of η. Let $\omega=\left(\omega_a^b\right)$ be the connection form of ∇ with respect to σ. Now, consider the frame $s_U=\sigma A$ over U, where

$A=(A_b^a)$ is a unknown matrix and $s_U=(s_a)$. The condition from problem is $\nabla_X s_a=\overline{\omega}_a^b(X)\,s_b=0$, $\forall X\in\Gamma(T\mathcal{F})$, where $\overline{\omega}=\left(\overline{\omega}_a^b\right)$ is the connection form of ∇ relative to s_U. Then,

$$\overline{\omega}(X)=0=A^{-1}\omega(X)\,A+A^{-1}dA(X), \tag{2.1}$$

condition equivalent with

$$\omega(X)\,A+dA(X)=0,\ \forall X\in\Gamma(T\mathcal{F}).$$

For $X=\frac{\partial}{\partial x^k}$, $A=(A_b^a)$, $\omega_a^b=\Gamma_{ba_1}^a dx^{a_1}$, $a_1=1,...,m$, $k=1,...,p$, $dx^{\widehat{k}}\left(\frac{\partial}{\partial x^k}\right)=\delta_k^{\widehat{k}}=0$, $\widehat{k}=p+1,...,m$ and therefore, the local components Γ_{bk}^a satisfy the equations

$$\Gamma_{bk}^a A_c^b+\frac{\partial A_c^a}{\partial x^k}=0. \tag{2.2}$$

In this system of P.D.E., $\widehat{x}=(x^{\widehat{k}})$ are some parameters and $x=\left(x^k\right)$ are independent variables. We transform this system in the Pfaff system

$$dA_c^a+\Gamma_{bk}^a A_c^b dx^k=0, \tag{2.3}$$

where d is the exterior differentiation operator. Using Frobenius theorem, we obtain the following compatibility conditions on U :

$$\left(\frac{\partial\Gamma_{bk}^a}{\partial x^h}-\frac{\partial\Gamma_{bh}^a}{\partial x^k}+\Gamma_{dk}^a\Gamma_{bh}^d-\Gamma_{dh}^a\Gamma_{bk}^d\right)A_c^b=0, \tag{2.4}$$

where $\det\left(A_c^b\right)\neq 0$, $1\leq h<k\leq p$ and $a,b,c,d=1,...,n$. Now, we can fix a,k,h. From these relations and $\det\left(A_c^b\right)\neq 0$, we get the following identities:

$$\frac{\partial\Gamma_{bk}^a}{\partial x^h}-\frac{\partial\Gamma_{bh}^a}{\partial x^k}+\Gamma_{dk}^a\Gamma_{bh}^d-\Gamma_{dh}^a\Gamma_{bk}^d=0. \tag{2.5}$$

On the other hand, one can prove that $\Omega(\nabla)\left(\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^h}\right)=0$ (where $\Omega(\nabla)$ is the curvature of ∇) coincide with the relations (2.5). Hence, there exists the frame s_U. By these considerations, by Lemma 2.2 and Theorem 2.1, we obtain finally the characterization of $\mathcal{F}$-flat connections:

Theorem 2.2. *Let $\eta=(E,\pi,M)$ be a vector bundle over the $\mathcal{F}$-foliated manifold M. A connection ∇ in η is $\mathcal{F}$-flat if and only if there exists an*

open cover $\mathcal{U} = \{U\}$ *of* M *endowed with local frame fields* $\{s_U\}$ *of* η, *parallel in the connection* ∇ *along the leaves of* $\mathcal{F}$.

Remark. Applying the method from [3] to system (2.2), we also obtain (2.5). Using this theorem together with Theorem 4 of [3], we have the following:

Corollary. *A connection* ∇ *in the vector bundle* $\eta = (E, \pi, M)$ *defines an* $\mathcal{F}$*-flat structure in* η *iff there exist an open cover* $\mathcal{U} = \{U\}$ *of* M *and the frame fields* $\{s_U\}$ *of* η, *parallel in the connection* ∇ *along the leaves of* $\mathcal{F}$.

Proposition 2.1. *Let* ∇ *be a connection in the vector bundle* $\eta = (E, \pi, M)$ *over an* $\mathcal{F}$*-foliated manifold* M *and* ∇^* *be the induced connection by* ∇ *in the dual vector bundle* $\eta^* = (E^*, \pi^*, M)$. *Then,* ∇ *is* $\mathcal{F}$*-flat iff* ∇^* *is* $\mathcal{F}$*-flat.*

Proof. Let ∇ be an $\mathcal{F}$-flat connection. By Lemma 2.2, there exists an open cover $\mathcal{U} = \{U\}$ with a frame field (s_a) of η on U, such that $\nabla_X s_a = 0$, $\forall X \in \Gamma(T\mathcal{F})$. Let $s^* = \left(\widetilde{s}^b\right)$ be the dual coframe field of (s_a). Considering $v = \lambda^b s_b \in \Gamma(E)$, one obtains:

$$\left(\nabla^*_X \widetilde{s}^a\right)(v) = X\left(\widetilde{s}^a(v)\right) - \widetilde{s}^a\left(\nabla_X v\right) = X\left(\lambda^a\right) - X\left(\lambda^a\right) = 0.$$

Therefore the curvature form of ∇^* satisfies the condition $\Omega(\nabla^*)(X, Y) = 0$, $X, Y \in \Gamma(T\mathcal{F})$. The converse affirmation is immediate.

