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MATEMATICĂ, Tomul LIII, 2007, f.2

A GOULD TYPE INTEGRAL WITH RESPECT TO A SUBMEASURE

BY

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Abstract. In this paper we introduce a Gould type integral for a bounded, real valued function with respect to a real submeasure of finite variation. Several natural properties of the integral are obtained, concerning its behavior with respect to the function, respectively, to the submeasure. Also, a characterization of the integrability with respect to a submeasure is proved using the integrability in the sense of GOULD [7] with respect to the variation of the submeasure.

Mathematics Subject Classification 2000: 28B20, 28C15, 49J53.
Key words: Gould type integral, submeasure, multisubmeasure.

1. Introduction. In [7], GOULD has introduced and studied a special type of integral for a real valued function with respect to a finitely additive set function taking values in a Banach space. In [9] and [10], PRECUPANU and CROITORU generalized his results for additive multimeasures with values in the family of all nonvoid, compact, convex sets of a Banach space. Following these two works, GAVRILUŢ has defined and studied in [3], [4], [5] and [6] a Gould type integral for a bounded, real valued function with respect to a multisubmeasure of finite variation.

In this paper we introduce and study a Gould type integral for a bounded, real valued function with respect to a submeasure of finite variation.

In the following, we shall use the terminology from [9].

Let T be an abstract, nonvoid set, $\mathcal{A}$ an algebra of subsets of T, $\mathcal{P}$ the set of all partitions of T with sets of $\mathcal{A}$ and $f : T \to \mathbb{R}$ a function. If $A \in \mathcal{A}$, we denote by $\mathcal{P}_A$ the set of all partitions of A.

Definition 1.1. If $P = \{A_i\}_{i=\overline{1,n}}$ and $P' = \{B_j\}_{j=\overline{1,m}}$ are two partitions of T, then P' is said to be *finer than* P (denoted by $P \leq P'$) if for every $j = \overline{1,m}$, there exists $i_j = \overline{1,n}$ so that $B_j \subseteq A_{i_j}$.

If $P = \{A_i\}_{i=\overline{1,n}}$ and $P' = \{B_j\}_{j=\overline{1,m}}$ are two partitions of T, by $P \wedge P'$ we mean the partition $\{A_i \cap B_j\}_{\substack{i=\overline{1,n} \\ j=\overline{1,m}}}$.

Definition 1.2. A set function $m : \mathcal{A} \mapsto \mathbb{R}$ is said to be *absolutely continuous* with respect to another set function $\nu : \mathcal{A} \mapsto \mathbb{R}$, denoted by $m \ll \nu$, if $\nu(A) = 0$ implies $m(A) = 0$, for every $A \in \mathcal{A}$.

All over this paper we assume that $m : \mathcal{A} \mapsto \mathbb{R}_+$ is a submeasure in Drewnowski's sense [2]. Let us also consider the following set functions associated to m:

$\overline{m}$ defined by

$$\overline{m}(A) = \sup\{\sum_{i=1}^{n} m(A_i)\}, \text{ for every } A \in \mathcal{A}, \tag{1}$$

where the supremum is extended over all finite partitions $\{A_i\}_{i=\overline{1,n}}$ of A, called *the variation* of m; and $\widetilde{m}$ defined by

$$\widetilde{m}(A) = \inf\{\overline{m}(B); A \subseteq B, B \in \mathcal{A}\}, \text{ for every } A \subseteq T. \tag{2}$$

Remark 1.3. $\widetilde{m}(A) = \overline{m}(A)$, for every $A \in \mathcal{A}$ and $\overline{m}$ is a finitely additive set function on $\mathcal{A}$.

Definition 1.4. We say that *a property* (P) *holds* m*-almost everywhere* if the property (P) is valid on $T \backslash A$, where $\widetilde{m}(A) = 0$.

Definition 1.5. A submeasure $m : \mathcal{A} \mapsto \mathbb{R}_+$ is said to be *of finite variation* if $\overline{m}(T) < \infty$.

2. The Gould type integral with respect to a submeasure. In the sequel, without any special assumptions, T will be an abstract nonvoid set, $m : \mathcal{A} \mapsto \mathbb{R}_+$ a submeasure of finite variation and $f : T \to \mathbb{R}$ a real valued, *bounded* function.

Definition 2.1. I) f is said to be $\overline{m}$*-totally-measurable on* $(T, \mathcal{A}, m)$ if for every $\varepsilon > 0$ there exists a partition $P_\varepsilon = \{A_i\}_{i=\overline{0,n}}$ of T such that:

$$\begin{cases} i)\ \overline{m}(A_0) < \varepsilon \text{ and} \\ ii)\ \sup\limits_{t,s\in A_i} |f(t) - f(s)| = osc(f, A_i) < \varepsilon, \text{ for every } i = \overline{1,n}. \end{cases}$$

II) f is said to be $\overline{m}-$ *totally-measurable on* $B \in \mathcal{A}$ if the restriction $f|_B$ of f to B is $\overline{m}$-totally measurable on $(B, \mathcal{A}_B, m_B)$, where $\mathcal{A}_B = \{A \cap B; A \in \mathcal{A}\}$ and $m_B = m|_{\mathcal{A}_B}$.

Remark 2.2. In [7], GOULD has considered for every finitely additive measure $\nu : \mathcal{A} \mapsto X$, where $(X, \|\ \|)$ is a Banach space, the set function $\widetilde{\nu}$, defined by

$$\widetilde{\nu}(A) = \sup\{\|\nu(B)\|; B \in \mathcal{A}, B \subset A\}.$$

It is easy to observe that $\widetilde{\overline{m}}(A) = \overline{m}(A)$,for every $A \in \mathcal{A}$. Consequently, the $\overline{m}$-totally measurability in our sense coincides with the $\overline{m}$-totally measurability in the sense of GOULD [7].

