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# HOMOLOGY OF FINITE INDEX SUBGROUPS OF FINITELY PRESENTED GROUPS

BY

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*To the memory of my friend Gheorghe Ionesei*

**Abstract.** Let $G$ be a finitely presented group and $p$ a prime number. We investigate index $p$ normal subgroups of $G$ through their first homology group. For such a group $K$ and a field $\mathbb{K}$ of characteristic $p$, we find that the betti number $b_1(K, \mathbb{K})$ is determined by certain algebraic subvarieties of $H^1(G, \mathbb{K})$.

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**1. Introduction.** Let $G$ be a finitely presented group and $n$ a natural number. The set of index $n$ subgroups of $G$ is an interesting invariant of the group. When $G$ has enough finite quotients, the set could turn out into quite a powerful invariant of the group; at the opposite end, when $G$ has no finite quotients, the invariant is trivial.

In this article we will focus on index $p$ normal subgroups, for $p$ a prime. Moreover, we will look at such subgroups from a rather simple perspective: We will only be interested in their abelianizations. Let $K$ be an index $p$ normal subgroup of $G$, and $\mathbb{K}$ a field of characteristic $\kappa$. Our goal here will be to determine the rank $b_1(K, \mathbb{K})$ of the homology $H_1(G, \mathbb{K})$ with coefficients in $\mathbb{K}$. We have to separate two cases: $\kappa$ is coprime with $p$ or zero, and respectively $\kappa = p$. The first case, where $\kappa = 0$, has been considered by many authors [13, 15, 28]. The betti number $b_1(K)$ is related with certain

Alexander type invariants of the group $G$ known as characteristic varieties. In [19] we established a similar relationship for $b_1(K, \mathbb{K})$ when $\kappa > 0$. The second case, $\kappa = p$, will be our main concern here.

For simplicity we will restrict to the class of finitely presented groups $G$ whose nilpotent quotients $G/G_k$ (by the lower central series) up to some $k$ are isomorphic to those of a free group. We call a group enjoying this property $k$-free. All finitely presented groups $G$ in the paper are assumed to be 2-free.

First note that index $p$ normal subgroups of a 2-free group $G$ are in bijection with index $p$ normal subgroups of $G/G_q$ since we can identify $H^1(G, \mathbb{Z}_p) = \mathrm{Hom}(G, \mathbb{Z}_p)$ with $\mathrm{Hom}(G/G_q, \mathbb{Z}_p)$ as $G/G_q$ is also 2-free and $b_1(G) = b_1(G/G_q)$. In fact the bijection extends to the betti numbers of the subgroups for large enough $q$.

**Theorem 1.1.** *Suppose $G$ is a 2-free group. Let $K$ be an index $p$ normal subgroup of $G$ and let $N$ be the corresponding index $p$ normal subgroup of $G/G_q$. Then $H_1(N) = \mathbb{Z}^{b_1(G)} \oplus (p\text{-torsion})$, in particular $b_1(N, \mathbb{Z}_p) \geq b_1(G)$. Moreover, for $q \geq p + 1$ we have $b_1(K, \mathbb{Z}_p) = b_1(N, \mathbb{Z}_p)$.*

These facts allow us to make the following definitions for $d$ a positive integer. Define $\mathcal{Q}_d(G, \mathbb{Z}_p)$ to be the set

$$\mathcal{Q}_d(G, \mathbb{Z}_p) = \{\lambda \in \mathrm{Hom}(G, \mathbb{Z}_p) \mid \dim_{\mathbb{Z}_p}(\mathrm{Tors}\, H_1(K_\lambda)) \otimes \mathbb{Z}_p \geq d\},$$

and $\mathcal{Q}_d^{(q)}(G, \mathbb{Z}_p)$ to be the set

$$\mathcal{Q}_d^{(q)}(G, \mathbb{Z}_p) = \{\lambda \in \mathrm{Hom}(G/G_q, \mathbb{Z}_p) \mid \dim_{\mathbb{Z}_p}(\mathrm{Tors}\, H_1(N_\lambda)) \otimes \mathbb{Z}_p \geq d\}.$$

Notice that $\mathcal{Q}_d(G, \mathbb{Z}_p) = \mathcal{Q}_d^{(q)}(G, \mathbb{Z}_p)$ when $q \geq p + 1$.

Our main goal here is to decribe these sets in terms of computable invariants of $G$. What we have in mind is for $\mathcal{Q}_d(G, \mathbb{Z}_p)$ and $\mathcal{Q}_d^{(q)}(G, \mathbb{Z}_p)$ to be given by polynomial equations whose coefficients are cohomologically defined. We achieve this goal for $G$ a $k$-free group (with $H_2(G)$ free abelian). In that case, the sets $\mathcal{Q}_d^{(k+1)}(G, \mathbb{Z}_p)$ coincide with the resonance varieties $\mathcal{R}_d(G, \mathbb{Z}_p)$, certain subvarieties of $H^1(G, \mathbb{Z}_p)$ defined from cup products and higher Massey products of classes in $H^1(G, \mathbb{Z}_p)$. In particular, for such a $G$, the betti number $b_1(K, \mathbb{Z}_p)$ may be determined from $\mathcal{R}_d(G, \mathbb{Z}_p)$.

**1.2.** We proceed now to explain why is it reasonable to expect that $\mathcal{Q}_d^{(q)}(G, \mathbb{Z}_p)$ are subvarieties of $H^1(G, \mathbb{Z}_p)$ defined in cohomological terms. To that end we further restrict the finitely presented groups we consider and we make precise the $k$-freeness notion. Most of the results hold without these restrictions, but the ideas and the techniques become more transparent in their presence. The conditions below are satisfied by many fundamental groups of complement spaces associated to geometric objects. The examples we have in mind are mainly links in 3-sphere.

**Definition 1.3.** We say that $G$ satisfies condition (A) if:

(1) The homology groups $H_1(G)$ and $H_2(G)$ are free abelian.

**Definition 1.4.** Let $\mathbb{F} \to G$ be a presenting homomorphism for $G$, where $\mathbb{F}$ is a free group of finite rank. The lower central series $G = G_1 \supseteq G_2 \supseteq \dots$ of $G$ is a sequence of normal subgroups generated by higher and higher commutators of elements in $G$: If $[G, H]$ is the group generated by commutators $[g, h] = ghg^{-1}h^{-1}$ with $g \in G$ and $h \in H$, then $G_{k+1} = [G, G_k], k \geq 1$. The quotients $G/G_k$ are called the nilpotent quotients of $G$. We say that $G$ is $k$-free if there exists $k \geq 2$ such that $\mathbb{F}/\mathbb{F}_k \cong G/G_k$.

The $k$-freeness condition is meant to ensure that certain cohomology operations, known as Massey products, are well-defined and uniquely determined up to order $k$. Under that condition on $G$, given cohomology classes $u_1, \dots, u_k \in H^1(G)$, the $k$-fold Massey product $\langle u_1, \dots, u_k \rangle \in H^2(G)$ is defined with zero indeterminacy. Moreover, the $k$-fold Massey operation is multilinear. That and the $k$-freeness of $G$ allows us to define maps $\mu_G : \mathbb{L}_k H^1(G) \to H^2(G)$ given by the $k$-fold Massey products. We denote the pair $(H^{\leq 2}(G), \mu_G)$ by $\mathcal{H}_k^2(G)$, and refer to it as *the Massey ring* of $G$.

**Definition 1.5.** We say that $G$ satisfies condition (B) if:

(2) The Massey product map $\mu_G : \mathbb{L}_k H^1(G) \to H^2(G)$ is surjective.

We can now state the result we anticipated above, which will be proven in Section 2. This is a generalization of Theorem 1. 15 from [20].

**Theorem 1.6.** *Let $G$ and $G'$ be $k$-free groups such that conditions* (A) *and* (B) *are satisfied. The following are equivalent:*

*(a)* $\mathcal{H}_k^2(G) \cong \mathcal{H}_k^2(G')$;

*(b)* $G/G_{k+1} \cong G'/G'_{k+1}$.

In other words, for a $k$-free group, the $(k+1)^{\text{st}}$ nilpotent quotient and the $k$-fold cohomology Massey products determine each other. Thus, the behaviour of homology of subgroups of the nilpotent quotients of $G$ should be captured by these Massey products. Our goal here is to express, for a characteristic $p$ field $\mathbb{K}$, the first $\mathbb{K}$-betti numbers of index $p$ normal subgroups $K$ in $k$-free groups $G$, and their nilpotent quotients $G/G_{k+1}$, in terms of certain algebraic subvarieties of the affine space $H^1(G,\mathbb{K})$. Clearly it suffices to work with $\mathbb{K}$ the prime field of $p$ elements.

The first series of invariants we introduce are the varieties defined by the Alexander ideals of $G$. They were considered by Novikov in [24] for $\mathbb{K}=\mathbb{C}$. Suppose $G$ is 2-free and $H_1(G)=\mathbb{Z}^n$.

**Definition 1.7.** Let $d$ be an integer, $0 \leq d \leq n$. The $d^{\text{th}}$ *characteristic variety* of $G$ (with coefficients in $\mathbb{K}$) is the subvariety of $(\mathbb{K}^*)^n$, defined as follows:

$$V_d(G,\mathbb{K}) = \{\lambda \in \operatorname{Hom}(G,\mathbb{K}^*) \mid \operatorname{rank}_{\mathbb{K}} M(G)_\lambda < n-d\},$$

where $M(G)_\lambda$ is the Alexander matrix of $G$ evaluated at the character $\lambda$. The varieties $V_d(G,\mathbb{K})$ are invariants of $G$, up to an ambient monomial isomorphism of $(\mathbb{K}^*)^n$.

We can now show how $b_1(K,\mathbb{K})$ can be determined from the characteristic varieties $V_d(G,\mathbb{K})$ when the characteristic $\kappa$ of $\mathbb{K}$ is a prime different from $p$ or $\kappa=0$. For $\kappa>0$ the result has been established by Matei and Suciu [19], and for $\kappa=0$ by Libgober [15], Hironaka [13] and Sakuma [28].

**Theorem 1.8.** *Let $K_\lambda = \ker(\lambda : G \to \mathbb{Z}_p)$ be a normal subgroup of prime index $p$, and let $\kappa$ be a prime, $\kappa \neq p$. Then*

$$b_1(K_\lambda,\mathbb{K}) = b_1(G,\mathbb{K}) + (p-1)\cdot \max\{d \mid \lambda \in V_d(G,\mathbb{K})\},$$

*where $\mathbb{K}=\mathbb{Z}_\kappa[\xi]$ and $\xi$ is a primitive $p^{th}$ root of unity. Moreover*

$$b_1(K_\lambda) = b_1(G) + (p-1)\cdot \max\{d \mid \lambda \in V_d(G,\mathbb{C})\}.$$

The second series of invariants we introduce are the resonance varieties. Suppose $G$ is $k$-free, for some $k \geq 2$, satisfies condition (B), and $H_1(G) = \mathbb{Z}^n$.

**Definition 1.9.** Let $d$ be an integer, $0 \leq d \leq n$. The $d^{\text{th}}$ *resonance variety* of $G$ (with coefficients in $\mathbb{K}$) is the subvariety of $\mathbb{K}^n$, defined as follows:

$$\mathcal{R}_d(G,\mathbb{K}) = \left\{ \lambda \in H^1(G,\mathbb{K}) \;\middle|\; \begin{array}{l} \exists \text{ subspace } W \subset H^1(G,\mathbb{K}) \text{ such that} \\ \dim W = d+1 \text{ and } \langle W, \lambda, \ldots, \lambda \rangle = \{0\} \end{array} \right\},$$

where $\langle W, \lambda, \ldots, \lambda \rangle$ stands for the vector space generated by the $k$-fold products $\langle v, \lambda, \ldots, \lambda \rangle$ with $v \in W$.

For $k = 2$ the resonance varieties were introduced by Matei and Suciu in [20]. Originally, the top variety $\mathcal{R}_1(G,\mathbb{C})$ was introduced by Falk [5] as an invariant of cohomology groups of complements to complex line arrangements. By Theorem 1.6 the resonance varieties of a $k$-free group $G$ are invariants, up to a linear isomorphism of $\mathbb{K}^n$, of the $k$ th nilpotent quotient $G/G_{k+1}$.

We will determine equations for $\mathcal{R}_d(X,\mathbb{K})$, in terms of the $(k-1)^{st}$ truncation $T = T^{k-1}(G)$ of the Alexander matrix $M(G)$ of $G$; the entries of $T$ are homogenous forms of degree $k-1$ in the coordinates of $\lambda$, that can be computed explicitely from a presentation of $G$.

**Theorem 1.10.** *For $G$ a $k$-free group with $H_2(G)$ torsion free,*

$$\mathcal{R}_d(G,\mathbb{K}) = \{\lambda \in \mathbb{K}^n \mid \operatorname{rank}_{\mathbb{K}} T(\lambda) < n-d\},$$

*where $T(\lambda)$ is the matrix $T$ evaluated at $\lambda$.*

The case $k = 2$ of the previous theorem was derived in [20].