Given an $\mathcal{F}$-flat connection ∇ in the vector bundle $\eta = (E, \pi, M)$, there is a frame field (s_a) of η on U parallel in connection ∇ a. l. $\mathcal{F}.$, $\nabla_X s_a = 0$, $X \in \Gamma(T\mathcal{F})$; let $\left(\widetilde{s}^b\right)$ be its dual frame. Now consider an arbitrary frame (σ_b) of η on U and $\left(\widetilde{\sigma}^b\right)$ its dual frame. If $s_a = \xi^b_a \sigma_b$ and $\widetilde{s}^b = \widetilde{\xi}^b_a \widetilde{\sigma}^a$, then $\widetilde{\xi}^b_c \xi^c_a = \delta^b_a$. We now consider

$$\nabla_X s_a = \nabla_X\left(\xi^b_a \sigma_b\right) = \left[d\xi^b_a(X) + \omega^b_c(X)\,\xi^c_a\right] s_b = 0,$$

where $\omega = \left(\omega^b_c\right)$ is the connection form of ∇ with respect to the frame field (σ_b) and $X \in \Gamma(T\mathcal{F})$. Then

$$d\xi^b_a(X) + \omega^b_c(X)\,\xi^c_a = 0. \tag{2.6}$$

Differentiating the relations $\widetilde{\xi}_c^b \xi_a^c = \delta_a^b$ and using the equalities (2.6), we obtain

$$[d\widetilde{\xi}_c^b(X) - \widetilde{\xi}_d^b \omega_c^d(X)]\xi_a^c = 0,$$

where $a, b, c, d = 1, ..., n$. Since $\det[\xi_a^c] \neq 0$ on U, we have

$$d\widetilde{\xi}_c^b(X) = \omega_c^d(X)\,\widetilde{\xi}_d^b. \tag{2.7}$$

Contracting (2.7) with (ξ_b^a), one obtains

$$\xi_b^a d\widetilde{\xi}_c^b(X) = \omega_c^d(X)\,\delta_d^a = \omega_c^a(X).$$

Consequently,

$$\omega_c^a(X) = \xi_b^a d\widetilde{\xi}_c^b(X),\ X \in \Gamma(T\mathcal{F}). \tag{2.8}$$

Let $\omega_c^a = \Gamma_{ck}^a dx^k + \Gamma_{c\widehat{k}}^a dx^{\widehat{k}}$ be the expression of ω_c^a in the local chart $(U; x, \widehat{x}) \in \mathcal{F}$. Putting $X = \partial/\partial x^k$ in (2.8), one obtains $\Gamma_{ck}^a = \omega_c^a\left(\partial/\partial x^k\right)$ and $\Gamma_{c\widehat{k}}^a$ are arbitrary functions, $k = 1, ..., p$; $\widehat{k} = p+1, ..., m$. So, we have

Theorem 2.3. *Let ∇ be an $\mathcal{F}$-flat connection in the vector bundle $\eta = (E, \pi, M)$, (s_a) a frame field of η on U, parallel in connection ∇ along the leaves of $\mathcal{F}$ and $\left(\widetilde{s}^b\right)$ its dual coreper. Let (σ_a) be an arbitrary frame field of η on U and $\left(\widetilde{\sigma}^b\right)$ be its dual coreper. Then,*

$$\Gamma_{ck}^a = \xi_b^a \frac{\partial \widetilde{\xi}_c^b}{\partial x^k}$$

and $\Gamma_{c\widehat{k}}^a$ are arbitrary real functions on U.

In particular, if $\mathcal{F}$ is the tangential foliation of M, we obtain the following

Corollary 2.1. *Let $\mathcal{F}$ be the tangential foliation of M. Consider ∇ a flat connection in the vector bundle $\eta = (E, \pi, M)$, (s_a) a frame field of η on U, parallel in connection ∇, and $\left(\widetilde{s}^b\right)$ its dual frame. If (σ_b) is an arbitrary frame field and $\left(\widetilde{\sigma}^b\right)$ is its dual frame, then*

$$\Gamma_{cb_1}^a = \xi_d^a \frac{\partial \widetilde{\xi}_c^d}{\partial x^{b_1}},\ b_1 = 1, ..., m,\ \text{where } s_a = \xi_a^b \sigma_b,\ \widetilde{s}^b = \widetilde{\xi}_a^b \widetilde{\sigma}^a.$$

Remark. Suppose that $E = TM$ and $\mathcal{F}$ is the tangent foliation of M. Then we obtain some well-known results from [9].