If $f : T \to \mathbb{R}$ is a real valued, bounded function, $m : \mathcal{A} \mapsto \mathbb{R}_+$ a submeasure of finite variation, $P = \{A_i\}_{i=\overline{1,n}}$ a partition of T and $t_i \in A_i, i = \overline{1,n}$ are arbitrarily, we denote by

$$\sigma(P, f, m) = \sum_{i=1}^{n} f(t_i) m(A_i). \tag{3}$$

If there is no confusion, we shortly denote it by $\sigma(P)$.

If $B \in \mathcal{A}$, let

$$\sigma_B(P) = \sum_{j=1}^{k} f(s_j) m(B_j), \tag{4}$$

for every partition $P = \{B_j\}_{j=\overline{1,k}}$ of B and every $s_j \in B_j, j = \overline{1,k}$.

Definition 2.3. I) f is said to be *m-integrable on* T if there exists $I \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists a partition P_ε of T so that for every partition $P = \{A_i\}_{i=\overline{1,n}}$,with $P \geq P_\varepsilon$ and every choice of points $t_i \in A_i, i = \overline{1,n}$, we have

$$|\sigma(P) - I| < \varepsilon, \tag{5}$$

that is, the net $(\sigma(P))_{P\in(\mathcal{P},\leq)}$ is convergent in $\mathbb{R}$.

In this case, I is called *the integral of f on T with respect to the submeasure m*, denoted by $\int_T f dm$.

II) f is said to be *m-integrable on $B \in \mathcal{A}$* if the restriction $f|_B$ of f to B is *m-integrable* on $(B, \mathcal{A}_B, m_B)$.

In the following we shall indicate a class of functions which are m-integrable.

Theorem 2.4. *Let $m : \mathcal{A} \to \mathbb{R}_+$ be a submeasure of finite variation and $f : T \to \mathbb{R}$ a $\overline{m}$-totally measurable function on T. Then f is m-integrable on T and, moreover,*

$$\int_T f\, dm = \int_T f d\overline{m}, \tag{6}$$

where $\int_T f d\overline{m}$ is the integral in the sense of Gould [7] *of f with respect to the variation $\overline{m}$ of m.*

Proof. Let $\varepsilon > 0$ and $M = \sup_{t\in T} |f(t)|$.

Because f is $\overline{m}$-totally-measurable on T, there exists a partition $P'_\varepsilon = \{A_0, A'_1, A'_2, \ldots, A'_v\}$ of T such that

$$\overline{m}(A_0) < \frac{\varepsilon}{8M} \text{ and } \sup_{t,s\in A'_l} |f(t) - f(s)| < \frac{\varepsilon}{8\overline{m}(T)}, \text{ for every } l = \overline{1, v}.$$

By the definition of $\overline{m}$ we get that there exists a partition $P''_\varepsilon = \{A''_1, A''_2, \ldots, A''_s\}$ of $T\backslash A_0$ such that

$$\overline{m}(T\backslash A_0) < \sum_{r=1}^{s} m(A''_r) + \frac{\varepsilon}{8M}. \tag{7}$$

Since $A''_r = A''_r \cap (T\backslash A_0)$, we obtain that $A''_r = \bigcup_{l=1}^{v}(A''_r \cap A'_l)$, hence $m(A''_r) \leq \sum_{l=1}^{v} m(A''_r \cap A'_l)$, for every $r = \overline{1, s}$.

From (7) it follows that

$$\overline{m}(T\backslash A_0) < \sum_{r=1}^{s}\sum_{l=1}^{v} m(A''_r \cap A'_l) + \frac{\varepsilon}{8M}.$$

We observe that the family $\{A''_r \cap A'_l\}_{\substack{r=\overline{1,s}\\ l=\overline{1,v}}}$ is a partition of $T\backslash A_0$. Let us rewrite it as $\{A_i\}_{i=\overline{1,n}}$.

Then $P_\varepsilon = \{A_0, A_1, \ldots, A_n\}$ is a partition of T and

$$(8)\qquad \begin{aligned} &\overline{m}(A_0) < \frac{\varepsilon}{8M}, \overline{m}(T\backslash A_0) < \sum_{i=1}^{n} m(A_i) + \frac{\varepsilon}{8M} \\ &\text{and } \sup_{t,s\in A_i} |f(t) - f(s)| < \frac{\varepsilon}{8\overline{m}(T)}, \text{ for every } i = \overline{1,n}. \end{aligned}$$

Now, we observe that f is $\overline{m}$-integrable on T. Indeed, since f is bounded, then, according to theorem 4.7 from [7] and remark 2.2, we get that f is $\overline{m}$-integrable on T (in the sense of Gould [7]).

We now prove that f is m-integrable on T and the equality (6). Because f is $\overline{m}$-integrable on T, then for every $\varepsilon > 0$, there exists a partition $\widetilde{P}_\varepsilon \in \mathcal{P}$ such that

$$|\sigma(P, f, \overline{m}) - \int_T f d\overline{m}| < \frac{\varepsilon}{2}, \text{ for every } P \in \mathcal{P}, \text{ with } P \geq \widetilde{P}_\varepsilon.$$

We consider the partition $\widetilde{\widetilde{P}}_\varepsilon = \widetilde{P}_\varepsilon \wedge P_\varepsilon$; let $P = \{B_j\}_{j=\overline{1,q}}$ be so that $P \geq \widetilde{\widetilde{P}}_\varepsilon$ and $t_j \in B_j$, $j = \overline{1,q}$ arbitrarily chosen. We observe that $P \geq \widetilde{P}_\varepsilon$ and $P \geq P_\varepsilon$, so,

$$(9)\qquad |\sum_{j=1}^{q} f(t_j)\overline{m}(B_j) - \int_T f d\overline{m}| < \frac{\varepsilon}{2}.$$