We have discussed before how the characteristic varieties of a 2-free group $G$ are related with the betti numbers of finite index subgroups in $G$. In a similar vein, the resonance varieties of a $k$-free group $G$ are related with the $\mathbb{Z}_p$-betti numbers of index $p$ subgroups in $G/G_{k+1}$. The following numerical invariants were introduced in [20]:

**Definition 1.11.** Let $G$ be a 2-free group. Given a prime $p$, and a positive integer $d$, define the $\nu$-invariants of $G$ to be:

$$\nu_{p,d}(G) = \#\big\{K \lhd G \;\big|\; [G:K] = p \text{ and } b_1(K,\mathbb{Z}_p) = b_1(G,\mathbb{Z}_p) + d\,\big\}.$$

We show that the $\nu$-invariants of $G/G_{k+1}$ can be computed from the projective resonance varieties of $G$:

**Theorem 1.12.** *Let $G$ be a $k$-free group with $H_2(G)$ torsion free, $H_1(G) = \mathbb{Z}^n$ and $p$ a prime number. Then for $p \leq k-1$ we have:*

$$\nu_{p,d}(G/G_{k+1}) = \begin{cases} \frac{p^n-1}{p-1}, & \text{if } d = (n-1)(p-1); \\ 0 & \text{otherwise.} \end{cases}$$

*and for $p \geq k$ we have:*

$$\nu_{p,d}(G/G_{k+1}) = \#(\mathcal{P}_{d-s}(G, \mathbb{Z}_p) \setminus \mathcal{P}_{d-s+1}(G, \mathbb{Z}_p)),$$

*where $s = (k-2)(n-1)$.*

The case $k = 2$ of the previous theorem was derived in [20].

We will also relate, for index $p$ normal subgroup of a 2-free group $G$ with the index $p$ normal subgroup in the nilpotent quotients of $G$.

**Theorem 1.13.** *Let $G$ be a 2-free group, such that $H_2(G)$ is free abelian. Let $n$ be the rank of the abelianization of $G$, and $p$ a prime number. For $\lambda : \mathbb{Z}^n \to \mathbb{Z}_p$ a homomorphism, denote by $K_\lambda$ the index $p$ subgroup in $G$ defined by $\lambda$, and by $N_{q,\lambda}$ the index $p$ subgroup in $G/G_q$ defined by $\lambda$. Then for $q \geq p+1$ we have:*

$$b_1(N_{q,\lambda}, \mathbb{Z}_p) = b_1(K_\lambda, \mathbb{Z}_p),$$

*and for $q \leq p$ we have:*

$$b_1(N_{q,\lambda}, \mathbb{Z}_p) \leq b_1(K_\lambda, \mathbb{Z}_p).$$

The previous theorem has the interesting consequence that, one can recover the $p$-torsion (tensored by $\mathbb{Z}_p$) in (abelianizations of) index $p$ normal subgroups of a 2-free group $G$ from the $p$-torsion in the corresponding index $p$ normal subgroups of a nilpotent quotient $G/G_q$, for large enough $q$. For such a normal subgroup group $N$ the rank $\dim_{\mathbb{Z}_p}(\operatorname{Tors} H_1(N) \otimes \mathbb{Z}_p)$ of the $p$-torsion part of $H_1(N)$ can be obtained subtracting $b_1(N, \mathbb{Z}_p)$, derived from the characteristic varieties over $\mathbb{C}$, from $b_1(N, \mathbb{Z}_p)$, derived from the resonance varieties over $\mathbb{Z}_p$.

The main technical tool used in this paper is Fox's free calculus. It is involved in the Massey products computations, and in computing the various Alexander matrices. In proving all the theorems concerning the finite index subgroups of $G$, Fox's algorithm that finds a presentation for the abelianization of such a subgroup is extensively used.

This work was part of author's Ph.D. thesis [18].

**2. Fox calculus and Massey products.** In this section we investigate, for a certain class of finitely presented groups, the relationship between the group structure and the cohomology of the group. More precisely, the main result of this chapter, Theorem 2.31, shows that for a $k$-free group the $(k+1)^{\text{st}}$ nilpotent quotient and the $k$-fold cohomology Massey products contain equivalent information. This result is a generalization of Theorem 1.15, from [20], which deals with the case $k = 2$. Also in this section we review the combinatorial group theory notions that will be needed here and elsewhere in the sequel. We take the terminology and the general results on Massey products from KRAINES [14], and MAY [21]. For Massey products of groups we use FENN and SJERVE [10, 11], and PORTER [27]. The technique we are going to use to compute Massey products is based on the Theorem 2.16, due to PORTER [27].

**2.1. Commutators, fox calculus and nilpotent quotients.** Let $G$ be a finitely generated group. The commutator $xyx^{-1}y^{-1}$ of two elements $x, y$ in $G$ is denoted by $[x, y]$. If $H$ and $K$ are subgroups of $G$ we denote by $[H, K]$ the subgroup of $G$ generated by the commutators $[h, k]$, with $h$ in $H$ and $k$ in $K$.

**Definition 2.2.** The sequence of groups $G = G_1 \supset G_2 \supset \ldots$, defined inductively by $G_{k+1} = [G, G_k]$ for $k \geq 1$, is called the *lower central series* of $G$.

The groups $G_k$ are normal, characteristic subgroups of $G$ and $G_{k+1}$ is normal in $G_k$. The *lower central series quotients* $G_k/G_{k+1}$ are abelian groups. For $G = \mathbb{F}$ the free group of rank $n$, these groups are well understood. The quotients $\mathbb{F}_k/\mathbb{F}_{k+1}$ of $\mathbb{F}$ are free abelian of rank $N_k = \frac{1}{k}\sum_{d|k} \mu(d) n^{k/d}$, where $\mu(d)$ is the Möbius function.

We need explicit integral bases for $\mathbb{F}_k/\mathbb{F}_{k+1}$. Let $x_1, \ldots, x_n$ be a basis of $\mathbb{F}$. A *basic commutator* in $\mathbb{F}$ is defined inductively as follows, see [6, 16]:

(a) Each basic commutator $c$ has length $\ell(c) \in \mathbb{N}$.

(b) The basic commutators of length 1 are the generators $x_1, \ldots, x_n$; those of length $> 1$ are of the form $c = [c_1, c_2]$, where $c_1$, $c_2$ are previously defined commutators and $\ell(c) = \ell(c_1) + \ell(c_2)$.

(c) Basic commutators of the same length are ordered arbitrarily; if $\ell(c) > \ell(c')$, then $c > c'$.

(d) If $\ell(c) > 1$ and $c = [c_1, c_2]$, then $c_1 < c_2$; if $\ell(c) > 2$ and $c = [c_1, [c_2, c_3]]$, then $c_1 \geq c_2$.

The basic commutators of the form $c = [x_{i_1}, [x_{i_2}, [\dots [x_{i_{k-1}}, x_{i_k}] \dots ]]]$ are called *simple*. We shall write them as $c = [x_{i_1}, x_{i_2}, \dots, x_{i_k}]$. For $k \leq 3$, all basic commutators are simple.

The following result of P. Hall is well-known, see [16]: The group $\mathbb{F}_k/\mathbb{F}_{k+1}$ has a basis consisting of the basic commutators of length $k$.

In particular, if $w \in \mathbb{F}$ and $c_1, \dots, c_r$ are the basic commutators of length $< k$, then $w^{(k)} := w \bmod \mathbb{F}_k$ may be written uniquely as $w^{(k)} = c_1^{e_1} c_2^{e_2} \cdots c_r^{e_r}$, for some integers $e_1, \dots, e_r$.

Notice that the graded group $\oplus_k \mathbb{F}_k/\mathbb{F}_{k+1}$ has a Lie algebra structure given by the commutator map and can be naturally identified with the free Lie algebra $\mathbb{L}(H) = \oplus_k \mathbb{L}_k(H)$, where $H = H_1(\mathbb{F})$.

**2.3.** We recall Fox's free differential calculus, see Fox [7]. Let $\mathbb{Z}\mathbb{F}$ be the group ring of the free group $\mathbb{F}$ on generators $x_1, \dots, x_n$. Denote the augmentation homomorphism by $\varepsilon : \mathbb{Z}\mathbb{F} \to \mathbb{Z}$.

**Definition 2.4.** For each $i$, $1 \leq i \leq n$ the *Fox derivative* $\partial_i = \frac{\partial}{\partial x_i} : \mathbb{Z}\mathbb{F} \to \mathbb{Z}\mathbb{F}$ is uniquely defined by $\partial_i(x_j) = \delta_{ij}$ and $\partial_i(uv) = \partial_i(u)\varepsilon(v) + u\partial_i(v)$.

For any index $I = (i_1, \dots, i_k)$ let $\partial_I : \mathbb{Z}\mathbb{F} \to \mathbb{Z}\mathbb{F}$ denote the higher order derivative $\partial_{i_1} \circ \partial_{i_2} \circ \cdots \circ \partial_{i_k}$, and $\varepsilon_I$ the composite $\varepsilon \circ \partial_I$.

**Definition 2.5.** For a word $w \in \mathbb{F}$ we call the integers $\varepsilon_I(w)$ the *Magnus coefficients* of $w$.

We record the following facts for later use, see [7] or [16]:

**Proposition 2.6.** *We have:*

*(a)* $\partial_i[u, v] = (1 - uvu^{-1})\partial_i u + (u - [u, v])\partial_i v$.

*(b)* $\alpha(\partial_i[x_{i_1}, x_{i_2}, \dots, x_{i_q}]) = (t_{i_1} - 1) \cdots (t_{i_{q-2}} - 1)\big((t_{i_{q-1}} - 1)\delta_{i,i_q} - (t_{i_q} - 1)\delta_{i,i_{q-1}}\big)$.

*(c)* $w \in \mathbb{F}_k$ *if and only if* $\varepsilon_I(w) = 0$ *for all indices* $I$ *of length* $< k$;

*(d)* $\varepsilon_I(uv) = \varepsilon_I(u) + \varepsilon_I(v)$, *if* $u, v \in \mathbb{F}_q$ *and* $|I| = q$;

*(e)* $w = \varepsilon(w) + \sum_{q \geq 1} \sum_I \varepsilon_I(w) \cdot \prod_{l=1}^{q} (x_{i_l} - 1)$, *where* $I = (i_1, \dots, i_q)$, *and* $w \in \mathbb{F}$.

Let $p$ pe a prime number. We are going to use later the $\mod p$ version of the Magnus coefficients defined by replacing $\varepsilon_I$ in 2.5 with the composite of $\varepsilon_I$ with the projection $\mathbb{Z} \to \mathbb{Z}_p$. Abusing the notation, we will denote the $\mathbb{Z}_p$-valued Magnus coefficients of a word $w$ by the same $\varepsilon_I(w)$.

**2.7.** A group $G$ is called *nilpotent* if $G_k = 1$ for some $k \geq 1$. The smallest $k$ with that property is called the nilpotency *class* of $G$.

**Definition 2.8.** The group $G/G_{k+1}$ is called the $k^{\text{th}}$ *nilpotent quotient* of $G$.

The first nilpotent quotient $G/G_2$ is an abelian group. In general, $G/G_{k+1}$ is a nilpotent group of class $\leq k$, which is a nilpotent-by-abelian central extension of finitely presented groups,

$$0 \to G_k/G_{k+1} \to G/G_{k+1} \to G/G_k \to 0. \tag{3}$$

The extension is classified by the $k$-invariant, $\bar{\chi} \in H^2(G/G_k; G_k/G_{k+1})$.

**Remark 2.9.** Recall from the general extension theory that the isomorphism type of a central extension $1 \to K \to G \to Q$, with $K$ finitely generated abelian, is determined by the groups $Q$, and $K$, and the $k$-invariant $\bar{\chi} \in H^2(Q; K)$. More precisely, let $G$ and $G'$ be two such groups. Then $G \cong G'$ if and only if there exist isomorphisms $\varphi : Q \to Q'$ and $\psi : K \to K'$ under which the $k$-invariants correspond: $\psi_*(\bar{\chi}) = \varphi^*(\bar{\chi}') \in H^2(Q; K)$.

From now on $G$ will always be a finitely presented group. If a presentation $\mathbb{F}/R = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$ of $G$ is known then the Hall commutators may be used to write down presentations for the nilpotent quotients of $G$. Indeed, we have the following presentation for $G/G_k = \mathbb{F}/R\mathbb{F}_k$:

$$G/G_k = \langle x_1, \ldots, x_n \mid r_1^{(k)}, \ldots, r_m^{(k)}, c_1, \ldots, c_N \rangle, \tag{4}$$

where $r_l^{(k)} = r_l \mod \mathbb{F}_k$, and $\{c_h\}_{1 \leq h \leq N}$ are the basic commutators of length $k$.

**2.10.** Later in this sectionwe will focus our attention to a class of groups which have, up to a certain range, the nilpotent quotients of a free group. We make now the explicit definition.

