ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" DIN IAŞI (S.N.)
MATEMATICĂ, Tomul LIII, 2007, f.2

A SEMIGROUP APPROACH TO A CLASS OF HAMILTON-JACOBI EQUATIONS WITH NONCONVEX HAMILTONIANS

BY

TEODOR HAVÂRNEANU*

Abstract. In this paper we study the existence of variational and semigroupal solutions for a class of Hamilton-Jacobi equations arising in a problem of differential games. The main tools in proving the results are the dynamic programming principle and Chernoff formula.

Mathematics Subject Classification 2000: 35F25, 90D45, 49C20.

Key words: Hamilton-Jacobi equations, dynamic programming principle, semigroups of operators, Chernoff formula.

1. Introduction. This paper in concerned with the study of the existence and uniqueness for the following Hamilton–Jacobi equation

(1.1)
$$\begin{cases} U_t(t,x) - F(x, U_x(t,x)) - (Ax, U_x(t,x)) = g(x), & (t,x) \in R^+ \times H \\ U(0,x) = \varphi_0(x), & x \in H, \end{cases}$$

where H is a Hilbert space with the norm $|\cdot|$ and scalar product $(\cdot,\cdot)$. The unknown function U is real valued and defined on $[0;+\infty) \times H$, φ_0, g are given functions on H and U_t, U_x stand for the derivatives with respect to variables t and x of the function U, respectively.

*The author acknowledges support from the Romanian Ministry of Education and Research through CEEX program Contract CERES-2-Cex06-11-56/25.07.2006.

In the following we assume that A is the infinitesimal generator of a C_0-semigroup on H which satisfies $\|e^{At}\| \leq M$ and F is given by

$$F(x,p) = \min_{z\in Z}\max_{y\in Y}\{(f(y,z),p) + h(x,y,z)\}, \tag{1.2}$$

where Y, Z are two compact sets of a topological space Ω and $f : Y \times Z \to H$, $h : H \times Y \times Z \to R$, $g : H \to R$ are bounded and uniformly continuous functions with bound constants M_f, M_g, M_h and, moduli of uniform continuity $\omega_f, \omega_g, \omega_h$, respectively.

We also assume that the max-min condition is fulfilled, i.e.

$$\begin{aligned}&\min_{z\in Z}\max_{y\in Y}\{(f(y,z),p) + h(x,y,z)\}\\ &= \max_{y\in Y}\min_{z\in Z}\{(f(y,z),p) + h(x,y,z)\} \text{ for every } x,p \in H.\end{aligned} \tag{1.3}$$

Given a Banach space $(X, \|\cdot\|)$ denote by $BUC(X)$ the space of bounded uniformly continuous real valued functions on X endowed with the norm

$$\|f\|_b = \sup\{|f(x)|;\ x \in X\}.$$

By $\mathrm{Lip}(X)$ we denote the space of all Lipschitz functions $f : X \to R$.

It is well known that Eq. (1.1) is related to certain differential game [7,9]. The value of this differential game can be viewed as the generalized solution of Eq. (1.1).

2. Definition and study of the differential game. Let us introduce the following sets

$$\begin{aligned}M(t) &= \{y : [t,+\infty) \to Y;\ y \text{ measurable}\},\\ N(t) &= \{z : [t,+\infty) \to Z;\ z \text{ measurable}\}.\end{aligned} \tag{2.1}$$

$M(t)$ and $N(t)$ will be named sets of controls employed by players I and II, respectively.

Fix $t \geq 0, x \in H$ and consider the differential equation

$$\begin{cases}\dot{x}(s) = Ax(s) + f(y(s),z(s)),\ s \geq t,\\ x(t) = x,\end{cases} \tag{2.2}$$

where A and f satisfy the conditions from the previous section and $y \in M(t), z \in N(t)$.

Following now [6, 8] we define the strategies of the player I (beginning at time t) as any mapping

$$\alpha : N(t) \to M(t)$$

such that for each $t \leq s$ and $z, \hat{z} \in N(t)$ we have

$$z(\tau) = \hat{z}(\tau) \quad \text{a.e. } t \leq \tau \leq s$$

implies

$$\alpha[z](\tau) = \alpha[\hat{z}](\tau) \quad \text{a.e. } t \leq \tau \leq s.$$

Similarly, any mapping

$$\beta : M(t) \to N(t)$$

with the property that for each $t \leq s$ and $y, \hat{y} \in M(t)$ satisfying $y(\tau) = \hat{y}(\tau)$ a.e. $t \leq \tau \leq s$ we have

$$\beta[y](\tau) = \beta[\hat{y}](\tau) \quad \text{a.e. } t \leq \tau \leq s,$$

is named a strategy of player II (beginning at time t).