3. Integrable $\mathcal{F}$-flat structures

3.1. Coordinates transformations of $\mathcal{F}$. We say that an $\mathcal{F}$-flat structure Λ on M (in TM) is *integrable* if Λ is defined by the family $\{\partial/\partial x^a\}$ of natural frames and Jacobian matrices $J_{UU'} = \left\{\partial x^{a'}/\partial x^a\right\}$ associated to $\mathcal{F}$. We shall give the analytical characterization of an integrable $\mathcal{F}$-flat structure. In general, the study of an integrable structure requires the study of a system of partial differential equations (P.D.E.). We determine the Jacobian matrix of the coordinate transformations under which $\mathcal{F}$ defines an integrable $\mathcal{F}$-flat structure on M. The coordinate transformations of $\mathcal{F}$ have the form $x^{i'} = x^{i'}(x^i, x^{\widehat{i}})$, $x^{\widehat{i}'} = x^{\widehat{i}'}(x^{\widehat{i}})$. Then the conditions on its Jacobian matrix can be written as

$$\left\{\begin{array}{l} \dfrac{\partial x^{i'}}{\partial x^i} = A_i^{i'}(\widehat{x}) \\ \dfrac{\partial x^{i'}}{\partial x^{\widehat{i}}} = B_{\widehat{i}}^{i'}(\widehat{x}). \end{array}\right. \tag{3.1}$$

Therefore, we start with arbitrary local functions $A_i^{i'}(\widehat{x})$, $B_i^{i'}(\widehat{x})$ and try to find $(x', \widehat{x}')$ solution of the system (3.1). In this system of P.D.E., $x^{i'}$ are the unknown C^∞ functions. In order to solve this system, we integrate the first group of equations (3.1). The solutions are $x^{i'} = A_i^{i'}(\widehat{x})\, x^i + b^{i'}(\widehat{x})$, where $b^{i'}(\widehat{x})$ denote some arbitrary functions (in particular $b^{i'}$ =constant). To determine $b^{i'}(\widehat{x})$, we require the functions $x^{i'}$ to verify the last equations. Then we obtain the system

$$\frac{\partial A_i^{i'}}{\partial x^{\widehat{i}}} x^i + \frac{\partial b^{i'}}{\partial x^{\widehat{i}}} = B_{\widehat{i}}^{i'}. \tag{3.2}$$

The obtained functions $x^{i'}$ verify the integrability conditions (i. e. commutability of the mixed derivatives). Then, using (3.1), we have

$$\frac{\partial^2 x^{i'}}{\partial x^i \partial x^j} = \frac{\partial A_i^{i'}}{\partial x^{\widehat{j}}} = \frac{\partial^2 x^{i'}}{\partial x^{\widehat{j}} \partial x^i} = \frac{\partial B_{\widehat{j}}^{i'}}{\partial x^i} = 0.$$

The remaining integrability conditions are identically verified in the C^∞ class. This system gives $A_i^{i'}$ =constant. Using (3.1) we obtain the following

conditions: $\frac{\partial b^{i'}}{\partial x^{\widehat{i}}} = B^{i'}_{\widehat{i}}$. It is clear that any $b^{i'}$ depends only on $\widehat{x} = (x^{\widehat{i}})$, i.e. $b^{i'} = b^{i'}(\widehat{x})$. Put $b = (b^{i'}(\widehat{x}))$. Using the notations $a^{i'}_i = A^{i'}_i$, we may write

$$\left\{ \begin{array}{c} x^{i'} = a^{i'}_i x^i + b^{i'}(\widehat{x}) \\ x^{\widehat{i'}} = x^{\widehat{i'}}(\widehat{x}) . \end{array} \right. \tag{3.3}$$

We can rewrite (3.3) in the matrix notation

$$\left\{ \begin{array}{c} x' = a \cdot x + b(\widehat{x}) \\ \widehat{x}' = \widehat{x}'(\widehat{x}) , \end{array} \right. \tag{3.4}$$

where $a = (a^{i'}_i) : U \cap U' \to GL(m, \mathbb{R})$ is a constant matrix-valued function on $U \cap U'$. Hence, if $\mathcal{F}$ defines an integrable $\mathcal{F}$-flat structure on M, then the Jacobian matrices of coordinate changes are constant with respect to $x = \left(x^i\right)$. The maximal atlas $\mathcal{F}$ of M whose changes of coordinates are given by (3.4) defines an integrable $\mathcal{F}$-flat structure on M. This assertion is immediate. By the previous considerations, we obtain finally the following

Theorem 3.1. *The foliated atlas $\mathcal{F}$ of M defines an integrable $\mathcal{F}$-flat structure on M if and only if the changes of coordinates of $\mathcal{F}$ have the form* (3.4), *where $a = (a^{i'}_i) \in GL(p, \mathbb{R})$ is a constant matrix and $b(\widehat{x}) = (b^{i'}(\widehat{x}))$ is an arbitrary matrix.*

We shall see that in this case, each leaf of $\mathcal{F}$ possesses a locally affine structure (Theorem 3.3).

Using this theorem we may determine the geometrical structure of TM (see Theorem 3.2 below).