**Definition 2.11.** Let $G$ be a finitely presented group and $k \geq 2$ an integer. The group $G$ is called *$k$-free* if a defining homomorphism $\phi : \mathbb{F} \to G$ induces an isomorphism $\mathbb{F}/\mathbb{F}_k \cong G/G_k$.

A group $G$ which is $k$-free, for some $k \leq 2$, is also $l$-free, for all $l \leq k$, so in particular is 2-free. Note that if $G$ is 2-free then $H_1(G)$ is torsion-free since $H_1(G) = G/G_2 \cong \mathbb{F}/\mathbb{F}_2 = \mathbb{Z}^n$.

**2.12. Massey products.** Let $G = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$ be a finitely presented group. Let $R$ be a commutative ring with unit. We denote by $H_*(G, R)$ and $H^*(G, R)$ the group homology and cohomology of $G$ with untwisted $R$-coefficients. When $R = \mathbb{Z}$, the coefficients will be omitted.

If $G$ is 2-free, the 1-cells of the 2-complex $X$ determine free generators $e_1, \dots, e_n$ for $H^1(G, R)$ and the 2-cells free generators $\gamma_1, \dots, \gamma_m$ for $H^2(G, R)$.

Let $C^*(G, R)$ be the standard cochain complex of $G$ with the usual associative cup product pairing, see [2]:

$$\cup : C^i(G, R) \otimes C^j(G, R) \to C^{i+j}(G, R).$$

All the Massey products considered here are of 1-dimensional cohomology classes. We now recall the definition.

**Definition 2.13.** Given $u_1, \dots, u_k \in H^1(G, R)$ we call a *defining system* for the *$k$-fold Massey product* $\langle u_1, \dots, u_k \rangle$ an array of 1-cochains $M = \{m_{ij} \in C^1(G, R) \mid 1 \leq i \leq k+1, i < j \leq k+1, (i, j) \neq (1, k+1)\}$ such that

1. $m_{i,i+1}$ is a cocycle representing $u_i$;
2. $\delta m_{ij} = \sum_{s=i+1}^{j-1} m_{is} \cup m_{sj}$.

The value of the product at $M$, denoted $\langle u_1, \dots, u_k \rangle_M$ is defined to be the cohomology class in $H^2(G, R)$ represented by the cocycle

$$\sum_{s=2}^{k} m_{1s} \cup m_{s,k+1}.$$

The product $\langle u_1, \dots, u_k \rangle$ is defined only if such an $M$ exists, and is taken to be the subset of $H^2(G, R)$ consisting of all elements of the form $\langle u_1, \dots, u_k \rangle_M$:

$$\langle u_1, \dots, u_k \rangle = \{\langle u_1, \dots, u_k \rangle_M \mid M \text{ a defining system}\}.$$

The *indeterminacy* $\mathrm{In}\langle u_1,\dots,u_k\rangle$ of a Massey product is the subset $\{a-b \mid a,b\in\langle u_1,\dots,u_k\rangle\}$ of $H^2(G,R)$.

The single Massey product $\langle u_1\rangle$ is just $u_1$. The 2-fold Massey product $\langle u_1,u_2\rangle$ is the cup product $u_1\cup u_2$. The triple product $\langle u_1,u_2,u_3\rangle$ is defined whenever the double products $\langle u_1,u_2\rangle$ and $\langle u_2,u_3\rangle$ are zero. In general, in order for $\langle u_1,\dots,u_k\rangle$ to be defined it is necessary, but not sufficient, that all the Massey subproducts $\langle u_i,\dots,u_j\rangle$, $1\le j-i\le k-2$ be defined and contain zero. If they contain zero only then $\langle u_1,\dots,u_k\rangle$ is defined.

Because of the indeterminacy, the Massey products are natural only up to inclusion, see [21]:

**Proposition 2.14.** *Let $f: G'\to G$ be a group homomorphism and $g: R\to R'$ be a ring homomorphism. If $\langle u_1,\dots,u_k\rangle$ is defined, then so is $\langle f^*g_\#(u_1),\dots,f^*g_\#(u_k)\rangle$ and $f^*g_\#\langle u_1,\dots,u_k\rangle\subseteq\langle f^*g_\#(u_1),\dots,f^*g_\#(u_k)\rangle$. Furthermore, if $f^*$ and $g_\#$ are isomorphism, then equality holds.*

**2.15.** We present next a key result of Porter [27] which makes the computation of Massey products in $H^2(G,R)$ manageable by expressing their values in terms of Magnus coefficients. The following theorem is a particular case of Theorem 2 in [27].

We first establish some notations. If $X$ is the 2-complex determined by the group presentation $G=\langle x_1,\dots,x_n \mid r_1,\dots,r_m\rangle$, we denote by $e_l$ the homology class determined by the generators $x_l$.

Given the cohomology clases $u_1,\dots,u_k$ in $H^1(X,R)$, we define the elements $\theta_{ij}(q)$ and the ideal $\Theta_{u_1,\dots,u_k}$ in $R$ as follows: for $1\le i\le j\le k$ and $1\le q\le m$ set

$$\theta_{ij}(q)=\sum_I u_i(l_1)u_{i+1}(l_2)\cdots u_j(l_{j-i+1})\ \varepsilon_I(r_q),$$

where the sum is over all sequences $I=\{l_1,\dots,l_{j-i+1}\}$ with $1\le l_s\le n$ and $u_i(l)$ denotes $u_i$ evaluated on the generator $e_l$. Set $\Theta_{u_1,\dots,u_k}$ equal to the ideal in $R$ generated by the $\theta_{ij}(q)$ with $(i,j)\ne(1,k)$.

Borrowing the notation from [11], let $\langle\langle u_1,\dots,u_k\rangle\rangle$ be the cohomology class in $H^2(X,R)$ defined by its evaluation on $r_q$ as follows:

$$(\langle\langle u_1,\dots,u_k\rangle\rangle,r_q)=(-1)^k\theta_{1,k}(q)$$

**Theorem 2.16.** *The Massey product $\langle u_1,\dots,u_k\rangle$ with coefficients $R/\Theta_{u_1,\dots,u_k}$ is defined and contains the cohomology class $\langle\langle u_1,\dots,u_k\rangle\rangle$.*

In the case $R = \mathbb{Z}$, the previous result is also proven in [10] and [11]. Next we derive a corollary of Theorem 2.16.

**Corollary 2.17.** *Let $X$ be the 2-complex associated with a $k$-free group $G$. Let $u_1, \ldots, u_k \in H^1(X, R)$ be arbitrary and $u_i = \sum_{j=1}^{n} a_{ij} e_j$. Then all the Massey products $\langle u_1, \ldots, u_l \rangle$, $l \leq k$ with $R$-coefficients are defined with zero indeterminacy, they vanish if $l < k$ and*

$$(\langle u_1, \ldots, u_k \rangle, r_q) = (-1)^k \sum_I a_{1,i_1} \ldots a_{k,i_k} \ \varepsilon_I(r_q), \tag{5}$$

*where $1 \leq q \leq m$ and the summation is over all sequences $I = \{i_1, \ldots, i_k\}$ with $1 \leq i_1, \ldots, i_k \leq n$.*

It follows from the corollary that the $k$-fold Massey operation of $X$ is linear in all the arguments if $G$ is $k$-free. The case $k = 2$ of the Corollary 2.17 gives a way to compute the cup products in $H^*(X, R)$ directly from the relators. Let us record another consequence of Theorem 2.16, see also [11].

**Lemma 2.18.** *The Massey products of $\mathbb{F}/\mathbb{F}_k$ of length $< k$ vanish and the $k$-fold products $\langle e_1, \ldots, e_k \rangle$ form a basis for $H^2(\mathbb{F}/\mathbb{F}_k) \cong \mathbb{F}_k/\mathbb{F}_{k+1}$.*

To illustrate the method, we give now an example of Massey product computations.

**Example 2.19.** Let $X$ be the complement in $\mathbb{S}^3$ of the 3-component Borromean link $L$. A presentation for the fundamental group $G = \pi_1(X)$ can be computed from the braid presentation $L = \hat{\beta}$, where $\beta$ is the pure braid $[A_{12}, A_{13}]$. We have:

$$G = \langle x_1, x_2, x_3 \mid [x_1 x_2 x_1^{-1}, x_3], [x_3^{-1}, x_1^{-1}], [x_3^{-1} x_1^{-1} x_3, x_1 x_2 x_1^{-1}] \rangle$$

It is easily seen that $G$ is a 3-free group. The elements $e_1, e_2, e_3$ in $H^1(G)$, the duals to the meridians $x_1, x_2, x_3$, freely generate $H^1(G) \cong \mathbb{Z}^3$. We apply Corollary 2.17 for $k = 3$. We get right away that all cup products vanish, $e_i \cup e_j = 0$, due to the fact that the relators of $G$ are 3-fold commutators. Next, the relators modulo 4-fold commutators are as follows: $r_1^{(4)} = [x_1, [x_2, x_3]], r_2^{(4)} = [x_2, [x_3, x_1]], r_3^{(4)} = [x_3, [x_1, x_2]]$. Hence their Magnus coefficients $\varepsilon_{ijk}(r_q)$ are $\pm 1$ if $i, j, k$ are all distinct, and 0 otherwise. We obtain $\langle e_i, e_j, e_k \rangle = \delta_{ij}\gamma_k - \delta_{jk}\gamma_i$.

**2.20.** Assume now that $G$ is $k$-free, $k \geq 2$. Consider the map in $H^2$ induced by the projection $G \xrightarrow{\alpha} G/G_k \xrightarrow{\phi^{-1}} \mathbb{F}/\mathbb{F}_k$. After identifications we obtain a map

$$\mu_G : \mathbb{L}_k H^1(G) \to H^2(G). \tag{6}$$

**Definition 2.21.** We call the homomorphism $\mu_G$ the *Massey product map* of $G$, by virtue of Lemma 2.18 and the naturality of the products. Notice that Proposition 2.14 and Corollary 2.17 imply that $\mu_G$ is well-defined linear maps with respect to the Massey products. We denote the pair $(H^2(G), \mu_G)$ in short by $\mathcal{H}^2_k(G)$, and refer to it as *the Massey ring* of $G$.

If $G$ and $G'$ are given $k$-free groups then we write $\mathcal{H}_k(G) \cong \mathcal{H}_k(G')$ if there exists an isomorphism $H^{\leq 2}(G) \to H^{\leq 2}(G')$ that commutes with $\mu_G$ and $\mu_{G'}$. The above considerations remain valid if we work over arbitrary coefficients, thus we can talk about $R$-versions of the maps $\mu_G$ and $\mu_X$.

**2.22.** For the rest of this section $G$ will be a group satisfying conditions (A) and (B). We establish now a series of notations. Let $\langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$ be a presentation of $G$ with presenting homomorphism $\phi : \mathbb{F} \to G$. Denote by $\alpha : G \to G/G_2$ the abelianization map and set $t_i = \alpha(\phi(x_i))$. Then $\{t_i \mid 1 \leq i \leq n\}$ form a basis for $H_1(G)$. The 2-freeness condition means that the relators $r_1, \ldots, r_m$ are in $[\mathbb{F}, \mathbb{F}]$. This implies that the Kronecker duals $\{e_i = t_i^* \mid 1 \leq i \leq n\}$ form a basis for $H^1(G)$. Moreover $H_2(G)$ is free abelian with basis $r_1, \ldots, r_m$ and $H^2(G)$ is free abelian with the $\gamma_1, \ldots, \gamma_m$, where $\gamma_j = r_j^*$ and $r_j$ is, abusing the notation, the image of the relator $r_j$ under the Hopf epimorphism $h : [\mathbb{F}, \mathbb{F}] \cap \ker \phi \to H_2(G)$.

By the Universal Coefficient Theorem the homology and cohomology of $G$ with coefficients in a commutative field $\mathbb{K}$ have the same ranks over $\mathbb{K}$ as the integral ones have over $\mathbb{Z}$. Moreover their basis correspond to the integral ones and therefore we are going to use the same notation for them.

**2.23.** Before we focus on the $k$-fold Massey products of $k$-free groups, let us consider the consequences of Theorem 2.16 for the $(k+1)$-fold Massey products.

Let $G$ be a $k$-free group, with $\mu_G$ surjective. Suppose $u_1, \ldots, u_{k+1}$ are in $H^1(G, R)$ and $\langle u_1, \ldots, u_{k+1} \rangle$ is defined. Recall from Corollary 2.17 that

$k$-fold Massey products are defined with no indeterminacy. Theorem 2.8 of [21] implies the following:

**Proposition 2.24.** *Suppose $\langle u_1, \dots, u_{k+1}\rangle \in H^2(G, R)$ is defined. Then*

$$\mathrm{In}\langle u_1, \dots, u_{k+1}\rangle = \bigcup_{v\in H^1(G,R)} \sum_{i=1}^{k} \langle u_1, \dots, u_{i-1}, v, u_{i+2}, \dots u_{k+1}\rangle.$$

*Moreover, then $\mathrm{In}\langle u_1, \dots, u_{k+1}\rangle$ is a $R$-module and for each $\gamma \in \langle u_1, \dots, u_{k+1}\rangle$, $\langle u_1, \dots, u_{k+1}\rangle = \gamma + \mathrm{In}\langle u_1, \dots, u_{k+1}\rangle$.*

Now, using Theorem 2.16, we can compute the coset $\langle u_1, \dots, u_{k+1}\rangle / \mathrm{In}$.