Using (8) we get that

$$\begin{aligned} &|\sum_{j=1}^{q} f(t_j)\overline{m}(B_j) - \sum_{j=1}^{q} f(t_j)m(B_j)| = |\sum_{j=1}^{q} f(t_j)(\overline{m}(B_j) - m(B_j))| \\ &\leq M\sum_{j=1}^{q}[\overline{m}(B_j) - m(B_j)] = M[\overline{m}(T) - \sum_{j=1}^{q} m(B_j)] \\ &= M[\overline{m}(T\backslash A_0) + \overline{m}(A_0) - \sum_{j=1}^{q} m(B_j)] \\ &< \frac{\varepsilon}{8} + M[\overline{m}(T\backslash A_0) - \sum_{j=1}^{q} m(B_j)] < \frac{\varepsilon}{8} + \frac{\varepsilon}{8} \\ &+M[\sum_{i=1}^{n} m(A_i) - \sum_{j=1}^{q} m(B_j)] \leq \frac{\varepsilon}{4} + M[\sum_{i=0}^{n} m(A_i) - \sum_{j=1}^{q} m(B_j)]. \end{aligned}$$

Since $(A_i)_{i=\overline{0,n}}$ are pairwise disjoint and $P \geq P_\varepsilon$, then for every $j = \overline{1,q}$, there exists an unique $i_j = \overline{0,n}$ so that $B_j \subseteq A_{i_j}$. Consequently, $m(B_j \cap A_{i_j}) = m(B_j)$. Now, supose that there exists $i_0 = \overline{0,n}$, $i_0 \neq i_j$ so that $B_j \cap A_{i_0} \neq \emptyset$. Then there is $a \in B_j \cap A_{i_0}$; hence, $a \in A_{i_j}$ and $a \in A_{i_0}$, a contradiction. Consequently, $B_j \cap A_i = \emptyset$, for every $i \neq i_j$, hence $m(B_j \cap A_i) = 0$, for every $i \neq i_j$. Therefore,

$$m(B_j) = \sum_{i=0}^{n} m(B_j \cap A_i), \text{ for every } j = \overline{1,q};$$

hence

$$|\sum_{j=1}^{q} f(t_j)\overline{m}(B_j) - \sum_{j=1}^{q} f(t_j)m(B_j)|$$
$$< \frac{\varepsilon}{4} + M[\sum_{i=0}^{n} m(A_i) - \sum_{j=1}^{q}\sum_{i=0}^{n} m(B_j \cap A_i)].$$

Because m is a submeasure on $\mathcal{A}$, we have $m(A_i) \leq \sum_{j=1}^{q} m(A_i \cap B_j)$, for every $i = \overline{0,n}$, which implies that $\sum_{i=0}^{n} m(A_i) - \sum_{j=1}^{q}\sum_{i=0}^{n} m(B_j \cap A_i) \leq 0$ and, consequently,

$$|\sum_{j=1}^{q} f(t_j)\overline{m}(B_j) - \sum_{j=1}^{q} f(t_j)m(B_j)| < \frac{\varepsilon}{4}. \tag{10}$$

From (9) and (10), we get that

$$|\sum_{j=1}^{q} f(t_j)m(B_j) - \int_T f d\overline{m}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} < \varepsilon,$$

which means that f is m-integrable on T and $\int_T f dm = \int_T f d\overline{m}$.

Remark 2.5. I) If $L = \inf_{t\in T} f(t) > 0$ and $\int_T f dm = 0$, then $m(T) = 0$. Indeed, as f is m-integrable on T and $\int_T f dm = 0$, then for every $\varepsilon > 0$, there exists $P_\varepsilon = \{A_i\}_{i=\overline{1,n}} \in \mathcal{P}$ so that $|\int_T f dm - \sum_{i=1}^{n} f(t_i)m(A_i)| < \varepsilon$, for every $t_i \in A_i, i = \overline{1,n}$. Then

$$\sum_{i=1}^{n} f(t_i)m(A_i) \leq |\int_T f dm - \sum_{i=1}^{n} f(t_i)m(A_i)| + |\int_T f dm| < \varepsilon.$$

On the other hand, $Lm(T) \leq \sum_{i=1}^{n} f(t_i)m(A_i)$, so $m(T) < \frac{\varepsilon}{L}$, for every $\varepsilon > 0$; this implies $m(T) = 0$, as claimed.

II) If $f = k$ on T, where k is a real constant, then f is m-integrable on T and

$$\int_T f dm = \int_T k d\overline{m} = k\overline{m}(T).$$

We get now the following:

Theorem 2.6. *If $f = 0$ m-almost everywhere, then f is $\overline{m}$ -totally-measurable, m-integrable on T and $\int_T f dm = 0$.*

Proof. We observe that f is $\overline{m}$-totally-measurable on T. Indeed, because $f = 0$ m-almost everywhere, there is a set $A \subset T$ so that $f = 0$ on $T \backslash A$ and $\widetilde{m}(A) = 0$. By the definition of $\widetilde{m}$, for every $\varepsilon > 0$, there exists $B_\varepsilon \in \mathcal{A}$ such that $A \subset B_\varepsilon$ and $\overline{m}(B_\varepsilon) < \varepsilon$. We consider the partition $\{B_\varepsilon, cB_\varepsilon\}$. Since $\overline{m}(B_\varepsilon) < \varepsilon$ and $f = 0$ on cB_ε, then f is $\overline{m}$-totally-measurable on T.

Further, f is m-integrable on T from theorem 2.4 and $\int_T f dm = \int_T f d\overline{m}$.

On the other hand, because $\overline{m}$ is finitely additive, then, according to theorem 2.8 [9] for the special case when the multimeasure is real valued, we obtain that $\int_T f d\overline{m} = 0$. So, f is m-integrable on T and $\int_T f dm = 0$.