Remark. Theorem 3.1 shows that the definition of an $\mathcal{F}$-flat structure does not affect some aspects from the geometry of $\tau\mathcal{F}$.

3.2. Structure of the tangent bundle of a differentiable manifold endowed with an integrable $\mathcal{F}$-flat structure

Lemma 3.1. *Let η_1, η_2 be two vector bundles over the $\mathcal{F}$-foliated manifold M. If η_1 and η_2 are $\mathcal{F}$-flates, then $\eta_1 \oplus \eta_2$ is $\mathcal{F}$-flat.*

Proof. Let $\{g'_{\alpha\beta}\}$, $\{g''_{\alpha\beta}\}$ $(\alpha, \beta \in I)$ be the transition functions of η_1 and η_2, respectively. Then the applications $\{g_{\alpha\beta}\}$ defined by

$$g_{\alpha\beta} = \left(\begin{array}{cc} g'_{\alpha\beta} & 0 \\ 0 & g''_{\alpha\beta} \end{array} \right)$$

are constant along the leaves of $\mathcal{F}$. Therefore, the vector bundle $\eta_1 \oplus \eta_2$ is $\mathcal{F}$-flat. □

Now we obtain some other conditions for $\mathcal{F}$-flatness. For our purpose, it is convenient to formulate an open problem: give the explicit expression of the vector bundle η endowed with an $\mathcal{F}$-flat structure. We give an answer for the tangent bundle TM of differentiable manifold endowed with an $\mathcal{F}$-flat structure. We know that the foliation $\mathcal{F}$ of M is parallelizable if there exists the vector fields $X_1, ..., X_p$ tangent to the leaves of $\mathcal{F}$ and they are everywhere linearly independent ($\mathcal{F}$ is defined by $X_1, ..., X_p$). An isomorphism of vector bundles will be denoted by the sign "=".

Theorem 3.2. *Let M be a differentiable manifold of dimension m, $\mathcal{F} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ be a foliation of dimension p and $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be the cover of M. Then the following conditions are equivalent:*

a) *M is endowed with an integrable $\mathcal{F}$-flat structure*

$$\Lambda = \left\{ \left(U; \frac{\partial}{\partial x^a_U},\ J_{U,U'} \right) \right\}_{U,U' \in \mathcal{U}},\quad a = 1, ..., m;$$

b) *The structure of TM is given by the isomorphism $TM = (M \times \mathbb{R}^p) \oplus \tau\mathcal{F}$;*

c) *Foliation $\mathcal{F}$ is parallelizable.*

Proof. *a) ⇒ b).* We use the previous theorem and Lemma 3.1. Fix the arbitrary functions $b^{i'}(\widehat{x})$ in (3.3) . The Jacobian matrix of (3.4) is

$$J_{UU'} = \begin{pmatrix} a_{UU'} = \left(a^{i'}_i \right) & \dfrac{\partial b^{i'}}{\partial x^{\widehat{i}}} \\ 0 & \dfrac{\partial x^{\widehat{i'}}}{\partial x^{\widehat{i}}} \end{pmatrix}_{U,U' \in \mathcal{U}},\ U \cap U' \neq \phi.$$

The transition functions of $T\mathcal{F}$ are given by the cocycles defined on $U \cap U'$ by $a_{UU'} = \left(a^{i'}_i \right)$, where $a^{i'}_i$ are constants. It follows that the vector bundle $T\mathcal{F}$ is trivial, $T\mathcal{F} = M \times \mathbb{R}^p$. Note that for any foliated manifold , the tangent bundle is $TM = T\mathcal{F} \oplus \tau\mathcal{F}$ and that the transition functions of $\tau\mathcal{F}$ are $\{\partial x^{\widehat{i'}} / \partial x^{\widehat{i}}\}_{U,U' \in \mathcal{U}}$. Therefore, we have $TM = (M \times \mathbb{R}^p) \oplus \tau\mathcal{F}$.

b) ⇒ a). Using the conditions *b)* and $TM = T\mathcal{F} \oplus \tau\mathcal{F}$, we obtain $T\mathcal{F} = M \times \mathbb{R}^p$, i.e. $T\mathcal{F}$ is flat. But the transition functions of $\tau\mathcal{F}$ are

$\{\partial\widehat{x^{i'}}/\partial\widehat{x^{i}}\}$, i.e. $\tau\mathcal{F}$ is $\mathcal{F}$-flat. Then, the transition functions of TM are constant with respect to $x=\left(x^{i}\right)$. Thus, by Lemma 3.1, TM is $\mathcal{F}$-flat. M is endowed with an integrable $\mathcal{F}$-flat structure.

b) $\Rightarrow$ *c*). Considering the condition *b*), one obtains $T\mathcal{F}=M\times\mathbb{R}^{p}$. This means that there is a global frame $(X_1,...,X_p)$ on M, which is tangent to $\mathcal{F}$.

The implication *c*) $\Rightarrow$ *b*) is clear. The theorem is proved. □

Remark. The concept of an integrable $\mathcal{F}$-flat structure contains a strong condition.