**Corollary 2.25.** *Let $X$ be the 2-complex associated with a $k$-free group $G$. Let $u_1, \dots, u_{k+1} \in H^1(X, R)$ be arbitrary and $u_i = \sum_{j=1}^{n} a_{ij} e_j$. Then*

$$(\langle\langle u_1, \dots, u_{k+1}\rangle\rangle, r_q) = (-1)^{k+1} \sum_{I} a_{1,i_1} \dots a_{k+1,i_{k+1}}\ \varepsilon_I(r_q), \tag{7}$$

*where $1 \le q \le m$ and the summation is over all sequences $I = \{i_1, \dots, i_{k+1}\}$ with $1 \le i_1, \dots, i_{k+1} \le n$.*

We will make use of the previous corollary in the special case of the triple products, and when $G$ is a 2-free group that has some non-vanishing cup products.

**2.26. The $(k+1)^{\text{st}}$ nilpotent quotient of a $k$-free group.** Recall that we work under the assumption that $G$ is a $k$-free group, for some $k \ge 2$ fixed, and that condition (B)is satisfied.

The five-term exact sequence for the extension $0 \to G_k \to G \xrightarrow{\alpha} G/G_k \to 0$ yields:

$$H_2(G) \xrightarrow{\alpha_*} H_2(G/G_k) \xrightarrow{\delta} G_k/G_{k+1} \to 0. \tag{8}$$

The boundary map $\delta$ corresponds, see [2], to the map $\chi$, the image of $\bar{\chi}$ under the epimorphism $H^2(G/G_k; G_k/G_{k+1}) \to \mathrm{Hom}(H_2(G/G_k), G_k/G_{k+1})$ provided by the universal coefficient theorem. After the identifications $G/G_k \cong \mathbb{F}/\mathbb{F}_k$ and $H_2(\mathbb{F}/\mathbb{F}_k) \cong \mathbb{F}_k/\mathbb{F}_{k+1} \cong \mathbb{L}_k H$ we obtain a map

$$\chi : \mathbb{L}_k H \to G_k/G_{k+1}$$

that we call the *classifying map* for the extension (3), where $H$ stands for $H_1(G) = G/G_2$. It is precisely the commutator map induced by $\phi : \mathbb{F} \to G$.

Under the identification $H_2(G/G_k) \cong \mathbb{L}_k H$, the next lemma interprets the map $\alpha_*$ in terms of the cohomology operations of $H^*(G)$.

**Lemma 2.27.** *The map $\alpha_* : H_2(G) \to \mathbb{L}_k H$ is the dual of the Massey product map $\mu_G : \mathbb{L}_k H^1(G) \to H^2(G)$.*

**Proof.** Follows from the commutativity of the diagram

$$\begin{array}{ccc} \mathbb{L}_k H^1(G/G_k) & \xrightarrow{\mathbb{L}_k \alpha^*} & \mathbb{L}_k H^1(G) \\ \downarrow{\scriptstyle \mu_H} & & \downarrow{\scriptstyle \mu_G} \\ H^2(G/G_k) & \xrightarrow{\alpha^*} & H^2(G) \end{array}$$

and the fact that the top and left arrows are isomorphisms. □

The following proposition extends a result for cup-products of Massey and Tralbi [17].

**Proposition 2.28.** *Assume that the Massey product map is surjective. Then $G_k/G_{k+1}$ is torsion-free, and the following is a split exact sequence:*

$$0 \to H_2(G) \xrightarrow{\mu^\top} \mathbb{L}_k H \xrightarrow{\chi} G_k/G_{k+1} \to 0. \tag{9}$$

**Proof.** The proof follows closely that in [17]. By Lemma 2.27, sequence (8) can be written as $H_2(G) \xrightarrow{\mu^\top} \mathbb{L}_k H \xrightarrow{\chi} G_k/G_{k+1} \to 0$. By the surjectivity of $\mu$, the map $\mu^\top$ is a monomorphism, whence the exactness of (9).

Since $\mu : \mathbb{L}_k H \to H^2(G)$ is an epimorphism between finitely generated free abelian groups, it admits a splitting. Hence $\mu^\top$ is a split injection, and so $\chi^\top$ is a split surjection. Since $\mathbb{L}_k H$ is torsion-free, $G_k/G_{k+1}$ is also torsion-free. □

Note that $H_2(G)$ is torsion-free as a consequence of (9).

The next result is an immediate consequence of Proposition 2.28:

**Proposition 2.29.** *Suppose $\mu$ is surjective. Then* $\ker \mu_G \cong G_k/G_{k+1}$, *and the exact sequence*

$$0 \to G_k/G_{k+1} \xrightarrow{\iota} \mathbb{L}_k H^1(G) \xrightarrow{\mu} H^2(G) \to 0 \tag{10}$$

*is the dual of sequence* (9).

**2.30.** We are now ready to establish the correspondence between the truncated cohomology ring of $G$ and the $(k+1)^{\text{st}}$ nilpotent quotient of $G$. The cup-products version of the equivalence $(b) \Leftrightarrow (\chi \sim \chi')$ below, with $\chi$ replaced by $\mu^{\top}$, was first established by Tralbi and Sakuma [31].

**Theorem 2.31.** *Let $G$ and $G'$ be $k$-free groups satisfying conditions* (A) *and* (B). *The following are equivalent:*

*(a)* $\mathcal{H}^2_k(G) \cong \mathcal{H}^2_k(G')$;

*(b)* $G/G_{k+1} \cong G'/G'_{k+1}$.

**Proof.** By Proposition 2.28 the abelian goup $G_k/G_{k+1}$ is torsion-free. Then, the universal coefficient map is an isomorphism, and so $\bar{\chi}$ and $\chi$ determine each other. Thus, the isomorphism type of $G/G_k$ is completely determined by the equivalence class of the classifying map $\chi$. The condition that the $k$-invariants of $G/G_k$ and $G'/G'_k$ correspond translates to $\psi \circ \chi = \chi' \circ \wedge^2\phi$, in short $\chi \sim \chi'$. We prove that both isomorphisms are equivalent to $\chi \sim \chi'$.

$(a) \Leftrightarrow (\chi \sim \chi')$. By Proposition 2.29, sequence (10) is exact, and $\chi = \iota^{\top}$. The equivalence follows from the definitions.

$(b) \Leftrightarrow (\chi \sim \chi')$. By Proposition 2.29, the $k^{th}$ lower central series quotients of $G$ and $G'$ are torsion-free. The equivalence follows from the discussion in Remark 2.9. □

In Section 3, having this theorem as the starting point, we shall construct, for a $k$-free group $G$, invariants of the Massey ring $\mathcal{H}_k(G)$ on one hand, and invariants of the nilpotent quotients $G/G_{k+1}$ on the other, which we will be related in Section 4.

**3. Characteristic and resonance varieties.** In the first part of this section we associate to a finitely presented group $G$ with torsion-free abelianization and to a field $\mathbb{K}$ a natural stratification by subvarieties of the character variety $\mathrm{Hom}(G, \mathbb{K}^*)$, which, up to a certain type of isomorphism, is an invariant of the group. In the case $\mathbb{K} = \mathbb{C}$, the field of complex numbers, the stratification was introduced by Novikov in [24] and more recently investigated by Libgober in [15] and Hironaka in [13]. For other fields the stratification is studied in the paper by Matei and Suciu [19]. In the second part of the section we associate to a $k$-free group $G$ satisfying certain conditions and to a commutative field $\mathbb{K}$, a stratification by subvarieties of $H^1(G, \mathbb{K})$, which, up to linear isomorphism, is an invariant of the

$k^{\text{th}}$ nilpotent quotient $G/G_{k+1}$. In the case $k = 2$, the stratification was studied by MATEI and SUCIU in [20], where the subvarieties of the stratification where called resonance varieties. These varieties where introduced by FALK [5] in his work on the cohomology of complements of complex hyperplane arrangements. The idea to generalize these invariants to higher $k$ owes a great deal to the work of TRALDI [29, 30]. The main result of this section is Theorem 3.14, where equations are found for the resonance varieties in terms of certain truncations of the Alexander matrix of $G$.

**3.1. Characteristic varieties.** Let $G$ be a finitely presented group and $\mathbb{K}$ a field. Assume that the first homology group $H_1(G)$ is torsion-free. For simplicity, denote $H_1(G)$ by $H$ and the abelianization homomorphism by $\alpha : G \to H$. Let $\mathrm{Hom}(G, \mathbb{K}^*)$ be the group of characters of $G$. Notice that $\mathrm{Hom}(G, \mathbb{K}^*)$ is actually isomorphic to $\mathrm{Hom}(H, \mathbb{K}^*)$. We think of $\mathrm{Hom}(G, \mathbb{K}^*)$ as an algebraic group with coordinate ring $\mathbb{K}H$. Let $n$ be the rank of $H$, and let $\psi : H \to \mathbb{Z}^n$ an isomorphism. Then extension to group rings $\bar{\psi}$ identifies $\mathbb{Z}H$ with the Laurent polynomials ring $\Lambda = \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. Moreover, $\mathrm{Hom}(G, \mathbb{K}^*)$ is identified with the affine torus $(\mathbb{K}^*)^n$ having as coordinate ring $\Lambda_{\mathbb{K}} = \mathbb{K}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$.

Let $\langle x_1, \dots, x_q \mid r_1, \dots, r_s \rangle$ be a presentation for $G$ with presenting homomorphism $\phi : \mathbb{F} \to G$. Denote by $\bar{\phi}$ the extension to group rings of $\alpha\phi : \mathbb{F} \to H$.

**Definition 3.2.** The *Alexander matrix* of $G$ is the $q \times s$ matrix of Fox derivatives having entries in the Laurent polynomials ring $\Lambda$:

$$M(G) = \left( \bar{\psi}\bar{\phi} \left( \frac{\partial r_i}{\partial x_j} \right) \right).$$

Define the $k^{th}$ *Alexander ideal* of $G$ to be the ideal $E_k(G, \psi)$ generated by the codimension $k$ minors of the Alexander matrix $M$. Clearly, $E_k(G, \psi) \subseteq E_l(G, \psi)$ if $k \leq l$. The Alexander ideals depend only on the isomorphism type of $X$, and on the identification $\psi : H_1(G) \to \mathbb{Z}^n$. Let $I$ be the augmentation ideal of the ring $\Lambda$. If $G$ has positive deficiency (i.e., admits a presentation with more generators than relations), then $E_1(G, \psi)$ is of the form $I \cdot (\Delta_{G,\psi})$, where $\Delta_{G,\psi} \in \Lambda$ is the (multi-variable) *Alexander polynomial* of $G$.

We now associate to $G$ subvarieties $V_k(G, \psi)$ of the algebraic torus $(\mathbb{K}^*)^n$, defined by the ideals $E_k(G, \psi)$. Clearly, $V_k(G, \psi) \supseteq V_\ell(G, \psi)$ if $k \leq \ell$.

**Proposition 3.3.** *The monomial isomorphism type of the subvariety $V_k(G,\psi)$ of the algebraic torus $(\mathbb{K}^*)^n$ depends only on the isomorphism type of $G$, and not on the identification $\psi : H_1(G) \to \mathbb{Z}^n$. We call $V_k(G,\mathbb{K}) = V_k(G,\psi)$ the $k^{\text{th}}$* characteristic variety *of $G$ with $\mathbb{K}$-coefficients.*

If $G$ has positive deficiency, the Alexander polynomial $\Delta_G = \Delta_{G,\psi}$ is well-defined up to a monomial change of basis in $(\mathbb{K}^*)^n$, and up to multiplication by a unit $\kappa t_1^{i_1}\cdots t_n^{i_n} \in \Lambda_{\mathbb{K}}$, where $\kappa \in \mathbb{K}^*$. Note that $V_1(G,\mathbb{K}) = \mathbf{1} \cup \{\Delta_X = 0\}$, where $\mathbf{1} = (1,\dots,1)$ is the origin of the complex torus $(\mathbb{K}^*)^n$.