We denote by $\Gamma(t)$ and $\Delta(t)$ the sets of all strategies beginning at time t for the player I and player II, respectively.

We associate with Eq. (2.2) the payoff functional

$$(\mathrm{P}_\lambda) \qquad P_\lambda(y, z) = \int_t^{+\infty} e^{-\lambda s} h(x(s), y(s), z(s)) ds,$$

where h satisfies the conditions from Section 1, $y \in M(t), z \in N(t)$, $x(t)$ is the "mild" solution of Eq. (2.2), and λ is a positive parameter.

The goal of player I is to maximize P_λ and the goal of player II is to minimize P_λ.

Let us define the lower value and the upper value of the differential game V_λ and U_λ, respectively.

$$\begin{aligned} V_\lambda(t, x) &= \inf_{\beta \in \Delta(t)} \sup_{y \in M(t)} P_\lambda(y, \beta[y]) \\ &= \inf_{\beta \in \Delta(t)} \sup_{y \in M(t)} \left\{ \int_t^{+\infty} e^{-\lambda s} h(x(s), y(s), \beta[y](s)) ds \right\}, \end{aligned} \tag{2.3}$$

where $x(\cdot)$ is the solution of Eq. (2.2) with $y(\cdot)$ and $z(\cdot) = \beta[y](\cdot)$.

(2.4)
$$\begin{aligned} U_\lambda(t,x) &= \sup_{\alpha\in\Gamma(t)} \inf_{z\in N(t)} P_\lambda(\alpha[z], z) \\ &= \sup_{\alpha\in\Gamma(t)} \inf_{z\in N(t)} \left\{ \int_t^{+\infty} e^{-\lambda s} h(x(s), y(s), \alpha[z](s), z(s)) ds \right\}, \end{aligned}$$

where $x(\cdot)$ is the solution of Eq. (2.2) with $y(\cdot) = \alpha[z](\cdot)$ and $z(\cdot)$.

Theorem 2.1. (Dynamic programming optimality condition) *For each* $0 \le t < t+\sigma$ *and* $x \in H$ *we have*

$$\begin{aligned} U_\lambda(t,x) &= \sup_{\alpha\in\Gamma(t)} \inf_{z\in N(t)} \left\{ \int_t^{t+\sigma} e^{-\lambda s} h(x(s), \alpha[z](s), z(s)) ds \right. \\ &\quad \left. + U_\lambda(t+\sigma,\ x(t+\sigma)) \right\} \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} V_\lambda(t,x) &= \inf_{\beta\in\Delta(t)} \sup_{y\in M(t)} \left\{ \int_t^{t+\sigma} e^{-\lambda s} h(x(s), y(s), \beta[y](s)) ds \right. \\ &\quad \left. + V_\lambda(t+\sigma,\ x(t+\sigma)) \right\}. \end{aligned} \tag{2.6}$$

Proof. We shall adopt the method from [7, 8] and we shall present the proof for reader's convenience.

Let

$$\begin{aligned} W(t,x) &= \sup_{\alpha\in\Gamma(t)} \inf_{z\in N(t)} \left\{ \int_t^{t+\sigma} e^{-\lambda s} h(x(s), \alpha[z](s), z(s)) ds \right. \\ &\quad \left. + U_\lambda(t+\sigma,\ x(t+\sigma)) \right\} \end{aligned} \tag{2.7}$$

and fix $\varepsilon > 0$.