In the general case, the matrix $b=(b^{i'}(\widehat{x}))$ from relations (3.4) is arbitrary. Now we find out under which condition $\mathcal{F}$ defines a locally affine (locally centro-affine) structure on each leaf of $\mathcal{F}$.

Theorem 3.3. *Let $\mathcal{F}$ be a foliation on a manifold M. Then:*

a) *$\mathcal{F}$ defines an integrable $\mathcal{F}$-flat structure on M iff $\mathcal{F}$ determines a locally affine structure $\mathcal{S}_F$ on each leaf F of $\mathcal{F}$.*

b) *$\mathcal{S}_F$, $F\in\mathcal{F}$, are locally centro-affine structures iff the local vector fields $\left\{x^{i}\frac{\partial}{\partial x^{i}}\right\}$ define a global vector field X on M.*

Proof. a) Denote by F the leaf of $\mathcal{F}$ containing $p\in U\cap U'$. F is determined locally by the equations $x^{\widehat{i}}=x^{\widehat{i}}(p)$, where p has the coordinates $(x^{i}(p),x^{\widehat{i}}(p))$. If $\mathcal{F}$ defines an $\mathcal{F}$-flat structure on M, then we have $x^{i'}=a_{i}^{i'}x^{i}+c^{i'}$, for the induced coordinates on F, where $c^{i'}$ are some constants. Hence $\mathcal{F}$ determines a locally affine structure on F. Conversely, if $\mathcal{F}$ induces a locally affine structure $\mathcal{S}_F$ on F, the coordinates transformations are given by $x^{i'}=a_{i}^{i'}x^{i}+c^{i'}$, $x^{\widehat{i'}}=x^{\widehat{i'}}(x^{\widehat{i}})$, $\det(a_{i}^{i'})\neq 0$, $c^{i'}=$constant. Since F is arbitrary, it follows that $\mathcal{F}$ determines an $\mathcal{F}$-flat structure on M.

b) Let $\widetilde{a}=\left(\widetilde{a}_{i'}^{i}\right)$ be the inverse matrix of $a=(a_{i}^{i'})$. Considering the first group of relations (3.3), one obtains:

$$x^{i}=\widetilde{a}_{i'}^{i}x^{i'}+\widetilde{b}^{i}\left(\widehat{x}'\right),\ \text{where}\ \widetilde{b}^{i}\left(\widehat{x}'\right)=\widetilde{a}_{i'}^{i}b^{i'}\left(x\left(\widehat{x}'\right)\right). \tag{3.5}$$

Using (3.4), (3.5), and the relations $\widetilde{a}_{i'}^{i}a_{i}^{k'}=\delta_{i'}^{k'}$, we have:

$$x^{i}\frac{\partial}{\partial x^{i}}=\left[\widetilde{a}_{i'}^{i}x^{i'}+\widetilde{b}^{i}\left(\widehat{x}'\right)\right]a_{i}^{k'}\frac{\partial}{\partial x^{k'}}=\left[\delta_{i'}^{k'}x^{i'}+a_{i}^{k'}\widetilde{b}^{i}\left(\widehat{x}'\right)\right]\frac{\partial}{\partial x^{k'}}. \tag{3.6}$$

Finally, one obtains

$$x^{i}\frac{\partial}{\partial x^{i}}=x^{i'}\frac{\partial}{\partial x^{i'}}+a_{i}^{k'}\widetilde{b}^{i}\left(\widehat{x}'\right)\frac{\partial}{\partial x^{k'}}. \tag{3.7}$$

The local vector fields $x^i\frac{\partial}{\partial x^i}$, $x^{i'}\frac{\partial}{\partial x^{i'}}$ agree on the intersection $U\cap U'$ iff $a_i^{k'}\widetilde{b}^i(\widehat{x}')=0$. Since $\det(a_i^{k'})\neq 0$, it follows that $\widetilde{b}^i(\widehat{x}')=0$. Hence, the coordinate transformations on F are $x^{i'}=a_i^{i'}x^i$. The structure $\mathcal{S}_F$ is a locally centro-affine structure.

Conversely, suppose that $\mathcal{S}_F$ is locally centro-affine. Then, $x^i\frac{\partial}{\partial x^i}=x^{i'}\frac{\partial}{\partial x^{i'}}$ and $\left\{x^i\frac{\partial}{\partial x^i}\right\}_{U,U'\in\mathcal{U}}$ defines a global vector field X on M, tangent to $\mathcal{F}$. □

Remarks. 1) Global vector field X is an infinitesimal automorphism of the foliated structure $\mathcal{F}$ (i.e. the corresponding flow consists of the isomorphisms of $\mathcal{F}$). The integral curves of X are represented by $x^i=\pm e^{t+c^i}$, $x^{\widehat{i}}=$ constant, where c^i are constants and $t\in\mathbb{R}$ is a parameter.