The varieties $V_k(G,\mathbb{K})$ may be interpreted as jumping loci for group cohomology, as follows. For $k$ a positive integer, let

$$W_k(G,\mathbb{K}) = \{\mathbf{t} = (t_1,\dots,t_n) \in (\mathbb{K}^*)^n \mid \dim_{\mathbb{K}} H^1(G,\mathbf{t}) \geq k\},$$

where $H^1(G,\mathbf{t})$ is the first cohomology group of $G$ with coefficients the $G$-module $\mathbb{K}$ with the action induced by the representation $\mathbf{t} : G \to \mathbb{K}^*$. Then we have the following:

**Proposition 3.4.** *For all $k$, the jumping loci $W_k(G,\mathbb{K})$ coincide with the characteristic varieties $V_k(G,\mathbb{K})$, except for $k = n$, in which case $W_n(G,\mathbb{K}) = V_n(G,\mathbb{K}) \cup \mathbf{1}$.*

The proof given in [13] for the case $\mathbb{K} = \mathbb{C}$ extends to an arbitrary field without any modification.

**3.5. Resonance varieties.** We define now the 'resonance' varieties of the cohomology products of a $k$-free group $G$. We then show that, under certain conditions on $G$, these varieties are the determinantal varieties of the appropriately truncated Alexander matrix of the group $G$.

**3.6.** Let $G$ be a $k$-free group with presentation $G = \langle x_1,\dots,x_n \mid r_1,\dots, r_m\rangle$. Suupose $G$ satisfies conditions (A) and (B). We thus have: $H^1(G) = \mathbb{Z}^n$, $H^2(G) = \mathbb{Z}^m$.

**Lemma 3.7.** *Let $G$ be as above and let $\mathbb{K}$ be a field.*

*(a) The $k$-fold Massey products with $\mathbb{K}$ coefficients may be computed from the integral ones: $\mu_{G,\mathbb{K}} = \mu_G \otimes \mathrm{id}_{\mathbb{K}}$.*

*(b) If $\mathcal{H}^2_k(G) \cong \mathcal{H}^2_k(G')$ then $\mathcal{H}^2_k(G,\mathbb{K}) \cong \mathcal{H}^2_k(G',\mathbb{K})$.*

**Proof.** Let $g : \mathbb{Z} \to \mathbb{K}$ be the homomorphism given by $g(1) = 1$. Recall that $k$-fold Massey products are defined with no indeterminacy, they are natural and multilinear. Thus, by Proposition 2.14, the coefficient map $g_\#$ commutes with Massey products, and by Corollary 2.17, the map $\mathrm{id} \otimes g$ commutes also. By the universal coefficient theorem, the map $\upsilon : H^*(G) \otimes \mathbb{K} \to H^*(G, \mathbb{K})$, $\upsilon([z] \otimes \kappa) = [z \otimes \kappa]$ is an isomorphism for $* \leq 2$. Since $\upsilon \circ (\mathrm{id} \otimes g) = g_\#$, the map $\upsilon$ also commutes with Massey products, and the conclusion follows. □

Identify $H^1(G, \mathbb{K})$ with $\mathbb{K}^n$, and $H^2(G, \mathbb{K})$ with $\mathbb{K}^m$.

**Definition 3.8.** Let $d$ be an integer, $0 \leq d \leq n$. The $d^{\text{th}}$ *resonance variety* of $G$ (with coefficients in $\mathbb{K}$) is the subvariety of $\mathbb{K}^n$, defined as follows:

$$\mathcal{R}_d(G, \mathbb{K}) = \left\{ \lambda \in H^1(G, \mathbb{K}) \;\middle|\; \begin{array}{l} \exists \text{ subspace } W \subset H^1(G, \mathbb{K}) \text{ such that} \\ \dim W = d + 1 \text{ and } \langle W, \lambda, \dots, \lambda \rangle = \{0\} \end{array} \right\},$$

where $\langle W, \lambda, \dots, \lambda \rangle$ stands for the vector space generated by the $k$-fold products $\langle v, \lambda, \dots, \lambda \rangle$ with $v \in W$.

**Remark 3.9.** The above definition makes sense because $k$-fold Massey products are defined with no indeterminacy, and they are natural and multilinear, see Corollary 2.17.

The resonance varieties form a descending filtration $\mathbb{K}^n = \mathcal{R}_0 \supset \mathcal{R}_1 \supset \cdots \supset \mathcal{R}_{n-1} \supset \mathcal{R}_n = \emptyset$. The *ambient type* of the $\mathbb{K}$-resonance varieties depends only on the 'Massey ring' $\mathcal{H}^2_k(G, \mathbb{K})$, and thus, by remark (b) above, only on $\mathcal{H}^2_k(G)$. More precisely, if $\mathcal{H}^2_k(G) \cong H^2_k(G')$, there exists a linear automorphism of $\mathbb{K}^n$ taking $\mathcal{R}_d(G, \mathbb{K})$ to $\mathcal{R}_d(G', \mathbb{K})$.

Thus, by Theorem 2.31, the resonance varieties $\mathcal{R}_d(G, \mathbb{K})$ are invariants of the $k^{\text{th}}$ nilpotent quotient $G/G_{k+1}$.

**3.10.** Let $G = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$ be a $k$-free group. Recall the projection map $\psi : \mathbb{F} \to G$, and the abelianization map, $\alpha : G \to \mathbb{Z}^n$, given by $\alpha(x_i) = t_i$.

Next, let $\psi : \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}] \to \mathbb{Z}[[s_1, \dots, s_n]]$ be the ring homomorphism given by $\psi(t_i) = 1 + s_i$ and $\psi(t_i^{-1}) = \sum_{q \geq 0} (-1)^q s_i^q$. Also, let $\psi^{(q)}$ be the graded $q^{\text{th}}$ piece of $\psi$. Since $G$ is a $k$-free group all the relators are words in $\mathbb{F}_k$, therefore the entries of $M$ are in the $(k-1)^{\text{st}}$ power $I^{k-1}$ of the ideal $I = (t_1 - 1, \dots, t_n - 1)$, and so $\psi^{(l)} M$ is the zero matrix, for any $l < k - 1$.

**Definition 3.11.** The $(k-1)^{\text{st}}$ *truncation* of the Alexander matrix of $G$ is the $m \times n$ matrix

$$T^{(k-1)} = \psi^{(k-1)} M.$$

When, $k = 2$ we refer to $T^{(1)}$ as the *linearized Alexander matrix.*

**Lemma 3.12.** *The entries of $T = T^{(k-1)}$ are integral homogenous forms of degree $k-1$ in the variables $s_1, \dots, s_n$, given by:*

$$T_{i,j} = \sum_{1 \le i_1, \dots, i_{k-1} \le n} \varepsilon_{i_1, \dots, i_{k-1}, j}(r_i) s_{i_1} \cdots s_{i_{k-1}}.$$

**Proof.** Follows easily from the 'Taylor series' expansion of Fox, see Proposition 2.6.

Similar truncations of the Alexander matrix where considered by Traldi [29, 30], in the case when $G$ is a link group.

**3.13.** We now find explicit equations for the varieties $\mathcal{R}_d(G, \mathbb{K})$. If $T$ is, as above, the truncated Alexander matrix of $G$, let $T_{\mathbb{K}}$ denote the corresponding matrix of linear forms over $\mathbb{K}$, and let $T(\lambda)$ be the matrix $T_{\mathbb{K}}$ evaluated at $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{K}^n$.

**Theorem 3.14.** *For $G$ a $k$-free group with $H_2(G)$ torsion free,*

$$\mathcal{R}_d(G, \mathbb{K}) = \{\lambda \in \mathbb{K}^n \mid \operatorname{rank}_{\mathbb{K}} T(\lambda) < n - d\}.$$

**Proof.** Let $\lambda = \sum_{i=1}^{n} \lambda_i e_i \in H^1(G, \mathbb{K})$. We are looking for $v = \sum_{i=1}^{n} v_i e_i$ such that $\langle v, \lambda, \dots, \lambda \rangle = 0$ in $H^2(G, \mathbb{K})$. From Corollary 2.17, we have that

$$\langle v, \lambda, \dots, \lambda \rangle = \sum_{q=1}^{m} \left( \sum_{1 \le i_1, \dots, i_{k-1}, j \le n} \lambda_{i_1} \cdots \lambda_{i_{k-1}} v_j \, \varepsilon_{i_1, \dots, i_{k-1}, j}(r_q) \right) \gamma_q.$$

We thus obtain a linear system of $m$ equations in $v_1, \dots, v_n$:

$$\sum_{j=1}^{n} \left( \sum_{1 \le i_1, \dots, i_{k-1} \le n} \lambda_{i_1} \cdots \lambda_{i_{k-1}} \varepsilon_{i_1, \dots, i_{k-1}, j}(r_q) \right) v_j = 0,$$

with coefficients matrix $T(\lambda)$.

Now $\lambda$ belongs to $\mathcal{R}_d(G,\mathbb{K})$ if and only if the space $W$ of solutions of the linear system $M(\lambda)\cdot v = 0$ is at least $(d+1)$-dimensional. That translates into the condition $\operatorname{rank} M(\lambda) < n-d$ of the statement, and we are done.□

We will be mainly interested in the coefficients fields $\mathbb{K}=\mathbb{C}$ and $\mathbb{K}=\mathbb{Z}_p$, for some prime $p$. By the above theorem, the $\mathbb{C}$-resonance varieties have *integral* equations. Although $\mu_{G,\mathbb{Z}_p}$ is the reduction mod $p$ of $\mu_G$, the variety $\mathcal{R}_d(G,\mathbb{Z}_p)$ is *not* necessarily the reduction mod $p$ of $\mathcal{R}_d(G,\mathbb{C})$.

**3.15.** The affine variety $\mathcal{R}_d(G,\mathbb{K})\subset\mathbb{K}^n$ is homogeneous, and so defines a projective variety $\mathcal{P}_d(G,\mathbb{K})\subset\mathbb{P}(\mathbb{K}^n)$. If $\mathcal{H}_k(G)$ is isomorphic to $\mathcal{H}_k(G')$, there is a projective automorphism $\mathbb{P}(\mathbb{K}^n)\to\mathbb{P}(\mathbb{K}^n)$ taking $\mathcal{P}_d(G,\mathbb{K})$ to $\mathcal{P}_d(G',\mathbb{K})$.

**Corollary 3.16.** $\mathcal{P}_d(G,\mathbb{K}) = \{\lambda\in\mathbb{P}(\mathbb{K}^n) \mid \operatorname{rank}_{\mathbb{K}} M(\lambda) < n-d-1\}$.

**4. Finite index subgroups.** Given a finitely presented group $G$, with $H_1(G)$ torsion-free, we study the abelianization of its prime index normal subgroups, relating it with its resonance varieties. We consider the distribution of normal subgroups according to their abelianization for the $(k+1)^{\text{st}}$ nilpotent quotient of a $k$-free group $G$, and we interpret it in terms of the resonance varieties of $G$. The main results of the section are Theorem 4.19 and Theorem 4.19 where the aforementioned interpretations are established. We also show, in Theorem 4.22, that the torsion of index type, in the abelianizations of prime index normal subgroups of $G$, is related with the analogue torsion of subgroups in the nilpotent quotients of $G$. In the last paragraph we give an account of our attempt to extend Theorem 4.19 to the next nilpotent quotient $G/G_{k+2}$.

**4.1. First homology of finite index subgroups.** We start by recalling the procedure introduced by Fox [9] for calculating the first homology of a finite index subgroup $K$ of a finitely presented group $G$ in terms of a presentation. Such a subgroup $K$ of index $p$ is determined by a homomorphism $\lambda: G\to S_p$, where $S_p$ is the symmetric group. When appropriate, we regard $S_p$ as the subgroup of $\mathrm{GL}(p,\mathbb{Z})$ of permutation matrices. Consider the Jacobian matrix of Fox derivatives $J=\phi\left(\partial r_i/\partial x_j\right)$ with entries in $\mathbb{Z}G$, where $\phi:\mathbb{F}\to G$ is the presenting homomorphism. Then we have the following:

**Theorem 4.2.** *Let $K$ be a index $p$ subgroup of $G = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$, and $\lambda : G \to S_p$ the associated representation. Then the $np \times mp$ matrix of integers $\lambda(J)$ is a presentation matrix, over $\mathbb{Z}$, for $H_1(K) \oplus \mathbb{Z}^{p-1}$.*

**4.3.** We restrict from now on to normal subgroups of prime index $p$. Such a subgroup $K$ is given by the kernel of a homomorphism $\lambda : G \to \mathbb{Z}_p$.

We now restrict $G$ to the class of 2-free groups. This condition ensures that the number of index $p$ normal subgroups depends only on $p$ and $n = b_1(G)$. Let $G = \mathbb{F}/R$ be a 2-free group. It is not hard to see that for all primes $p$ the number of index $p$ normal subgroups of $G$ is equal to $\frac{p^n-1}{p-1}$.

**4.4. Nilpotent quotients invariants from finite index subgroups.** In this section, for the $q^{\text{th}}$ nilpotent quotient of a $(q-1)$-free group, we show how to count the normal subgroups of prime index, according to their abelianization.