Using the definition 2.3 of m-integrability, we easily get:

Theorem 2.7. *Suppose that $m : \mathcal{A} \mapsto \mathbb{R}_+$ is a submeasure of finite variation and $f, g : T \mapsto \mathbb{R}$ are two m-integrable functions on T.*

i) *If $\alpha \in \mathbb{R}$, then αf is m-integrable on T and*

$$\int_T \alpha f dm = \alpha \int_T f dm; \tag{11}$$

ii) *If $\alpha \in \mathbb{R}_+$, then f is αm-integrable on T and*

$$\int_T f d(\alpha m) = \alpha \int_T f dm; \tag{12}$$

iii)

$$|\int_T f dm| \leq \sup_{t \in T} |f(t)| \ \overline{m}(T); \tag{13}$$

iv) *$f+g$ is m-integrable on T and*

$$\int_T (f+g)dm = \int_T fdm + \int_T gdm; \tag{14}$$

v) *If $f \leq g$ on T,then*

$$\int_T fdm \leq \int_T gdm; \tag{15}$$

vi) *If $f : T \mapsto \mathbb{R}_+$, then $\int_T fdm \geq 0$.*

Remark 2.8. We note that, if $\overline{m}(A) = 0$, where $A \in \mathcal{A}$, then every function $f : T \to \mathbb{R}$ is m-integrable on A and $\int_A fdm = 0$. As a consequence, in theorem 2.7 v) we may replace the condition "$f \leq g$ on T" by "$f \leq g$ on $T\backslash A$, with $A \in \mathcal{A}$ and $\overline{m}(A) = 0$".

Further, we prove the additivity with respect to the submeasure:

Theorem 2.9. *Let $m_1, m_2 : \mathcal{A} \mapsto \mathbb{R}_+$ be two submeasures of finite variation and $f : T \mapsto \mathbb{R}$ a m_1-integrable and m_2-integrable function on T. Then*

i) *f is m-integrable on T with respect to the submeasure $m : \mathcal{A} \mapsto \mathbb{R}_+$, $m = m_1 + m_2$ and*

$$\int_T fd(m_1 + m_2) = \int_T fdm_1 + \int_T fdm_2; \tag{16}$$

ii) *If $f : T \mapsto \mathbb{R}_+$ and $m_1(A) \leq m_2(A)$, for every $A \in \mathcal{A}$, then*

$$\int_T fdm_1 \leq \int_T fdm_2. \tag{17}$$

Proof. i) We easily check that m is a submeasure of finite variation. Let $\varepsilon > 0$ and consider $P_1 \wedge P_2 \in \mathcal{P}$, where P_1 and P_2 are the partitions of T from the definition of m_1-integrability and m_2-integrability of f. Then

$$|\int_T fdm_1 - \sigma(P, f, m_1)| < \frac{\varepsilon}{2} \text{ and } |\int_T fdm_2 - \sigma(P, f, m_2)| < \frac{\varepsilon}{2},$$

for every $P \in \mathcal{P}$, with $P \geq P_1 \wedge P_2$.

Let $P = \{C_k\}_{k=\overline{1,p}} \in \mathcal{P}, P \geq P_1 \wedge P_2$ and $\theta_k \in C_k, k = \overline{1,p}$ be arbitrarily chosen. Since $P \geq P_1$ and $P \geq P_2$ we get

$$\begin{aligned}
&|\sum_{k=1}^{p} f(\theta_k)m(C_k) - (\int_T f dm_1 + \int_T f dm_2)| \\
&= |\sum_{k=1}^{p} f(\theta_k)m_1(C_k) + \sum_{k=1}^{p} f(\theta_k)m_2(C_k) - (\int_T f dm_1 + \int_T f dm_2)| \\
&\leq |\sum_{k=1}^{p} f(\theta_k)m_1(C_k) - \int_T f dm_1| + |\sum_{k=1}^{p} f(\theta_k)m_2(C_k) - \int_T f dm_2| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

which implies that f is m-integrable and $\int_T f d(m_1 + m_2) = \int_T f dm_1 + \int_T f dm_2$.

ii) Because f is m_1-integrable and m_2-integrable on T, then for every $\varepsilon > 0$, there exists $P_1 \in \mathcal{P}$ such that for every $P \in \mathcal{P}, P \geq P_1$,

$$|\int_T f dm_1 - \sigma(P, f, m_1)| < \frac{\varepsilon}{2}$$

and there exists $P_2 \in \mathcal{P}$ so that for every $P \in \mathcal{P}, P \geq P_2$,

$$|\int_T f dm_2 - \sigma(P, f, m_2)| < \frac{\varepsilon}{2}.$$

Let $P_0 = P_1 \wedge P_2 \in \mathcal{P}$ and $P \in \mathcal{P}, P = \{C_k\}_{k=\overline{1,p}} \geq P_0, \theta_k \in C_k, k = \overline{1,p}$ arbitrarily chosen. Since $P \geq P_1, P \geq P_2$, using the hypothesis, we obtain:

$$\begin{aligned}
&\int_T f dm_1 - \int_T f dm_2 \leq |\int_T f dm_1 - \sum_{k=1}^{p} f(\theta_k)m_1(C_k)| \\
&+ \sum_{k=1}^{p} f(\theta_k)m_1(C_k) - \sum_{k=1}^{p} f(\theta_k)m_2(C_k) + |\sum_{k=1}^{p} f(\theta_k)m_2(C_k) - \int_T f dm_2| \\
&< \varepsilon + \sum_{k=1}^{p} \{f(\theta_k)[m_1(C_k) - m_2(C_k)]\} < \varepsilon.
\end{aligned}$$

Hence $\int_T f dm_1 \leq \int_T f dm_2$.

Remark 2.10. Obviously, all the above results remain valid if we consider the integrability on a certain set $B \in \mathcal{A}$, instead of the entire space T.