2) Let $Y=Y^i\frac{\partial}{\partial x^i}+Y^{\widehat{i}}\frac{\partial}{\partial x^{\widehat{i}}}$ be a vector field on M and $X=x^i\frac{\partial}{\partial x^i}$ be as in the previous result. Then

$$[X,Y]=\left(x^k\frac{\partial Y^i}{\partial x^k}-Y^i\right)\frac{\partial}{\partial x^i}+x^i\frac{\partial Y^{\widehat{k}}}{\partial x^i}\frac{\partial}{\partial x^{\widehat{k}}}.$$

From this expression one obtains that $[X,Y]$ is an infinitesimal automorphism of $\mathcal{F}$ iff $\{Y^{\widehat{k}}\}$ are constant along the leaves of $\mathcal{F}$.

4. Special cases. The below results can be proved by Theorem 3.1; we can also find them directly.

4.1. Integrable $\mathcal{F}$-flat structures and integrable almost tangent structures. Let M be a differentiable manifold of dimension $m=2n$, $\mathcal{A}$ be the maximal atlas of M, and $(U;x=(x^i),\widehat{x}=(\widehat{x}^j))$, $(U';x'=(x^{i'}),\widehat{x}'=(\widehat{x}^{j'}))$ be two charts of $\mathcal{A}$, $i,j,i',j'=1,...,n$. $\mathcal{A}$ defines an integrable almost tangent structure on M if the coordinate transformations are of the type

$$\left\{\begin{array}{c} x'=(D\widehat{x}')_{\widehat{x}}\cdot x+b(\widehat{x}) \\ \widehat{x}'=\widehat{x}'(\widehat{x}), \end{array}\right. \tag{4.1}$$

where $\widehat{x}'=\widehat{x}'(\widehat{x})$ is a local diffeomorphism in $\mathbb{R}^n$ and $b(\widehat{x})$ is a function on $U\cap U'\neq\phi$ ([5]). The Jacobian matrices of these functions do not contain $\widehat{x}$ iff $\partial^2\widehat{x}^{i'}/\partial\widehat{x}^j\partial\widehat{x}^k=0$. By integration, we get $\widehat{x}'=a\widehat{x}+c$, where $a=(a_k^{i'})\in GL(n,\mathbb{R})$ and $c=(c^{i'})$ is a constant matrix. By (4.1) it follows that the coordinate transformations are of the type

$$\left\{\begin{array}{c} x'=a\cdot x+b(\widehat{x}) \\ \widehat{x}'=a\cdot\widehat{x}+c. \end{array}\right. \tag{4.2}$$

Therefore, such charts are adapted for an $\mathcal{F}$-flat structure on M, where $\mathcal{F}$ is the foliation of dimension n defined by the atlas $\mathcal{A}$. Since transition functions of transverse bundle $\tau\mathcal{F}$ are constant matrices $\{a\}$, $\tau\mathcal{F}$ is flat. Thus, we have established the following

Proposition 4.1. *Let M be a differentiable manifold of dimension $m = 2n$, endowed with an integrable almost tangent structure $\mathcal{A}$ and $\mathcal{F}$ be the underlying foliation defined by $\mathcal{A}$. Then $\mathcal{A}$ determines an (integrable) $\mathcal{F}$-flat structure on M iff $\tau\mathcal{F}$ is flat.*

4.2. Integrable $\mathcal{F}$-flat structures on $\tau\mathcal{F}$. Let $\mathcal{F}$ be a foliation on M as in the first section. Now, consider a class $\widehat{u} \in \tau\mathcal{F}$, where $u = u^k \frac{\partial}{\partial x^k} + u^{\widehat{k}} \frac{\partial}{\partial x^{\widehat{k}}} \in \widehat{u}$. Then, $(x = \left(x^i\right), \widehat{x} = (x^{\widehat{i}}), u = (u^{\widehat{k}}))$ are coordinates in $\tau_1^{-1}(U)$, where $\tau_1 : \tau\mathcal{F} \to M$ is the projection of $\tau\mathcal{F}$. The coordinates transformations on $\tau_1^{-1}(U \cap U')$, in matrix notation, are given by

$$x' = x'(x, \widehat{x}), \ \widehat{x}' = \widehat{x}'(\widehat{x}), \ \widehat{u}' = \frac{\partial \widehat{x}'}{\partial \widehat{x}} \widehat{u}, \tag{4.3}$$

where $x' = (x^{i'}), \widehat{x}' = (x^{\widehat{i'}}), \widehat{u}' = (u^{\widehat{k'}})$. We note that $\mathcal{A} = \left\{\left(\tau_1^{-1}(U); x, \widehat{x}, \widehat{u}\right)\right\}$ is an atlas of $\tau\mathcal{F}$ which defines an integrable G_T-structure ([7], [1], [2]). It follows that

$$\frac{\partial}{\partial \widehat{u}} = \frac{\partial \widehat{x}'}{\partial \widehat{x}} \cdot \frac{\partial}{\partial \widehat{u}'}.$$

Thus, there is a foliation $\mathcal{F}_1$ on $\tau\mathcal{F}$ defined by the equations $dx = 0$, $d\widehat{x} = 0$. So, $\mathcal{F}_1$ defines an $\mathcal{F}$-flat structure ($\mathcal{F} = \mathcal{F}_1$) iff $\partial^2 x^{\widehat{k'}} / \partial x^{\widehat{i}} \partial x^{\widehat{k}} = 0$, i.e.

$$x' = x'(x, \widehat{x}), \ \widehat{x}' = \widehat{a}' \cdot x + \widehat{b}(x), \ \widehat{u}' = \widehat{a}' \cdot \widehat{u},$$

where $\widehat{a}' = (a^{\widehat{i'}})$ is constant, $\det(\widehat{a}') \neq 0$.