Then there exists $\gamma \in \Gamma(t)$ such that

$$\begin{aligned} W(t,x) &\le \inf_{z\in N(t)} \left\{ \int_t^{t+\sigma} e^{-\lambda s} h(x(s), \gamma[z](s), z(s)) ds \right. \\ &\quad \left. + U_\lambda(t+\sigma,\ x(t+\sigma)) \right\} + \varepsilon. \end{aligned} \tag{2.8}$$

Also, for each $w \in H$, there exists $\gamma_w \in \Gamma(t+\sigma)$ such that

$$U_\lambda(t+\sigma, w) \le \inf_{z\in N(t+\sigma)} \left\{ \int_{t+\sigma}^{+\infty} e^{-\lambda s} h(x(s), \gamma_w[z](s), z(s)) ds \right\} + \varepsilon. \tag{2.9}$$

Let us define now $\alpha \in \Gamma(t)$ by

$$\alpha[z](s) = \begin{cases} \gamma[z](s), & t \le s \le t+\sigma \\ \gamma_{x(t+\sigma)}[z](s), & t+\sigma < s, \end{cases}$$

for every $z \in N(t)$, where $x(\cdot)$ is the solution of Eq. (2.2) for $y = \gamma[z]$ and z with the initial condition x.

Therefore, using (2.8) and (2.9), we have

$$W(t,x) \le \inf_{z\in N(t)} \left\{ \int_t^{+\infty} e^{-\lambda s} h(x(s), \alpha[z](s), z(s)) ds \right\} + 2\varepsilon,$$

and taking supremum over $\alpha \in \Gamma(t)$ it results

$$W(t,x) \le U_\lambda(t,x) + 2\varepsilon. \tag{2.10}$$

On the other hand, there exists $\alpha \in \Gamma(t)$ such that

$$U_\lambda(t,x) \le \inf_{z\in N(t)} \left\{ \int_t^{+\infty} e^{-\lambda s} h(x(s), \alpha[z](s), z(s)) ds \right\} + \varepsilon. \tag{2.11}$$

Then we have (with α from (2.11))

$$W(t,x) \ge \inf_{z\in N(t)} \left\{ \int_t^{t+\sigma} e^{-\lambda s} h(x(s), \alpha[z](s), z(s)) ds + U_\lambda(t+\sigma,\ x(t+\sigma)) \right\},$$

and there exists $z_1 \in N(t)$ such that

$$W(t,x) \ge \int_t^{t+\sigma} e^{-\lambda s} h(x(s), \alpha[z_1](s), z_1(s)) ds + U_\lambda(t+\sigma,\ x(t+\sigma)) - \varepsilon. \tag{2.12}$$

Now, for each $z \in N(t+\sigma)$ we define $\tilde{z} \in N(t)$ by

$$\tilde{z}(s) = \begin{cases} z_1(s), & t \le s < t+\sigma \\ z(s), & t+\sigma \le s \end{cases}$$

and then we define $\underset{\sim}{\alpha} \in \Gamma(t+\sigma)$ by

$$\underset{\sim}{\alpha}[z](s) = \alpha[\tilde{z}](s) \text{ for } s \ge t+\sigma.$$

Therefore

$$U_\lambda(t+\sigma, x(t+\sigma)) \ge \inf_{z\in N(t+\sigma)} \left\{ \int_{t+\sigma}^{+\infty} e^{-\lambda s} h(x(s), \underset{\sim}{\alpha}[z](s), z(s)) ds \right\}$$

and there exists $z_2 \in N(t+\sigma)$ such that

$$U_\lambda(t+\sigma, x(t+\sigma)) \geq \int_{t+\sigma}^{+\infty} e^{-\lambda s} h(x(s), \underset{\sim}{\alpha}[z_2](s), z_2(s))ds - \varepsilon. \tag{2.13}$$

From (2.12) and (2.13), defining $z \in N(t)$ by

$$z(s) = \begin{cases} z_1(s), & t \leq s < t+\sigma \\ z_2(s), & t+\sigma \leq s, \end{cases}$$

we obtain

$$W(t,x) \geq \int_t^{+\infty} e^{-\lambda s} h(x(s), \alpha[z](s), z(s))ds - 2\varepsilon.$$

Using now (2.11) it results

$$W(t,x) \geq U_\lambda(t,x) - 3\varepsilon \text{ for each } \varepsilon > 0.$$

This last relation together with (2.10) give us

$$U_\lambda(t,x) = W(t,x).$$

For (2.6) the proof is similar.
Let us consider now the infinite horizon problem (for $t=0$)

$$U_\lambda^0(x) \equiv \sup_{\alpha \in \Gamma_0} \inf_{z \in N_0} \left\{ \int_0^{+\infty} e^{-\lambda s} [g(x(s)) + h(x(s), \alpha[z](s), z(s))]ds \right\}$$