**4.5.** Let $G = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$ be a $(q-1)$-free group, for some fixed $q \geq 3$. We are going to use extensively Fox's procedure from Theorem 4.2, so more notation and terminology are in order. Let $K \lhd G$ be a normal subgroup of index $p$, defined by a representation $\lambda : G \to \mathbb{Z}_p$,$\lambda(x_i) = \lambda_i$. Let $\bar{\lambda} : \mathbb{Z}G \to \mathbb{Z}\mathbb{Z}_p$ be the linear extension of $\lambda$ to group rings. More precisely, we view here $\mathbb{Z}_p$ as a multiplicative group, with generator $\zeta$. Then $\bar{\lambda}(x_i) = \zeta^{\lambda_i}$. Finally, let $\beta : \mathbb{Z}\mathbb{Z}_p \to \mathrm{Mat}(p, \mathbb{Z})$ be the ring homomorphism defined by the (left) regular representation of $\mathbb{Z}_p$.

**Definition 4.6.** For a given representation $\lambda : G \to \mathbb{Z}_p$, the *twisted Alexander matrix of* $G$ is the $pm \times pn$ matrix

$$A_\lambda = \left(\bar{\lambda}\phi\partial_i(r_k)\right)^\beta$$

obtained from $\left(\bar{\lambda}\phi\partial_i(r_k)\right)$ by replacing each entry $e$ with $\beta(e)$.

The matrix $A_\lambda$ is a relation matrix for for the group $H_1(K_\lambda) \oplus \mathbb{Z}^{p-1}$.

**4.7.** We now apply the above procedure to a particular class of groups: the nilpotent quotients $G/G_q$, $q \geq 3$, of a $(q-1)$-free group $G = \mathbb{F}/R$.

Let $\lambda : G/G_q \to \mathbb{Z}_p$ be a non-trivial representation. To describe explicitly the presentation matrix $A_\lambda$, we need to examine more closely the Fox derivatives of the relators $c_h$ and $r_k^{(q)}$ in the presentation (4) for $G/G_q$.

If $c$ is a non-simple basic commutator, then Proposition 2.6 (a), (c) gives $\bar{\lambda}\phi(\partial_i c) = 0$. If $c = [x_{i_1}, x_{i_2}, \dots, x_{i_q}]$ is a simple commutator, then it follows from Proposition 2.6 (c) that $\bar{\lambda}\phi(\partial_i c)$ is either zero or of the form $e = \pm(\zeta^{a_1} - 1)\cdots(\zeta^{a_{q-2}} - 1) \in \mathbb{Z}\mathbb{Z}_p$, for some integers $1 \le a_j \le p-1$.

Recall that the truncation $r_k^{(q)}$ is a product of basic commutators of length $< q$. The same argument shows that $\bar{\lambda}\phi\big(\partial_i r_k^{(q)}\big)$ is a linear combination of elements in $\mathbb{Z}\mathbb{Z}_p$ of the form $(\zeta^{a_{i_1}} - 1)\cdots(\zeta^{a_{i_j}} - 1)$, for $j < q-2$.

The following lemma shows the typical simplifications that we will perform on $\big(\bar{\lambda}\phi(\partial c_h)\big)^\beta$ and $\big(\bar{\lambda}\phi(\partial r_k^{(q)})\big)^\beta$.

**Lemma 4.8.** *The integral $p\times p$ matrix $e^\beta$ of $e = (\zeta^{a_1} - 1)\cdots(\zeta^{a_k} - 1) \in \mathbb{Z}\mathbb{Z}_p$ has diagonal form*

$$\big(\underbrace{p^{r-1}, \dots, p^{r-1}}_{p-l-1}, \underbrace{p^r, \dots, p^r}_{l}, 0\big),$$

*where $r = \lceil \frac{k-1}{p-1} \rceil$, and $l = k-1-(r-1)(p-1)$. Moreover, there is a sequence of row and column operations, independent of the particular $e$, that brings $e^\beta$ to that diagonal form.*

**Proof.** We describe the sequence of row and column operations from the statement. Denote the rows of a matrix by $R_1, R_2, \dots$ and the columns by $C_1, C_2, \dots$, so that we simplify the discussion. We first add all the rows and all the columns to the last row, and respectively to the last column, to obtain a row and a column of zeroes. Then at each step $s$, from 1 to $p-1$, we perform a row reduction: subtract $R_s$ from $R_{s+1}, \dots, R_{p-1}$, followed by a column reduction: add $\binom{q}{s} \cdot C_q$ to $C_s$, for all $s+1 \le q \le p-1$. □

**Proposition 4.9.** *Let $K$ be an index $p$ normal subgroup of $\mathbb{F}(n)/\mathbb{F}(n)_q$. Then:*

$$H_1(K) = \mathbb{Z}^n \oplus (\mathbb{Z}/p^{r-1}\mathbb{Z})^{(n-1)(p-l-1)} \oplus (\mathbb{Z}/p^r\mathbb{Z})^{(n-1)l},$$

*where $r = \left\lceil \frac{q-2}{p-1} \right\rceil$, and $l = q-2-(r-1)(p-1)$.*

**Proof.** In this case, only commutator relators are present, so Lemma 4.8, applied to each entry $\bar{\lambda}\phi(\partial c_h)$, shows that the matrix $A_\lambda$ is equivalent the following diagonal matrix:

$$D = \big(\underbrace{p^{r-1}, \dots, p^{r-1}}_{(n-1)(p-l-1)}, \underbrace{p^r, \dots, p^r}_{(n-1)l}, \underbrace{0, \dots, 0}_{n+p-1}\big). \tag{11}$$

□

**Theorem 4.10.** *Let $G = \mathbb{F}(n)/R$ be a 2-free group. Let $K$ be an index $p$ normal subgroup of $G/G_q$. Set $r = \left\lceil \frac{q-2}{p-1} \right\rceil$. Then:*

$$H_1(K) = \mathbb{Z}^n \oplus \bigoplus_{i=0}^{r} (\mathbb{Z}/p^i\mathbb{Z})^{d_i},$$

*for some positive integers $d_0, \ldots, d_r$ such that $d_0 + \cdots + d_r = (n-1)(p-1)$ and $d_r \leq l(n-1)$.*

**Proof.** Let $K = \ker(\lambda : G/G_q \to \mathbb{Z}_p)$. Consider the relation matrix $A_\lambda$, corresponding to the presentation $G/G_q = \mathbb{F}/R\mathbb{F}_q$ from (4). Partition $A_\lambda$ into two blocks, $A_\lambda = \binom{B_\lambda}{C_\lambda}$, where $B_\lambda$ corresponds to the relators $R$, and $C_\lambda$ corresponds to the basic commutators.

Assume that the row and column operations of Lemma 4.8 have already been performed. Then, after moving all the zero columns to the right, $A_\lambda$ is equivalent to $\begin{pmatrix} B'_\lambda & 0 \\ D' & 0 \end{pmatrix}$, where $D = (D'\ 0)$ is the diagonal matrix (11). Since the number of zero diagonal elements of $D$ is $n + p - 1$, the rank of $K$ is $n$. Since the non-zero diagonal elements of $D$ are either $p^{r-1}$ or $p^r$, the elementary divisors of $K$ are among $p, p^2, \ldots, p^r$. □

**4.11.** In view of Theorem 4.10, we define the following numerical invariants of isomorphism type for the nilpotent quotients of a group.

**Definition 4.12.** Let $G$ be a 2-free group, and $G/G_q$ be the $(q-1)^{st}$ nilpotent quotient of $G$. Given a prime $p$, and a positive integer $d$, define

$$\nu_{p,d}(G/G_q) = \#\big\{K \lhd G/G_q \;\big|\; [G/G_q : K] = p \text{ and } \dim_{\mathbb{Z}_p}(\operatorname{Tors} H_1(K)) \otimes \mathbb{Z}_p = d\big\}.$$

We are going to refer to the numbers $\nu_{p,d}$ as *$\nu$-invariants*. We can define the analogue invariants for an arbitrary 2-free group $G$; we shall denote them by $\nu_{p,d}(G)$.

**Example 4.13.** If $q = 3$, then $H_1(K) = \mathbb{Z}^n \oplus \mathbb{Z}_p^d$, for some $0 \leq d \leq n-1$. So we have invariants $\nu_{p,0}(G/G_3), \ldots, \nu_{p,n-1}(G/G_3)$ for the second nilpotent quotient of $G$. Since $\sum_{d=0}^{n-1} \nu_{p,d} = \frac{p^n-1}{p-1}$, it is enough to compute $\nu_{p,1}, \ldots, \nu_{p,n-1}$.

**Example 4.14.** If $q = 4$, and $p \geq 3$, then $H_1(K) = \mathbb{Z}^n \oplus \mathbb{Z}_p^d$, for some $0 \leq d \leq 2n-2$. If $p = 2$, then $H_1(K) = \mathbb{Z}^n \oplus \mathbb{Z}_2^{d_1} \oplus \mathbb{Z}_4^{d_2}$, for some $0 \leq d = d_1 + d_2 \leq n-1$.

**4.15.** We now restrict our attention to $G/G_3$. From (4), for $q = 3$ we obtain the presentation:

$$G/G_3 = \langle x_1, \dots, x_n \mid r_1^{(3)}, \dots, r_m^{(3)}, c_1, \dots, c_l \rangle, \tag{12}$$

where $l = 2\binom{n+1}{3}$, and the basic commutators $c_h$ are of the form $[x_i, [x_j, x_k]]$, with $j < k$ and $i \geq j$.

**Theorem 4.16.** *Given an epimorphism $\lambda : G/G_3 \to \mathbb{Z}_p$, with kernel $K_\lambda$, we have*

$$\dim_{\mathbb{Z}_p}(\operatorname{Tors} H_1(K_\lambda)) \otimes \mathbb{Z}_p = n - 1 - \operatorname{rank}_{\mathbb{Z}_p} T(\lambda).$$

**Proof.** Recall from the proof of Theorem 4.10 that the relation matrix of the abelian group $H_1(K_\lambda)$ has the following form: $A_\lambda = \begin{pmatrix} B'_\lambda & 0 \\ C'_\lambda & 0 \end{pmatrix}$. We have already seen in Proposition 4.9 that $C'_\lambda$ is equivalent to a diagonal matrix $D' = \begin{pmatrix} I_{(n-1)(p-2)} & 0 \\ 0 & p \cdot I_{n-1} \end{pmatrix}$.

Recall also that $r_k^{(3)} = \prod_{i<j} [x_i, x_j]^{\varepsilon_{i,j}(r_k)}$. A computation using formula (a) in Proposition 2.6 shows:

$$\bar{\lambda}\phi\left(\frac{\partial r_k^{(3)}}{\partial x_l}\right) = \sum_{i=1}^{n} \varepsilon_{i,l}(r_k)(\zeta^{\lambda_i} - 1), \tag{13}$$

for $1 \leq l \leq n$ and $1 \leq k \leq m$.

Consider $e = \sum_{\sigma=1}^{p-1} a_\sigma(\zeta^\sigma - 1) \in \mathbb{Z}\mathbb{Z}_p$. Set $a = \sum_{\sigma=1}^{p-1} a_\sigma$. It is readily seen that the matrix $e^\beta$ is equivalent to:

$$\begin{pmatrix} * & \cdots & * & p \cdot a & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ * & \cdots & * & p \cdot a & 0 \\ * & \cdots & * & \sum_{\sigma=1}^{p-1} a_\sigma \cdot \sigma & 0 \end{pmatrix}. \tag{14}$$

Now (3.12), together with (13) and (14), imply that $B'_\lambda$ is equivalent to $\begin{pmatrix} * & 0 \\ * & T(\lambda)' \end{pmatrix}$, where $T(\lambda)'$ is some codimension 1 minor of $T(\lambda)$. Hence, $A_\lambda$ is equivalent to:

$$\begin{pmatrix} * & 0 & 0 \\ * & T(\lambda)' & 0 \\ I_{(n-1)(p-2)} & 0 & 0 \\ 0 & p \cdot I_{n-1} & 0 \end{pmatrix}.$$

The theorem now follows from the following fact: An integral matrix of the form $\begin{pmatrix} Q \\ p\cdot I_n \end{pmatrix}$ is equivalent to $\begin{pmatrix} I_r & 0 \\ 0 & p\cdot I_d \end{pmatrix}$, where $r = \operatorname{rank}_{\mathbb{Z}_p} Q \otimes \operatorname{id}_{\mathbb{Z}_p}$, and $d = n - r$. □

**Corollary 4.17.** $\nu_{p,d}(G/G_3) = \#\{K_\lambda \in \Sigma_p(G/G_3) \mid \operatorname{rank}_{\mathbb{Z}_p} T(\lambda) = n - d - 1\}$.

**4.18.** The following theorem relates the distribution of index $p$ normal subgroups of $G/G_3$, according to their abelianization, to the number of points on the $n$-dimensional projective space over $\mathbb{Z}_p$, according to the stratification by the resonance varieties.