Similarly, $\frac{\partial}{\partial x} = \frac{\partial x'}{\partial x} \frac{\partial}{\partial x'}$ and $\tau\mathcal{F}$ can be endowed with a foliation $\mathcal{F}_2$ defined by $d\widehat{x} = 0$, $d\widehat{u} = 0$. In this case, the coordinates transformations for $\mathcal{F}_2$ are

$$x' = \overline{a} \cdot x + \overline{b}(\widehat{x}), \ \widehat{x}' = \widehat{x}'(\widehat{x}), \ \widehat{u}' = \frac{\partial \widehat{x}'}{\partial \widehat{x}} \widehat{u}, \tag{4.4}$$

where $\overline{a}$ =constant, $\det \overline{a} \neq 0$, $\overline{b}(\widehat{x})$ is an arbitrary matrix (similar proof with that from Theorem 3.1). We have arrived at

Proposition 4.2. *The foliations $\mathcal{F}_1, \mathcal{F}_2$ define (integrable) $\mathcal{F}$-flat structures on $\tau\mathcal{F}$ ($\mathcal{F} = \mathcal{F}_1$ or $\mathcal{F} = \mathcal{F}_2$) iff the coordinate transformations of $\mathcal{F}_1$, $\mathcal{F}_2$ are of the form* (4.3) *and* (4.4), *respectively.*

Remark. In the previous result, $T\mathcal{F}_1$ and $T\mathcal{F}_2$ are flat vector bundles.

Now we give some examples.

1) Let G be a Lie group, g be its Lie algebra, and P be a principal G-bundle over a manifold M. The vertical tangent bundle of P is trivial $VP = P \times g$. Denote by $\mathcal{F}$ the vertical foliation of P and by $\tau\mathcal{F}$ the transverse bundle of $\mathcal{F}$. It is known that $\tau\mathcal{F}$ is $\mathcal{F}$-flat ([3]). Using Theorem 3.1 and Lemma 3.1, we get the isomorphism $TP = (P \times g) \oplus \tau\mathcal{F}$, where TP is the tangent bundle of P.

2) Consider a given differentiable manifold M. Let $\mathcal{R}$ be the bundle of frames of M, $T\mathcal{R}$ its tangent vector bundle and $\mathcal{F}$ the vertical foliation of $\mathcal{R}$. Algebra $gl(m, \mathbb{R})$ of general linear group $GL(m, \mathbb{R})$ is isomorphic (as a vector space) with $\mathbb{R}^{m^2}$, $m = \dim M$. Then, $T\mathcal{R}$ is $\mathcal{F}$-flat and $T\mathcal{R} = \left(\mathcal{R} \times \mathbb{R}^{m^2}\right) \oplus \tau\mathcal{F}$.

Other examples can be read in [3].

5. Connections associated to integrable $\mathcal{F}$-flat structures

5.1. We suppose now that $\mathcal{F}$ defines an $\mathcal{F}$-flat structure on M. Then, if $h = (U; x, \widehat{x})$, $h' = (U'; x', \widehat{x}')$ are two charts of $\mathcal{F}$, where $U \cap U' \neq \phi$, we have the possibility to define an operator $\nabla : \Gamma(T\mathcal{F}) \times \Gamma(TM) \to \Gamma(TM)$ in the following way. Denote by $x^{a'} = x^{a'}(x^a)$ and $x^a = x^a(x^{a'})$ the changes of coordinates associated to charts $h, h' \in \mathcal{F}$. For a vector field $Z = Z^a \frac{\partial}{\partial x^a} = Z^{a'} \frac{\partial}{\partial x^{a'}} \in \Gamma(TM|_{U \cap U'})$, we can write

$$\overset{U}{\nabla}_X Z = X(Z^a) \frac{\partial}{\partial x^a} \text{ and } \overset{U'}{\nabla}_X Z = X\left(Z^{a'}\right) \frac{\partial}{\partial x^{a'}},$$

where $X = X^i \frac{\partial}{\partial x^i} \in \Gamma(T\mathcal{F}/_U)$. Because $\mathcal{F}$ defines an $\mathcal{F}$-flat structure, the changes of coordinates do not depend of (x^i), $(x^{i'})$ and then $X\left(\frac{\partial x^{a'}}{\partial x^a}\right) = X\left(\frac{\partial x^a}{\partial x^{a'}}\right) = 0$. Thus we can write:

$$\overset{U}{\nabla}_X Z = [X\left(\frac{\partial x^a}{\partial x^{a'}}\right) Z^{a'} + \frac{\partial x^a}{\partial x^{a'}} X\left(Z^{a'}\right)] \frac{\partial x^{b'}}{\partial x^a} \cdot \frac{\partial}{\partial x^{b'}} = X\left(Z^{b'}\right) \frac{\partial}{\partial x^{b'}} = \overset{U'}{\nabla}_X Z.$$

