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MAXIMAL DIVISIBLE SUBGROUPS IN MODULAR GROUP ALGEBRAS OF TORSION ABELIAN GROUPS OVER CERTAIN COMMUTATIVE RINGS

BY

PETER DANCHEV

Abstract. We give the isomorphism description of the maximal divisible subgroup of the group of normalized units in commutative group algebras formed by certain torsion abelian groups over special decomposable or indecomposable rings with prime characteristic. This extends recent results of ours (Rad. Mat., 2004) and (Bull. Georgian Nat. Acad. Sci. - Math., 2006) as well as a result due to Kuneva (Compt. rend. Acad. bulg. Sci., 2006).

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1. Introduction. Recently, we started in [1] the study of the structure of the divisible part of the group of all units with augmentation 1 in modular group algebras of p-mixed abelian groups over fields of characteristic p. There we established the description up to isomorphism of that divisible part (which separates as a direct factor of the whole group) by giving an explicit formula. In [6], it was demonstrated by Kuneva an attempt to generalize the aforementioned main theorem from [1] for a more general coefficient ring which is indecomposable but, unfortunately, there is no any important novelty in her approach; even more the proof only coped the idea from [1].

Moreover, in [3] we enlarged the previous investigation of [1] by computing in some cases (especially for reduced coefficient rings) the rank of the maximal divisible subgroup of the group of normalized units modulo its basis, thus obtaining its isomorphic classification.

The purpose of the present brief note is to describe the divisible part in some special situations of torsion not necessarily p-mixed abelian groups and decomposable or indecomposable commutative rings of prime characteristic.

To this aim, throughout the paper, let G be an abelian group with maximal divisible subgroup dG and p-component of torsion G_p and let R be a commutative ring with identity element of prime characteristic p with unit group $U(R)$, with nil-radical $N(R)$ and with a subring L which contains the same identity as R. As usual, RG denotes the group algebra of G over R with group of normalized invertible elements $V(RG)$ and its p-primary component $S(RG)$. Besides, we write $dV(RG)$ and $dS(RG)$ to denote the divisible parts of $V(RG)$ and $S(RG)$, respectively. Traditionally, $|M|$ designates the cardinality of an arbitrary set M.

2. Chief results. Before stating and proving our central theorem, we need a few preliminaries. And so, as aforementioned, we have established in [1] the isomorphism structure of $dV(RG)$ when G is p-mixed, that is the only torsion is p-torsion, and R is a field. The method for proof was inspired by the truthfulness of the equality $V(RG) = GS(RG)$. However, when R is decomposable, it is no longer valid as showed in ([7], [8], [4] and [5]). Nevertheless, this ratio is fulfilled even when the group basis G is not necessarily p-mixed, specifically we shall further demonstrate that it is true for certain torsion groups over special indecomposable rings. Even more, in [4], and more generally in [5], we found a criterion only in terms associated both with R and G when $V(RG) = GS(RG)$ holds. We shall use in the sequel this necessary and sufficient condition as well as we shall show an independent approach based on the trivial units in commutative group algebras and some specific decompositions.

So, we have at our disposal all the information necessary to proceed by arguing our attainments. First of all, we state a key technicality.

Lemma 1. *The following are equivalent:*

(i) $V(RG)/G$ *is a* p*-group;*

(ii) $V(RG) = GS(RG)$*;*

(iii) $V(RG)/G \cong S(RG)/G_p$.

Proof. It is straightforward because G being p-pure in $V(RG)$ insures that $(V(RG)/G)_p = S(RG)G/G \cong S(RG)/G_p$. □

Lemma 2. *The following implication holds:*

(j) $V(RG)/G = (V(RG)/G)_p \Rightarrow G$ *is balanced in* $V(RG)$.