and

$$V_\lambda^0(x) \equiv \inf_{\beta \in \Delta_0} \sup_{y \in M_0} \left\{ \int_0^{+\infty} e^{-\lambda s} [g(x(s)) + h(x(s), y(s), \beta[y](s))]ds \right\}$$

where $x(\cdot)$ is the solution of the equation

$$\begin{cases} \dot{x}(s) = Ax(s) + f(\alpha[z](s), z(s)), \ s > 0 \\ x(0) = x \end{cases} \tag{2.14}$$

and

$$\begin{cases} \dot{x}(s) = Ax(s) + f(y(s), \beta[y](s)), \ s > 0 \\ x(0) = x, \end{cases} \tag{2.15}$$

for $U^0_\lambda(\cdot)$ and $V^0_\lambda(\cdot)$, respectively.

Proposition 2.1. *$U^0_\lambda(\cdot)$ and $V^0_\lambda(\cdot)$ are bounded and uniformly continuous on H. Moreover, if g and h are Lipschitzian, then $U^0_\lambda(\cdot)$ and $V^0_\lambda(\cdot)$ are Lipschitzian.*

Proof. We shall prove the assertion only for $U^0_\lambda(\cdot)$.

$$|U^0_\lambda(x)| \leq \int_0^{+\infty} e^{-\lambda s}(|g(x(s))| + |h(x(s), \alpha[z](s), z(s))|)ds \leq \frac{c}{\lambda}, \ \forall x \in H,$$

where c is a constant which bounds the sum $|g| + |h|$.
Let us prove now that $U^0_\lambda(\cdot)$ is uniformly continuous on H.
Fix $x_1, x_2 \in H$ and $\varepsilon > 0$. Let $\alpha \in \Gamma_0$ such that

$$U^0_\lambda(x_1) \leq \inf_{z \in N_0} \int_0^{+\infty} e^{-\lambda s}[g(x_2(s)) + h(x_2(s), \alpha[z](s), z(s))]ds + \varepsilon$$

and let $z \in N_0$ such that

$$U^0_\lambda(x_2) \geq \int_0^{+\infty} e^{-\lambda s}[g(x_2(s)) + h(x_2(s), \alpha[z](s), z(s))]ds - \varepsilon.$$

Therefore

$$U^0_\lambda(x_1) \leq \int_0^{+\infty} e^{-\lambda s}[g(x(s)) + h(x(s), \alpha[z](s), z(s))]ds + \varepsilon.$$

Here $x_1(\cdot)$ is the solution of the problem

$$\begin{cases} \dot{x}_1(s) = Ax_1(s) + f(\alpha[z](s), z(s)), \ s > 0 \\ x_1(0) = x_1, \end{cases}$$

and $x_2(\cdot)$ is the solution of the problem

$$\begin{cases} \dot{x}_2(s) = Ax_2(s) + f(\alpha[z](s), z(s)), \ s > 0 \\ x_2(0) = x_2. \end{cases}$$

Then we have

$$|x_1(s) - x_2(s)| \leq M|x_1 - x_2|, \ \forall s \geq 0, \text{ where } \|e^{At}\| \leq M, \ t \geq 0.$$

Using these estimations one can easily obtain

$$U_\lambda^0(x_1) - U_\lambda^0(x_2) \le \int_0^{+\infty} e^{-\lambda s}[g(x_1(s)) - g(x_2(s))]ds$$
$$+ \int_0^{+\infty} e^{-\lambda s}[h(x_1(s), \alpha[z](s), z(s)) - h(x_2(s), \alpha[z](s), z(s))]ds + 2\varepsilon$$
$$\le \frac{1}{\lambda}\omega_g(|x_1 - x_2|) + \frac{1}{\lambda}\omega_h(|x_1 - x_2|) + 2\varepsilon,$$

where ω_g, ω_h are moduli of continuity for g and h, respectively.
On the other hand, let $\alpha \in \Gamma_0$ such that

$$U_\lambda^0(x_2) \le \inf_{z \in N_0} \left\{ \int_0^{+\infty} e^{-\lambda s}[g(x_2(s)) + h(x_2(s), \alpha[z](s), z(s))]ds \right\} + \varepsilon.$$