Then $\overset{U}{\nabla}_X Z = \overset{U'}{\nabla}_X Z$ on $U \cap U'$ and therefore the set $\{\overset{U'}{\nabla}\}_{U\in\mathcal{U}}$ defines an operator through the equality: $(\nabla_X Z)(p) = (\overset{U}{\nabla}_X Z)(p)$, $p \in U$. The operator ∇ satisfies the conditions of the definition of a connection in TM, for $X \in \Gamma(T\mathcal{F})$. Let g be a Riemannian metric in TM. Then $TM \simeq T\mathcal{F} \oplus T^{\perp}\mathcal{F}$, where $T^{\perp}\mathcal{F}$ is the orthogonal complement to $T\mathcal{F}$ in TM. Now let $\overline{\nabla}$ be a connection in TM and $X = X_1 + X^{\perp} \in \Gamma(TM)$, where $X_1 \in \Gamma(T\mathcal{F})$, $X^{\perp} \in \Gamma\left(T^{\perp}\mathcal{F}\right)$. The operator $D : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ defined by $D_X Z = \nabla_{X_1} Z + \overline{\nabla}_{X^{\perp}} Z$ is a connection in TM.

5.2. Curvature and the torsion of connection D. Let $\overset{U}{\theta} = \left(\theta_a^b\right)$ be the connection matrix and $\overset{U}{\Omega}$ be the curvature matrix of D relative to the frame field $\left(\frac{\partial}{\partial x^a}\right)$. We first observe that $D_X\left(\frac{\partial}{\partial x^a}\right) = \theta_a^b(X) = 0$, $\forall X \in \Gamma(T\mathcal{F})$. Then, if $\theta_a^b = \Gamma_{ak}^b dx^k + \Gamma_{a\hat{k}}^b dx^{\hat{k}}$, we obtain $\theta_a^b\left(\frac{\partial}{\partial x^i}\right) = \Gamma_{ai}^b = 0$, $i = 1, ..., p$, that is $\theta_a^b = \Gamma_{a\hat{k}}^b dx^{\hat{k}}$. This result, together with the equations of structure of D, $\overset{U}{\Omega} = d\overset{U}{\theta} + \overset{U}{\theta}\Lambda\overset{U}{\theta}$, gives $\overset{U}{\Omega}(X, Y) = 0$, $X, Y \in \Gamma(T\mathcal{F})$. Hence we get $\Omega(D)(X, Y) = 0$, $\forall X, Y \in \Gamma(T\mathcal{F})$, where $\Omega(D)$ denotes the 2-form of curvature of D.

We now consider the torsion $T(D)$ of D. If $X = X_1 + X^{\perp}$, $Y = Y_1 + Y^{\perp} \in \Gamma(TM)$, $X_1, Y_1 \in \Gamma(T\mathcal{F})$, $X^{\perp}$, $Y^{\perp} \in \Gamma\left(T^{\perp}\mathcal{F}\right)$, then

$$T(D)(X, Y) = D_X Y - D_Y X - [X, Y] = T(\nabla)(X_1, Y_1) + T\left(\overline{\nabla}\right)(X^{\perp}, Y^{\perp})$$

$$+\nabla_{X_1} Y^{\perp} - \nabla_{Y_1} X^{\perp} + \overline{\nabla}_{X^{\perp}} Y - \overline{\nabla}_{Y^{\perp}} X - \left[X, Y^{\perp}\right] - \left[X^{\perp}, Y\right],$$

where $T(\nabla)(X_1, Y_1) = \nabla_{X_1} Y_1 - \nabla_{Y_1} X_1 - [X_1, Y_1] = 0$ and $T\left(\overline{\nabla}\right)\left(X^{\perp}, Y^{\perp}\right) = \overline{\nabla}_{X^{\perp}} Y^{\perp} - \overline{\nabla}_{Y^{\perp}} X^{\perp} - \left[X^{\perp}, Y^{\perp}\right]$.

From this result, for $X = X_1$, $Y = Y_1 \in \Gamma(T\mathcal{F})$, we find $T(D)(X, Y) = 0$ (i.e. D is symmetrical on leaves). Thus, we finally obtain

Theorem 5.1. *Let Λ be the integrable $\mathcal{F}$-flat structure defined by the foliation $\mathcal{F}$ of M. Then:* 1) *There is a connection $D_X = \nabla_{X_1} + \overline{\nabla}_{X^{\perp}}$, $X = X_1 + X^{\perp}$, associated to Λ.* 2) *The connection D is flat and symmetrical along the leaves of F.*

Using the method of [3], one can find an $\mathcal{F}$-flat structure Λ, starting from an $\mathcal{F}$-flat connection. But Λ is generally non-integrable.

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