Proof. It is obvious that G is p-isotype in $V(RG)$. Since by Lemma 1 we deduce that $V(RG) = GS(RG)$, the proof goes on in the same manner as in [2]. □

Corollary 3. *Suppose that G is p-mixed and R is indecomposable. Then G is balanced in $V(RG)$.*

Proof. As already noted above, $V(RG) = GS(RG)$ holds. Thus Lemma 2 works. □

We are now prepared to argue the following affirmation.

Theorem 4. *Let R and G satisfy at most one of the following conditions:*

$$G = G_p \times C, |C| < \aleph_0, R = L + N(R), |L| < \aleph_0 \tag{1}$$

$$G = G_p \times C, |C| = 3, p = 2, U(R) = 1 + N(R) \tag{2}$$

and for each pair (r, f) of elements of R the relation $r^2+f^2+rf+1 \in N(R)$ implies that $1 + r \in N(R), 1 + f \in N(R)$ or $1 + r \in N(R)$, $f \in N(R)$ or $1 + f \in N(R), r \in N(R)$.

Then $dV(RG) = dS(RG)$ and thereby

$$dV(RG) \cong \coprod_{\lambda} \mathbf{Z}(p^{\infty}),$$

where λ is calculated only in terms of R and G as in [9].

Proof. Foremost we deal with point (2). In virtue of [5] we have that $V(RG) = GS(RG)$, whence the method posed in [1] works. However, to be more precise, we observe with the last equality at hand that $V(RG) = C \times S(RG)$. Consequently, $dV(RG) = dS(RG)$ since C is reduced being finite; note also that $dG = d(G_p) = (dG)_p$.

On the other hand, concerning cause (1), with the aid of simple arguments, we derive that $G = G_p \times C$ forces $V(RG) = S(RG)V(RC)$. Besides, it is a plain exercise to see that $V(RC) = V((L + N(R))C) = V(LC +$

$N(R)C) \subseteq V(LC)S(RC)$ and thus $V(RG) = S(RG)V(LC) = S(RG) \times M$ for some $M \leq V(LC)$ since $V(LC)$ is finite hence torsion; notice that, in view of [4], $V(LC) = C$ whenever $|C| = 2$ and either $|L| = 2$ or $|L| = 3$ and thus $V(RG) = S(RG) \times C$. Therefore, $dV(RG) = dS(RG)$ because M is finite whence reduced and so $dM = 0$. We are finished.

Henceforth, we apply in both situations [9] to get the claim in all generality. □

Remark 5. There are two similar approaches in attacking the statement as well, provided that $V(RG) = GS(RG)$. In fact, again in accordance with [5] we have that $V(RG) = GS(RG)$. Thus, owing to Lemma 2, we may write $dV(RG) \cong dG \times dV(RG)/dG \cong dG \times d(V(RG)/G) \cong d(S(RG)/G_p)$. Hereafter, we employ [3] to finish the classification of $dV(RG)$.

It is also worthwhile noticing that our strategy used above can be successfully applied even in the case when C need not be torsion, whence G is p-splitting but not necessarily torsion (about other independent approach for p-splitting groups, compare with [1]). Indeed, $G = G_p \times C$ secures that $V(RG) = S(RG; G_p) \times V(RC)$, hence $dV(RG) = dS(RG; G_p) \times dV(RC)$, where $S(RG; G_p) = 1 + I(RG; G_p)$ and $I(RG; G_p)$ is the relative augmentation ideal of RG with respect to G_p. Since $dS(RG; G_p)$ can be characterized by the methods described in [3], one must classify $dV(RC)$. In particular, one must found a criterion when $V(RC)$ is reduced that is $dV(RC) = 0$. However, this is the theme of some other research study.

Owing to a similar variant of the scheme of computations from [1], we also emphasize that it can be established an explicit formula for dA where $A = BC$ is an arbitrary abelian group with subgroups B and C, one of which is finite, respectively.

Finally, it is worth noting that in case (1) R may have nontrivial idempotents whereas in case (2) it is demonstrably without such elements.

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13, General Kutuzov Street,
block 7, floor 2, apartment 4,
4003 Plovdiv,
BULGARIA
pvdanchev@yahoo.com