Theorem 2.11. *If $m : \mathcal{A} \to \mathbb{R}_+$ is a submeasure of finite variation, $f : T \to \mathbb{R}$ is m-integrable on T and $B \in \mathcal{A}$ is an arbitrary set, then f is m-integrable on B.*

Proof. Let $\varepsilon > 0$. Since f is m-integrable on T, there exists a partition $P_\varepsilon = \{A_i\}_{i=\overline{1,n}}$ of T so that for every partition $P \in \mathcal{P}$, with $P \geq P_\varepsilon$ and every $t_i \in A_i, i = \overline{1,n}$, we have

$$|\int_T f dm - \sum_{i=1}^{n} f(t_i)m(A_i)| < \frac{\varepsilon}{2}.$$

Let $B \in \mathcal{A}$ and $P_\varepsilon^B = \{B \cap A_i\}_{i=\overline{1,n}} \in \mathcal{P}_B$. We consider $P_1 = \{C_k\}_{k=\overline{1,p}} \in \mathcal{P}_B, P_2 = \{B_j\}_{j=\overline{1,q}} \in \mathcal{P}_B$, so that $P_1 \geq P_\varepsilon^B, P_2 \geq P_\varepsilon^B$ and we also take $u_k \in C_k, k = \overline{1,p}, s_j \in B_j, j = \overline{1,q}$ arbitrary.

Let $P_1' = \{C_1, C_2, \ldots, C_p, A_1\backslash B, A_2\backslash B, \ldots, A_n\backslash B\}$ and $P_2' = \{B_1, B_2, \ldots, B_q, A_1\backslash B, A_2\backslash B, \ldots, A_n\backslash B\}$. We observe that P_1' and P_2' are partitions of T and $P_1' \geq P_\varepsilon, P_2' \geq P_\varepsilon$. Let also $a_i \in A_i\backslash B, i = \overline{1,n}$ be arbitrarily.

We have:

$$\begin{aligned}
&|\sigma_B(P_1) - \sigma_B(P_2)| = |\sum_{k=1}^{p} f(u_k)m(C_k) - \sum_{j=1}^{q} f(s_j)m(B_j)| \\
&= \Big|\sum_{k=1}^{p} f(u_k)m(C_k) + \sum_{i=1}^{n} f(a_i)m(A_i\backslash B) \\
&-(\sum_{j=1}^{q} f(s_j)m(B_j) + \sum_{i=1}^{n} f(a_i)m(A_i\backslash B))\Big| \\
&= |\sigma(P_1') - \sigma(P_2')| \leq |\sigma(P_1') - \int_T f dm| + |\sigma(P_2') - \int_T f dm| < \varepsilon.
\end{aligned}$$

Then $(\sigma_B(P))_{P\in(\mathcal{P},\leq)}$ is a Cauchy net in $\mathbb{R}$, hence it converges in $\mathbb{R}$. The proof is thus finished.

In the sequel we prove that, even m is a submeasure, the function $I : \mathcal{A} \to \mathbb{R}$, defined by

$$I(A) = \int_A f dm, \text{ for every } A \in \mathcal{A}, \tag{18}$$

is a finitely additive measure on $\mathcal{A}$.

Theorem 2.12. *Let $m : \mathcal{A} \to \mathbb{R}_+$ be a submeasure of finite variation and $B, C \in \mathcal{A}$, with $B \cap C = \emptyset$. If f is m-integrable on B and C, then f is m-integrable on $B \cup C$, and, moreover,*

$$\int_{B\cup C} f dm = \int_B f dm + \int_C f dm. \tag{19}$$

Proof. Let $\varepsilon > 0$. Since f is $m-$integrable on B and C, we get that there is a partition $P_\varepsilon^B = \{B_i\}_{i=\overline{1,n}} \in \mathcal{P}_B$ so that for every $P \in \mathcal{P}_B$, with $P \geq P_\varepsilon^B$, we have

$$|\int_B f dm - \sigma_B(P)| < \frac{\varepsilon}{2},$$

and there exists a partition $P_\varepsilon^C = \{C_j\}_{j=\overline{1,q}} \in \mathcal{P}_C$ so that for every $P \in \mathcal{P}_C$, with $P \geq P_\varepsilon^C$,

$$|\int_C f dm - \sigma_C(P)| < \frac{\varepsilon}{2}.$$

Now, let be the partition $P_\varepsilon^{B\cup C} = \{B_1,...B_n,C_1,...C_q\} \in \mathcal{P}_{B\cup C}$. If $P = \{A_k\}_{k=\overline{1,p}} \in \mathcal{P}_{B\cup C}$ is such that $P \geq P_\varepsilon^{B\cup C}$ and $s_k \in A_k$, $k \in \{1,...p\}$ are arbitrarily chosen, then

$$\sum_{k=1}^{p} f(s_k)m(A_k) = \sum_{k=1}^{r} f(s_k)m(B'_k) + \sum_{k=r+1}^{p} f(s_k)m(C'_k),$$

where $\{B'_k\}_{k=\overline{1,r}} = P'_B \in \mathcal{P}_B$ and $P'_B \geq P_\varepsilon^B$, $\{C'_k\}_{k=\overline{r+1,p}} = P'_C \in \mathcal{P}_C$ and $P'_C \geq P_\varepsilon^C$.

Consequently,

$$\begin{aligned}
&|\sum_{k=1}^{p} f(s_k)m(A_k) - (\int_B f dm + \int_C f dm)| \\
&= |\sum_{k=1}^{r} f(s_k)m(B'_k) + \sum_{k=r+1}^{p} f(s_k)m(C'_k) - (\int_B f dm + \int_C f dm)|
\end{aligned}$$

$$\leq |\sum_{k=1}^{r} f(s_k)m(B'_k) - \int_B f dm| + |\sum_{k=r+1}^{p} f(s_k)m(C'_k) - \int_C f dm|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

The proof is thus finished.