**Theorem 4.19.** *For $G$ a 2-free group with $H_2(G)$ torsion free,*

$$\nu_{p,d}(G/G_3) = \#(\mathcal{P}_d(G,\mathbb{Z}_p) \setminus \mathcal{P}_{d+1}(G,\mathbb{Z}_p)).$$

**Proof.** Follows from Corollary 3.16 and Corollary 4.17. □

The previous theorem extends with no conceptual difficulty to the more general case of the $(k+1)^{\text{st}}$ nilpotent quotient $G/G_{k+1}$ of a $k$-free group $G$, and the corresponding resonance varieties. The computations are similar but more tedious, and the formulae more combersome, but the idea of the proof remains the same.

**Theorem 4.20.** *Let $G$ be a $k$-free group with $H_2(G)$ torsion free, $H_1(G) = \mathbb{Z}^n$ and $p$ a prime number. Then for $p \leq k-1$ we have:*

$$\nu_{p,d}(G/G_{k+1}) = \begin{cases} \frac{p^n-1}{p-1}, & \text{if } d = (n-1)(p-1); \\ 0 & \text{otherwise.} \end{cases}$$

*and for $p \geq k$ we have:*

$$\nu_{p,d}(G/G_{k+1}) = \#(\mathcal{P}_{d-s}(G,\mathbb{Z}_p) \setminus \mathcal{P}_{d-s+1}(G,\mathbb{Z}_p)),$$

*where $s = (k-2)(n-1)$.*

**Proof.** The proof goes as in the case $k = 2$. The shift $s = (k-2)(n-1)$ appears because now the twisted Alexander matrix $A_\lambda$ in the proof of Theorem 4.16, will be equivalent, in virtue of Lemma 4.8, to:

$$\begin{pmatrix} * & 0 & 0 & 0 \\ * & 0 & T(\lambda)' & 0 \\ I_{(n-1)(p-2)} & 0 & 0 & 0 \\ 0 & p\cdot I_s & 0 & 0 \\ 0 & 0 & p\cdot I_{n-1} & 0 \end{pmatrix}.$$

The first part follows the fact that if $e = (\zeta - 1)^{\alpha_1} \cdots (\zeta^{p-1} - 1)^{\alpha_{p-1}} \in \mathbb{Z}\mathbb{Z}_p$ is such that $\alpha_1 + \cdots + \alpha_{p-1} \geq p$ then $e^\beta \equiv 0 \mod p$. □

**4.21.** Motivated by the facts established in the previous paragraph, we may associate to a finitely presented group $G$ affine and projective subvarieties over the field $\mathbb{Z}_p$, by counting the abelianization types of index $p$ normal subgroups in $G$. Let $d$ be a positive integer. Define $\mathcal{Q}_d^{(q)}(G, \mathbb{Z}_p)$ to be the set:

$$\mathcal{Q}_d^{(q)}(G, \mathbb{Z}_p) = \{\lambda \in \mathrm{Hom}(G/G_q, \mathbb{Z}_p) \mid \dim_{\mathbb{Z}_p}(\mathrm{Tors}\, H_1(K_\lambda)) \otimes \mathbb{Z}_p \geq d\}.$$

For the case that interests us the most, when $G$ is a $k$-free group, with $H_2(G)$ is free abelian, the sets $\mathcal{Q}_d^{(k+1)}(G, \mathbb{Z}_p)$ coincide with the resonance varieties $\mathcal{R}_d(G, \mathbb{Z}_p)$. We call $\mathcal{Q}_d^{(k+2)}(G, \mathbb{Z}_p)$, the *secondary resonance varieties* of $G$, and we denote them simply by $\mathcal{Q}_d(G, \mathbb{Z}_p)$. We will only prove here that $\mathcal{Q}_d^{(4)}(G, \mathbb{Z}_p)$ are given by polynomial equations, so at least in that case they are rightfully called varieties. We believe that the same is true in general, for all $k \geq 2$.

We show next that the sets $\mathcal{Q}_d^{(q)}(G, \mathbb{Z}_p)$ contain significant information about the $p$-torsion part of abelianizations of index $p$ subgroups in $G$.

**Theorem 4.22.** *Let $G$ be a 2-free group, such that $H_2(G)$ is free abelian. Let $n$ be the rank of the abelianization of $G$, and $p$ a prime number. For $\lambda : \mathbb{Z}^n \to \mathbb{Z}_p$ a homomorphism, denote by $K_\lambda$ the index $p$ subgroup in $G$ defined by $\lambda$, and by $N_{q,\lambda}$ the index $p$ subgroup in $G/G_q$ defined by $\lambda$. Then for $q \geq p + 1$ we have:*

$$b_1(N_{q,\lambda}, \mathbb{Z}_p) = b_1(K_\lambda, \mathbb{Z}_p),$$

*and for $q \leq p$ we have:*

$$b_1(N_{q,\lambda}, \mathbb{Z}_p) \leq b_1(K_\lambda, \mathbb{Z}_p).$$

**Proof.** By Fox's algorithm, the twisted Alexander matrices $A_\lambda$ of $G$, and $A_{\lambda,q}$ of $G/G_q$ are presentation matrices for $H_1(K_\lambda) \oplus \mathbb{Z}^{p-1}$, and respectively $H_1(N_{q,\lambda}) \oplus \mathbb{Z}^{p-1}$. We could think of $A_\lambda(G/G_q)$ as obtained from $A_\lambda(G)$ by adjoining rows coming from the $q$-fold commutator relators. Then

as in Proposition 4.9, the $q$-fold commutators produce, after Lemma 4.8 is applied, a diagonal matrix

$$D = \big(\underbrace{p^{r-1}, \ldots, p^{r-1}}_{(n-1)(p-l-1)}, \underbrace{p^r, \ldots, p^r}_{(n-1)l}, \underbrace{0, \ldots, 0}_{n+p-1}\big),$$

where $r = \left\lceil \frac{q-2}{p-1} \right\rceil$ and $l = q - 2 - (r-1)(p-1)$.

Now $q > p+1$ implies $r = \left\lceil \frac{p-1}{p-1} \right\rceil > 1$, so $D \otimes \mathbb{Z}_p$ is the diagonal matrix $\big(p, \ldots, p, 0, \ldots, 0\big)$, having $(n-1)(p-1)$ diagonal entries of $p$. The same is true about $D \otimes \mathbb{Z}_p$ also when $q = p+1$, because in that case we have $l = p-1$.

Now let us look at what happens with an entry $\bar{\lambda}\phi(\partial_i r_k)$ of $\mathcal{A}_\lambda$. Write $r_k = r_k^{(q)} \cdot c$, where $r_k^{(q)}$ is the truncation modulo $q$-fold commutators and $c \in \mathbb{F}(n)_q$. Then we have that $\big(\bar{\lambda}\phi(\partial_i r_k)\big)^\beta = \big(\bar{\lambda}\phi(\partial_i r_k^{(q)})\big)^\beta + \big(\bar{\lambda}\phi(\partial_i c)\big)^\beta$, since $r_k^{(q)} \in \mathbb{F}(n)_2$, as $G$ is 2-free. If $q \geq p+1$, by the discussion on $D$ above we have, after Lemma 4.8 is applied, that $\big(\bar{\lambda}\phi(\partial_i r_k)\big)^\beta \equiv \big(\bar{\lambda}\phi(\partial_i r_k^{(q)})\big)^\beta \mod p$. The statement follows from this by comparing the ranks $\mod p$ of the integral matrices $A_\lambda$ and $A_{\lambda,q}$.

In the case $q \leq p$, the $q$-fold commutators do not contribute to $A_\lambda$, after the row and columns operations of Lemma 4.8, a diagonal matrix $D$ that has all the entries zero $\mod p$ , and thus we can only infer an inequality.□

The previous theorem says that one can recover the $\mathbb{Z}_p$-rank of the $p$-torsion in abelianizations of index $p$ normal subgroups of a 2-free group $G$ from the $p$-torsion in the corresponding index $p$ normal subgroups of a nilpotent quotient $G/G_q$, for large enough $q$, provided we know the rank of the free part of the abelianization. The latter rank can be determined by counting $p$-torsion points on the characteristic varieties of $G$ over $\mathbb{C}$, see [13, 28]. Moreover, is it possible sometimes to detect the presence of $p$-torsion in the homology of index $p$ normal subgroups of a 2-free group $G$, just by considering nilpotent quotients $G/G_q$ for low values of $q$.

**4.23.** We turn now to the secondary resonance varieties of a 2-free group $G$. Let $H_1(G) = \mathbb{Z}^n$ and $H_2(G) = \mathbb{Z}^m$. We show that $\mathcal{Q}_d(G, \mathbb{Z}_p)$ are the determinantal varieties of a matrix which is build up from the linear and quadratic truncations of the Alexander matrix of $G$. Let $S = S(\lambda)$ be the $2m \times 2n$ matrix with entries in the polynomial ring $\mathbb{Z}_p[\lambda_1, \ldots, \lambda_n]$,

having the following block upper diagonal form:

$$S = \begin{pmatrix} T^{(1)} & 2 \cdot T^{(2)} + U^{(1)} \\ 0 & T^{(1)} \end{pmatrix}, \tag{15}$$

where $T^{(1)} = T^{(1)}(\lambda)$ and $T^{(2)} = T^{(2)}(\lambda)$ are the corresponding truncations of the Alexander matrix of $G$ and $U^{(1)} = U^{(1)}(\lambda_1, \ldots, \lambda_n)$ is $T^{(1)}(\lambda_1^2, \ldots, \lambda_n^2)$. Recall that $T^{(1)}(\lambda)$ and $T^{(2)}(\lambda)$ are the $m \times n$ matrices defined by:

$$T^{(1)}_{i,j} = \sum_{l=1}^{n} \varepsilon_{l,j}(r_i)\lambda_l,$$

and respectively:

$$T^{(2)}_{i,j} = \sum_{1 \le l_1, l_2 \le n} \varepsilon_{l_1,l_2,j}(r_i)\lambda_{l_1}\lambda_{l_2}.$$

**Theorem 4.24.** *For $G$ a 2-free group with $H_2(G)$ torsion free, $p \geq 3$ and $0 \leq d \leq 2n$*

$$\mathcal{Q}_d(G, \mathbb{Z}_p) = \{\lambda \in \mathbb{Z}_p^n \mid \operatorname{rank}_{\mathbb{Z}_p} S(\lambda) < 2n - d\}.$$

**Proof.** We follow once again the steps of Fox's algorithm that gives a presentation matrix for the abelianization $H_1(K_\lambda)$ of a index $p$ subgroup in $G/G_4$ given by $\lambda \in \mathbb{Z}_p^n$. First we split the truncation modulo 4-fold commutators of the relators $r_j$ as $r_j^{(4)} = r_j^{(3)} \cdot w_j$. We then have that $\left(\bar{\lambda}\phi(\partial_i r_j^{(4)})\right)^\beta = \left(\bar{\lambda}\phi(\partial_i r_j^{(3)})\right)^\beta + \left(\bar{\lambda}\phi(\partial_i w_j)\right)^\beta$. We have already seen in Theorem 4.16 and Theorem 4.20 that, after row and columns operations, the truncations $r_j^{(3)}$, and $r_j^{(4)}$ when $r_j^{(3)} = 0$, yield the truncated Alexander matrices $T^{(1)}$ and $T^{(2)}$ respectively. Similarly it is seen that the remainders $w_j$ of the 3-truncations of $r_j^{(4)}$ will produce $T^{(2)}$. We will show that the $r_j^{(3)}$ give the rest of the entries in $S$.