Corollary 2.13. *If $m : \mathcal{A} \to \mathbb{R}_+$ is a submeasure of finite variation, f is m-integrable on T, and $I : \mathcal{A} \to \mathbb{R}$ is the function defined by* (18)*, then:*

i) $I \ll m$;

ii) *If m is σ-subadditive on $\mathcal{A}$, then I is σ-additive on $\mathcal{A}$.*

Proof. Let $\alpha = \sup_{t\in T} |f(t)|$. If $\alpha = 0$, then $f \equiv 0$, so $I \equiv 0$ on T. In what follows, we suppose that $\alpha \neq 0$.

i) Consider an arbitrary set $A \in \mathcal{A}$, with $m(A) = 0$. Since f is m-integrable on A, then, according to theorem 2.7 iii), $|I(A)| = |\int_A f dm| \leq \alpha\, \overline{m}(A)$.

If $P = \{A_i\}_{i=\overline{1,n}}$ is an arbitrary partition of A, then $m(A_i) = 0$, because $A_i \subset A$, for every $i = \overline{1,n}$ and $m(A) = 0$; this implies $\overline{m}(A) = \sup_{P\,\in\,\mathcal{P}_A} \{\sum_{i=1}^{n} m(A_i)\} = 0$, so $|I(A)| = 0$, and hence $I(A) = 0$.

ii) Let $\varepsilon > 0$. We consider a pairwise disjoint sequence of sets $(A_k)_k \subset \mathcal{A}$, with $\bigcup_{k=1}^{\infty} A_k \in \mathcal{A}$.

Since m is σ-subadditive, then, according to [1] or [8], $\overline{m}$ is σ-additive. So, for every $\varepsilon > 0$, there exists $n_0 \in \mathbf{N}$ such that $\overline{m}(\bigcup_{k=n+1}^{\infty} A_k) < \frac{\varepsilon}{\alpha}$, for every $n \geq n_0$.

Then, from i) we have

$$|I(\bigcup_{k=1}^{\infty} A_k) - \sum_{k=1}^{n} I(A_k)| = |I(\bigcup_{k=1}^{\infty} A_k) + I(\bigcup_{k=n+1}^{\infty} A_k) - \sum_{k=1}^{n} I(A_k)|$$
$$= |I(\bigcup_{k=n+1}^{\infty} A_k)| \leq \sup_{t\in T} |f(t)|\ \ \overline{m}(\bigcup_{k=n+1}^{\infty} A_k) < \alpha \frac{\varepsilon}{\alpha} = \varepsilon,$$

for every $n \geq n_0$. It means that I is σ-additive on $\mathcal{A}$, as claimed.

Theorem 2.14. *Let $m : \mathcal{A} \to \mathbb{R}_+$ be a submeasure of finite variation, $B, C \in \mathcal{A}$, with $B \subseteq C$ and $f : T \to \mathbb{R}_+$ a $m-$integrable function on C. Then*

$$\int_B f dm \leq \int_C f dm. \tag{20}$$

Proof. Let $\varepsilon > 0$. By the $m-$integrability of f on B and C there is $P_\varepsilon^1 = \{B_i\}_{i=\overline{1,n}} \in \mathcal{P}_B$ so that for every $P \in \mathcal{P}_B$, with $P \geq P_\varepsilon^1$,

$$|\int_B f dm - \sigma_B(P)| < \frac{\varepsilon}{2}, \tag{21}$$

and there exists $P_\varepsilon^2 = \{C_j\}_{j=\overline{1,q}} \in \mathcal{P}_C$ such that for every $P \in \mathcal{P}_C$, with $P \geq P_\varepsilon^2$,

$$|\int_C f dm - \sigma_C(P)| < \frac{\varepsilon}{2}. \tag{22}$$

Let us consider $\widetilde{P}_\varepsilon^1 = \{B_1, B_2, ...B_n, C\backslash B\}$. It is easy to check that $\widetilde{P}_\varepsilon^1 \in \mathcal{P}_C$. Let also $\widetilde{P}_\varepsilon^1 \wedge P_\varepsilon^2 \in \mathcal{P}_C$ and pick an arbitrary partition $P = \{D_k\}_{k=\overline{1,p}} \in \mathcal{P}_C$, with $P \geq \widetilde{P}_\varepsilon^1 \wedge P_\varepsilon^2$. We note that the family $P_\varepsilon^{''} = \{D_k \cap B\}_{k=\overline{1,p}}$ is also a partition of B and $P_\varepsilon^{''} \geq P_\varepsilon^1$.

Indeed, since $P \geq \widetilde{P}_\varepsilon^1 \wedge P_\varepsilon^2$ and $\widetilde{P}_\varepsilon^1 \wedge P_\varepsilon^2 = \{B_1, B_2, ...B_n, C\backslash B\} \wedge \{C_j\}_{j=\overline{1,q}} = \{\{B_i \cap C_j\}_{\substack{i=\overline{1,n} \\ j=\overline{1,q}}}, \{C_j\backslash B\}_{j=\overline{1,q}}\}$, we obtain that $P_\varepsilon^{''} \geq P_\varepsilon^1$.

Consequently, for P holds (22) and for $P_\varepsilon^{''}$ holds (21).

Let $t_k \in D_k \cap B, k = \overline{1,p}$ be arbitrarily chosen. Then $t_k \in D_k$, for every $k = \overline{1,p}$,

$$|\int_C f dm - \sum_{k=1}^p f(t_k)m(D_k)| < \frac{\varepsilon}{2} \text{ and } |\int_B f dm - \sum_{k=1}^p f(t_k)m(D_k \cap B)| < \frac{\varepsilon}{2},$$

which imply that

$$\begin{aligned}
&\int_B f dm - \int_C f dm \leq |\int_B f dm - \sum_{k=1}^p f(t_k)m(D_k \cap B)| \\
&+ \sum_{k=1}^p f(t_k)m(D_k \cap B) - \sum_{k=1}^p f(t_k)m(D_k) + |\int_C f dm - \sum_{k=1}^p f(t_k)m(D_k)| \\
&< \varepsilon + \sum_{k=1}^p f(t_k)m(D_k \cap B) - \sum_{k=1}^p f(t_k)m(D_k) \leq \varepsilon,
\end{aligned}$$

for every $\varepsilon > 0$. This yields $\int_B f dm \leq \int_C f dm$, as claimed.

In what follows, we give a characterization of the m-integrability using the integrability in the sense of Gould [7] with respect to the variation $\overline{m}$ of m.

Theorem 2.15. *Let $m : \mathcal{A} \to \mathbb{R}_+$ be a submeasure of finite variation and $f : T \to \mathbb{R}$. Then f is m-integrable on T if and only if f is $\overline{m}$-integrable on T. Moreover,*

$$\int_T f\, dm = \int_T f d\overline{m}. \tag{23}$$

Proof. Let $\varepsilon > 0$ and $M = \sup_{t\in T} |f(t)|$.

The if part.

By the definition of $\overline{m}$ we get that there exists a partition $P_\varepsilon = \{A_1, \ldots, A_n\}$ of T such that

$$\overline{m}(T) < \sum_{i=1}^{n} m(A_i) + \frac{\varepsilon}{2M}. \tag{24}$$

Because f is m-integrable on T, there exists a partition $\widetilde{P}_\varepsilon \in \mathcal{P}$ such that

$$|\sigma(P) - \int_T f dm| < \frac{\varepsilon}{2}, \text{ for every } P \in \mathcal{P}, \text{ with } P \geq \widetilde{P}_\varepsilon.$$

Now, we consider the partition $\widetilde{\widetilde{P}}_\varepsilon = \widetilde{P}_\varepsilon \wedge P_\varepsilon$; let $P = \{B_j\}_{j=\overline{1,q}}$ be so that $P \geq \widetilde{\widetilde{P}}_\varepsilon$ and $t_j \in B_j$, $j = \overline{1,q}$ arbitrarily chosen. Since $P \geq \widetilde{P}_\varepsilon$, we have

$$|\sum_{j=1}^{q} f(t_j)m(B_j) - \int_T f dm| < \frac{\varepsilon}{2}. \tag{25}$$

But

$$\begin{aligned} &|\sum_{j=1}^{q} f(t_j)\overline{m}(B_j) - \sum_{j=1}^{q} f(t_j)m(B_j)| \\ &= |\sum_{j=1}^{q} f(t_j)(\overline{m}(B_j) - m(B_j))| \qquad (26) \\ &\leq M \sum_{j=1}^{q} [\overline{m}(B_j) - m(B_j)] = M[\overline{m}(T) - \sum_{j=1}^{q} m(B_j)] \end{aligned}$$

$$< \frac{\varepsilon}{2} + M[\sum_{i=1}^{n} m(A_i) - \sum_{j=1}^{q} m(B_j)].$$

Since $P = \{B_j\}_{j=\overline{1,q}} \geq P_\varepsilon$, as in the proof of theorem 2.4, we have

$$m(B_j) = \sum_{i=1}^{n} m(B_j \cap A_i), \text{ for every } j = \overline{1,q}.$$

Consequently, by (26) we get

$$\begin{aligned} &|\sum_{j=1}^{q} f(t_j)\overline{m}(B_j) - \sum_{j=1}^{q} f(t_j)m(B_j)| \\ &< \frac{\varepsilon}{2} + M[\sum_{i=1}^{n} m(A_i) - \sum_{j=1}^{q}\sum_{i=1}^{n} m(B_j \cap A_i)] \\ &= \frac{\varepsilon}{2} + M\{\sum_{i=1}^{n}[m(A_i) - \sum_{j=1}^{q} m(B_j \cap A_i)]\}. \end{aligned} \tag{27}$$

Because m is a submeasure on $\mathcal{A}$,

$$m(A_i) = m(\bigcup_{j=1}^{q}(A_i \cap B_j)) \leq \sum_{j=1}^{q} m(A_i \cap B_j)$$

and then from (27) we obtain

$$|\sum_{j=1}^{q} f(t_j)\overline{m}(B_j) - \sum_{j=1}^{q} f(t_j)m(B_j)| < \frac{\varepsilon}{2}. \tag{28}$$

From (25) and (28), we get that

$$\begin{aligned} &|\sum_{j=1}^{q} f(t_j)\overline{m}(B_j) - \int_T f dm| \leq |\sum_{j=1}^{q} f(t_j)\overline{m}(B_j) - \sum_{j=1}^{q} f(t_j)m(B_j)| \\ &+|\sum_{j=1}^{q} f(t_j)m(B_j) - \int_T f dm| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Consequently, f is $\overline{m}$-integrable on T and $\int_T f dm = \int_T f d\overline{m}$.

The only if part.

Because f is $\overline{m}$-integrable on T, there exists a partition $\widetilde{P}_\varepsilon \in \mathcal{P}$ such that

$$|\sigma(P, f, \overline{m}) - \int_T f d\overline{m}| < \frac{\varepsilon}{2}, \text{ for every } P \in \mathcal{P}, \text{ with } P \geq \widetilde{P}_\varepsilon.$$

We consider the partition $\widetilde{\widetilde{P}}_\varepsilon = \widetilde{P}_\varepsilon \wedge P_\varepsilon$, where P_ε is the same partition as in (24). Let $P = \{B_j\}_{j=\overline{1,q}}$ be a partition of T so that $P \geq \widetilde{\widetilde{P}}_\varepsilon$ and $t_j \in B_j$, $j = \overline{1,q}$ arbitrarily chosen.