In order to see that we get back to the proof of Theorem 4.16, to determine what happens with some of the entries we neglected there while apllying the row and column operations of Lemma 4.8. Consider $e = \sum_{\sigma=1}^{p-1} a_\sigma(\zeta^\sigma - 1) \in \mathbb{Z}\mathbb{Z}_p$. Set $a = \sum_{\sigma=1}^{p-1} a_\sigma$. Then, after we add all the rows and all the columns to the last row, and respectively to the last column, and remove the row and column of zeroes obtained this way, the

matrix $e^\beta$ writes as:

$$e^\beta = \begin{pmatrix} -a & a_1 & \cdots & a_{p-3} & a_{p-2} \\ a_{p-1} & -a & \cdots & a_{p-4} & a_{p-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_3 & a_4 & \cdots & -a & a_1 \\ a_2 & a_3 & \cdots & a_{p-1} & -a \end{pmatrix}.$$

For this matrix we subtract the first row from each of the others, and then, for $2 \leq q \leq p-2$, add the $q^{th}$ column, muliplied by $q$, to the first column. That produces $u = a_1 + 2a_2 + \cdots + (p-1)a_{p-1}$ in the top entry of the first column and $p \cdot a$ in the other entries. Then, for $3 \leq q \leq p-2$, we add the $q^{th}$ column, muliplied by $\binom{q}{2}$, to the first column. We thus get $v = a_1 + \binom{3}{2}a_2 + \cdots + \binom{p-1}{2}a_{p-2}$ in the top first entry of the second column, $-a - a_1 + \binom{3}{2}(a_1 - a_2) + \cdots + \binom{p-1}{2}(a_{p-3} - a_{p-2})$ in the second entry, $a_{p-1} - a_1 + \binom{3}{2}(-a - a_2) + \binom{4}{2}(a_1 - a_3) + \cdots + \binom{p-1}{2}(a_{p-4} - a_{p-2})$ in the third, and so on. After simplifications we obtain in the $q^{th}$ entry of the second column, $(q-1) \cdot u$, where $2 \leq q \leq p-1$. Next we subtract the second row, multiplied by $q$, from the $q^{th}$ row, for $3 \leq q \leq p-1$, to get the following matrix:

$$\begin{pmatrix} u & v & a_2 & \dots & a_{p-2} \\ p \cdot a & u & a_1 - a_2 & \cdots & a_{p-3} - a_{p-2} \\ 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & * & \cdots & * \end{pmatrix}.$$

Now, as in the proof of Theorem 4.16, $u = a_1 + 2a_2 + \cdots + (p-1)a_{p-1}$ leads to the matrix $T^{(1)}$. From $v - \frac{1}{2}u = \frac{1}{2}(1^2a_1 + \cdots + (p-1)^2a_{p-1})$, it is easily seen that $v$ leads to the matrix $\frac{1}{2}U^{(1)}$. Finally notice that Proposition 4.9 implies that for $q = 4$ and $p \geq 3$ the 4-fold commutators yield the diagonal matrix $D = (p, p, 1, \dots, 1, 0, \dots, 0)$. Combining that with the form we arrived at for the part of $A_\lambda$ coming from the relators, we see that the statement follows. □

**5. Links.** In this section, we investigate the invariants considered above, in the case of groups of pure links in the 3-sphere, by relating the

resonance varieties with the classical higher linking invariants of Milnor. Similar results were obtained by Hillman, Matei and Morishita in [12] by $p$-adic methods.

**5.1. Milnor numbers and resonance varieties.** The general knot theory facts we are going to use are taken from [1, 3, 23]. A link in $\mathcal{S}^3$ is a union of embedded circles $L = \{L_1, \dots, L_n\}$. Let $X = \mathcal{S}^3 \setminus L$ be the complement of a link $L$ and $G = \pi_1(X)$ its fundamental group. The link group $G$ is in general presented by $m$ generators and $m-1$ relators, for some $m \geq n$, but $G$ is normally generated by only $n$ generators. A link is called non-split if there is no 2-sphere separating the link into proper, non-empty sublinks. For such links, $X$ is a $K(G,1)$, and $G$ has the homology and cohomology groups of $X$. If the link $L$ splits into $s$ unsplittable sublinks then $H_1(G) = H^1(G) = \mathbb{Z}^n$ and $H_2(G) = H^2(G) = \mathbb{Z}^{n-s}$.

We are going to use here the so-called Artin presentation of a link group, see Birman [1]. By a theorem of Alexander, any link is isotopic to a closed-up braid. If $L = \hat{\beta}$ is the closure, denoted by $\hat{}$, of $\beta \in B_m$ then $G$ is presented by:

$$G = \langle x_1, \dots, x_m \mid \beta(x_1) = x_1, \dots, \beta(x_m) = x_m \rangle,$$

where $B_m$ acts on the free group $\mathbb{F}(x_1, \dots, x_m)$ by the Artin representation. Let us remark that because any braid $\beta \in B_m$ leaves $x_1 \cdots x_m$ invariant, one of the above relations in the presentation of $G$ is the consequence of the others.

**5.2.** We restrict ourselves to a special class of links, namely the close-up pure braids.

**Definition 5.3.** We call $L = \hat{\beta}$ *pure link* if $\beta$ is a pure braid.

Notice first that a pure link $L = \hat{\beta}$, with $\beta \in P_n$, has $n$ components. The action of $\beta$ on $x_i$ can we written as $\beta(x_i) = l_i x_i l_i^{-1}$, for some word $l_i$ in the free group $\mathbb{F} = \mathbb{F}(x_1, \dots, x_n)$, which will be referred to as the *longitude* of the component $L_i$, see [23] for details. Thus, the group $G$ of a pure link has a presentation of the form:

$$G = \langle x_1, \dots, x_n \mid [l_1, x_1], \dots, [l_n, x_n] \rangle. \tag{16}$$

It follows from this that the group of a pure link is 2-free.

If $\{c_h\}_{1\leq h\leq l}$ are the basic commutators of length $k$, the nilpotent quotient $G/G_k$ may be presented as follows:

$$G/G_k = \langle x_1, \ldots, x_n \mid [l_1^{(k)}, x_1], \ldots, [l_n^{(k)}, x_n], c_1, \ldots, c_l \rangle. \tag{17}$$

**5.4.** We now give a necessary and sufficient condition for a pure link to have a $k$-free group.

**Proposition 5.5.** *Let $k \geq 2$ be an integer and $L = \beta$ be a pure link, $\beta \in P_n$. Then the group $G$ of $L$ is $k$-free if and only if $\beta \in \Gamma_{k-1}P_n$, the $(k-1)^{st}$ lower central series term of $P_n$.*

**Proof.** Suppose $\beta \in \Gamma_{k-1}P_n$. Then, in Lemma 2.2. of [26] it is shown that the longitudes $l_i$ of $\beta$ belong to the $(k-1)^{st}$ lower central series term of $\mathbb{F}_n$. Which in turn implies that the relators $[l_i, x_i]$ are in $\mathbb{F}_k$, thus $G/G_k \cong \mathbb{F}/\mathbb{F}_k$, and $G$ is $k$-free.

Suppose $G$ is $k$-free. Then the presentation (16) is such that $[l_i, x_i]$ are in $\mathbb{F}_k$. From Lemma 17 of [25] it follows that $l_i \in \mathbb{F}_{k-1}$. Recall now from Birman [1] that every $\beta \in P_n$ can be written uniquely in the form $\beta = \beta_2\beta_3 \cdots \beta_n$, where each $\beta_j$ is a pure braid which belongs to the rank $j-1$ free subgroup $U_j$ of $P_n$ generated by $\langle A_{ij} \mid 1 \leq i < j \rangle$. Let $\lambda_j : \mathbb{F}(n) \to \mathbb{F}(j)$ be the homorphism defined by $\lambda_j(x_i) = x_i$ if $i \leq j$ and by $\lambda_j(x_i) = 1$ if $i > j$. It follows from the discussion in [1] that if $\lambda_{j-1}(l_j)$ is given in $\mathbb{F}(j-1)$ by a word $w(x_1, \ldots, x_{j-1})$ then $\beta_j$ is given in $U_{j-1}$ precisely by $w(A_{1j}, \ldots, A_{j-1,j})$. But $l_j \in \mathbb{F}_{k-1}$ implies $\beta_j \in \Gamma_{k-1}U_j$, and so $\beta \in \Gamma_{k-1}P_n$. □

We thus see that there exist $k$-free link groups $G$ with $b_1(G) = n$ for any given $k \geq 2$ and $n \geq 2$.

**5.6.** Next we recall the definition of the Milnor $\bar{\mu}$-invariants of a link $L$. For simplicity we restrict to the case of a pure link, which will be all we need here. Let $L$ be a pure link of $n$ components.

**Definition 5.7.** Let $(i_1, \ldots, i_s, j)$ be a sequence of integers between 1 and $n$. The *Milnor number* $\mu_L(i_1, \ldots, i_s, j)$ is the integer equal to the Magnus coefficient $\varepsilon_{i_1,\ldots,i_s}(l_j)$ of the $j^{\text{th}}$ longitude $l_j$ of $L$. Let $\Delta(i_1, \ldots, i_r)$ be the greatest common divisor of $\mu(j_1, \ldots, j_s)$ over all cyclic permutations $(j_1, \ldots, j_s)$ of subsequences of $(i_1, \ldots, i_r)$, $2 \leq s < r$. The *Milnor invariant* $\bar{\mu}_L(i_1, \ldots, i_r)$ is the residue class of $\mu(i_1, \ldots, i_r)$ modulo $\Delta(i_1, \ldots, i_r)$.

MILNOR, [22] shows that $\bar{\mu}(i_1, \ldots, i_r)$ are link invariants. The numbers $\bar{\mu}(i,j) = \mu(i,j)$ are equal to the linking numbers $\mathrm{lk}(L_i, L_j)$. If all Milnor numbers of length $< r$ are zero then $\bar{\mu}(i_1, \ldots, i_r)$ is the integer $\mu(i_1, \ldots, i_r)$ defined with no indeterminacy.

It is readily seen from Proposition 5.5, Proposition 2.6, and the definitions that all Milnor numbers of length $< k$ are zero if the group of the pure link $L$ is $k$-free.

**5.8.** Now assume that $G$ is $k$-free and satisfies condition (B). It follows from the discussion in (5.1) that $L$ has to be non-split.

Let $\{e_1, \ldots, e_n\}$ be the basis for $H^1(X)$, dual to the meridian basis $\{t_1, \ldots, t_n\}$ of $H_1(X)$. The relators of the group $G$ determine natural generators for $H^2(X)$. Let $r_i = [l_i, x_i] \bmod [R, \mathbb{F}]$. Then $\{r_1, \ldots, r_n\}$ generate $H_2(X)$, and their duals, $\{\gamma_1, \ldots, \gamma_n\}$, generate $H^2(X)$. We have: $r_1 + \cdots + r_n = 0$ in $H_2(X)$, and $\gamma_1 + \cdots + \gamma_n = 0$ in $H^2(X)$. A natural geometric generating set for $H^2(X)$ is given by $\gamma_{ij}$, the Lefschetz dual to an arc in $X$ connecting $L_i$ with $L_j$. We have that $\gamma_{ij} = \gamma_i - \gamma_j$. By the Alexander-Lefschetz duality the cup product $e_i \cup e_j$ is equal to $\mu(ij)\gamma_{ij}$.

Let $R$ be a commutative ring with unity. It is readily seen that the facts of the previous paragraph hold also for the homology and cohomology with $R$-coefficients. We keep the same notations for the various bases in that case.

A presentation for the cohomology ring $H^2(X, R)$ is given by:

$$H^2(X) = \left( e_i, \gamma_j \;\middle|\; e_i e_j = \mu(ij)(\gamma_i - \gamma_j),\ e_i\gamma_j = \gamma_j\gamma_l = 0 \right). \tag{18}$$

Now, since $G$ is $k$-free, it follows from Section 2, that Massey products in $H^2(X, R)$ of length $< k$ vanish and that the $k$-fold Massey products are defined with zero indeterminacy, and they can all be computed from the products $\langle e_{i_1}, e_{i_2}, \ldots, e_{i_k} \rangle$. In [27] and [32] certain Massey products of $X$ were shown to be related with the Milnor $\bar{\mu}$-invariants. In the particular case here, the relationship becomes:

$$\langle e_{i_1}, e_{i_2}, \ldots, e_{i_k} \rangle = \bar{\mu}(i_1, i_2, \ldots, i_k)(\gamma_{i_1} - \gamma_{i_k}), \tag{19}$$

where the equality holds in $H^2(X, R)$.

**5.9.** In Section 3, we showed that the resonance varieties of a 2-complex $X$ with $k$-free fundamental group $G$ are the determinantal varieties of the $k$-truncated Alexander matrix of $G$. We shall see now that, in the case when $X$ is a complement of a link $L$, these truncations are related with certain matrices written in terms of the Milnor invariants of $L$, which were first considered by Traldi in [29, 30].

**Proposition 5.10.** *Let $\mathbb{K}$ be a field. Let $L$ be a $n$-component link, such that its group is $k$-free and satisfies condition $(C_k)$. Then the resonance variety, $R_d(X, \mathbb{K})$, of $X = \mathcal{S}^3 \setminus L$ is given by the vanishing locus of the $(n-d) \times (n-d)$ minors of the matrix $T^{(k-1)}(L)$ over the polynomial ring $\mathbb{K}[\lambda_1, \dots, \lambda_n]$, with diagonal entries:*

$$T^{(k-1)}_{i,i} = -\sum \bar{\mu}_L(i_1, \dots, i_{k-1}, i)\, \lambda_{i_1} \cdots \lambda_{i_{k-1}},$$

*where the summation is taken over all $1 \leq i_1, \dots, i_{k-1} \leq n$ such that $i_{k-1} \neq i$, and other entries given by:*

$$T^{(k-1)}_{i,j} = \sum \bar{\mu}_L(i_1, \dots, i_{k-2}, j, i)\, \lambda_i\, \lambda_{i_1} \cdots \lambda_{i_{k-2}},$$

*where the summation is taken now over all $1 \leq i_1, \dots, i_{k-2} \leq n$.*

**Proof.** The statement follows from the definitions and equation (19) relating the Massey products with the Milnor invariants, exactly like in Theorem 3.14. □

In the particular case of a 2-free group $G$, the above equations take the following simple form:

$$T^{(1)}_{i,j} = \mu(i,j)\lambda_i - \delta_{i,j}\Big(\sum_{k=1}^{n} \mu(k,j)\lambda_k\Big). \tag{20}$$

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