Since $P \geq \widetilde{P}_\varepsilon$, we have

$$|\sum_{j=1}^{q} f(t_j)\overline{m}(B_j) - \int_T f d\overline{m}| < \frac{\varepsilon}{2}. \tag{29}$$

Now, as in the if part we obtain

$$|\sum_{j=1}^{q} f(t_j)\overline{m}(B_j) - \sum_{j=1}^{q} f(t_j)m(B_j)| < \frac{\varepsilon}{2}. \tag{30}$$

Consequently, from (29) and (30) we have

$$|\sum_{j=1}^{q} f(t_j)m(B_j) - \int_T f d\overline{m}| \leq |\sum_{j=1}^{q} f(t_j)\overline{m}(B_j) - \sum_{j=1}^{q} f(t_j)m(B_j)|$$
$$+|\sum_{j=1}^{q} f(t_j)\overline{m}(B_j) - \int_T f d\overline{m}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which means that f is m-integrable on T and $\int_T f dm = \int_T f d\overline{m}$, as claimed.

Theorem 2.16. *Let $m : \mathcal{A} \mapsto \mathbb{R}_+$ be a submeasure of finite variation and $f : T \mapsto \mathbb{R}$ a bounded function. Then the following statements are equivalent:*

i) *f is m-integrable;*

ii) *f is $\overline{m}$-integrable;*

iii) *f is $\overline{m}$-totally-measurable.*

Proof. By theorem 2.15, i) is equivalent to ii).

Now, we prove that ii) is equivalent to iii). Indeed, according to theorem 4.7 [7] and remark 2.2, we get that f is $\overline{m}$-integrable if and only if f is $\overline{m}$ -totally-measurable on T.

Theorem 2.17. *Let $m : \mathcal{A} \mapsto \mathbb{R}_+$ be a submeasure of finite variation and $f : T \mapsto \mathbb{R}$ a m-integrable function on T. Then $|f|$ is m-integrable and, moreover,*

$$|\int_T f dm| \leq \int_T |f|\, dm.$$

Proof. According to theorem 2.16, it is sufficient to prove that $|f|$ is $\overline{m}$-totally-measurable on T.

Let $\varepsilon > 0$. Since f is m-integrable on T, then, according to theorem 2.16, it is $\overline{m}$-totally-measurable on T. Hence, there exists a partition $P_\varepsilon = \{A_0, A_1, \ldots, A_n\}$ of T such that $\overline{m}(A_0) < \varepsilon$ and $\sup_{t,s\in A_i} |f(t) - f(s)| < \varepsilon$, for every $i = \overline{1,n}$.

Because

$$\sup_{t,s\in A_i} \big||f(t)| - |f(s)|\big| \leq \sup_{t,s\in A_i} |f(t) - f(s)| < \varepsilon, \text{ for every } i = \overline{1,n},$$

then $|f|$ is $\overline{m}$ -totally-measurable and hence m-integrable on T.

Since $-|f| \leq f \leq |f|$, from theorem 2.7 i) and v) we get that

$$\int_T (-|f|)\, dm = -\int_T |f|\, dm \leq \int_T f dm \leq \int_T |f|\, dm.$$

Consequently, $|\int_T f dm| \leq \int_T |f|\, dm$.

Theorem 2.18. Let $m : \mathcal{A} \mapsto \mathbb{R}_+$ *be a submeasure of finite variation and $f, g : T \mapsto \mathbb{R}$ two m-integrable functions on T. Then the functions* $\sup(f, g)$ *and* $\inf(f, g)$ *are m-integrable on T.*

Proof. According to theorems 2.7 i), iv) and 2.17 we have that $f + g$, $f - g$ and $|f - g|$ are m-integrable on T. Hence, $\sup(f, g) = \frac{1}{2}(f+g+|f - g|)$ and $\inf(f, g) = \frac{1}{2}(f + g - |f - g|)$ are m-integrable on T.

Acknowledgement. The authors are indebted to Prof. Dr. A. M. Precupanu for the careful reading of the paper and for the useful suggestions.

REFERENCES

1. Dinculeanu, N. – *Teoria Măsurii şi Funcţii Reale*, Ed. Did. şi Ped., Bucureşti, 1964.
2. Drewnowski, L. – *Topological rings of sets, continuous set functions, Integration, I, II*, Bull. Acad. Polon. Sci. Sér. Math. Astron. Phys. 20 (1972), 269-276, 277-286.
3. Gavriluţ, A. – *Properties of regularity for multisubmeasures*, An. Şt. Univ. "Al. I. Cuza" Iaşi, Tomul L, s.I a, 2004, f. 2, 373-392.
4. Gavriluţ, A. – *Proprietăţi de regularitate a multifuncţiilor de mulţime*, Casa de Editură "Venus", Iaşi, 2006.
5. Gavriluţ, A. – *A Gould type integral with respect to a multisubmeasure,* Math. Slovaca, in print.
6. Gavriluţ, A. – *On some properties of the Gould type integral with respect to a multisubmeasure,* An. Şt. Univ. "Al. I. Cuza" Iaşi, in print.
7. Gould, G. G. – *Integration over vector-valued measures*, Proc. London. Math. Soc. 15, 1965, 193-205.
8. Halmos, P. – *Measure Theory*, D. Van Nostrand Company Inc., New-York, 1950.
9. Precupanu, A.; Croitoru, A. – *A Gould type integral with respect to a multimeasure, I*, An. Şt. Univ. "Al. I. Cuza" Iaşi, Tomul XLVIII, s. I a, Matematică, 2002, f. 1, 165-200.
10. Precupanu, A.; Croitoru, A. – *A Gould type integral with respect to a multimeasure, II*, An. Şt. Univ. "Al. I. Cuza" Iaşi, Tomul XLIX, s. I a, Matematică, 2003, f. 1, 183-207.

Received: 30.I.2007

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