Choose now $z \in N_0$ such that

$$U_\lambda^0(x_1) \ge \int_0^{+\infty} e^{-\lambda s}[g(x_1(s)) + h(x_1(s), \alpha[z](s), z(s))]ds - \varepsilon,$$

where $x_1(\cdot), x_2(\cdot)$ are the solutions of (2.14) with $x_1(0) = x_1$ and $x_2(0) = x_2$, respectively.
Now it is easy to see that

$$U_\lambda^0(x_2) - U_\lambda^0(x_1) \le \int_0^{+\infty} e^{-\lambda s}[g(x_2(s)) - g(x_1(s))]ds$$
$$+ \int_0^{+\infty} e^{-\lambda s}[h(x_2(s), \alpha[z](s), z(s)) - h(x_1(s), \alpha[z](s), z(s))]ds + 2\varepsilon$$
$$\le \frac{1}{\lambda}\omega_g(|x_1 - x_2|) + \frac{1}{\lambda}\omega_h(|x_1 - x_2|) + 2\varepsilon.$$

Therefore

$$|U_\lambda^0(x_1) - U_\lambda^0(x_2)| \le \frac{1}{\lambda}[\omega_g(|x_1 - x_2|) + \omega_h(|x_1 - x_2|)], \text{ for } x_1, x_2 \in H.$$

If g and h are Lipschitzian then we easily obtain

$$|U_\lambda^0(x_1) - U_\lambda^0(x_2)| \le \frac{1}{\lambda}(L_g + L_h)|x_1 - x_2|, \text{ for } x_1, x_2 \in H$$

where L_g and L_h are Lipschitz constants for g and h, respectively.

Let $g \in BUC(H)$ (or Lipschitzian) and $\lambda > 0$. We define (see [2, 3])

(2.16)
$$(R^s(\lambda)g)(x) \equiv \sup_{\alpha\in\Gamma_0}\inf_{z\in N_0}\left\{\int_0^{+\infty} e^{-\lambda s}[g(x(s)) + h(x(s), \alpha[z](s), z(s))]ds\right\}$$

and

(2.17)
$$(R^d(\lambda)g)(x) \equiv \inf_{\beta\in\Delta_0}\sup_{y\in M_0}\left\{\int_0^{+\infty} e^{-\lambda s}[g(x(s)) + h(x(s), y(s), \beta[y](s))]ds\right\},$$

where $x(\cdot)$ from (2.16) and (2.17) is the solution of the Cauchy problem (2.14) and (2.15), respectively.

Proposition 2.2. *For $\lambda > 0$ and $g \in BUC(H)$ the following assertions are true:*

(i) $R^s(\lambda)g$ and $R^d(\lambda)g$ belong to $BUC(H)$.

(ii) $R^s(\lambda)g = R^s(\mu)[(\mu-\lambda)R^s(\lambda)g + g]$ for $0 < \lambda \le \mu < +\infty$,
$R^d(\lambda)g = R^d(\mu)[(\mu-\lambda)R^d(\lambda)g + g]$ for $0 < \lambda \le \mu < +\infty$.

Moreover, if $g \in \mathrm{Lip}(H)$ then $R^s(\lambda)g$ and $R^d(\lambda)g$ belong to $\mathrm{Lip}(H)$.

Proof. The fact that $R^s(\lambda)g$ and $R^d(\lambda)g$ belong to $BUC(H)$ (or $\mathrm{Lip}(H)$) results from Proposition 2.1. We shall show the resolvent equality only for $R^s(\lambda)$, the other equality is similarly proved.

From the definition of $R^s(\mu)g$ we have

$$\begin{aligned}
&R^s(\mu)[(\mu-\lambda)R^s(\lambda)g + g](x)\\
&= \sup_{\alpha\in\Gamma_0}\inf_{z\in N_0}\left\{\int_0^{+\infty} e^{-\mu t}\left\{[g(x(t)) + h(x(t), \alpha[z](t), z(t))]\right.\right.\\
&+(\mu-\lambda)\sup_{\tilde\alpha\in\Gamma_0}\inf_{\tilde z\in N_0}\int_0^{+\infty} e^{-\lambda s}[g(\tilde x(s)) + h(\tilde x(s), \tilde\alpha[\tilde z](s), \tilde z(s))]ds;\\
&\left.\dot{\tilde x}(s) = A\tilde x(s) + f(\tilde\alpha[\tilde z](s), \tilde z(s)), \tilde x(0) = x(t)\right\}dt;\\
&\left.\dot x(t) = Ax(t) + f(\alpha[z](t), z(t)), x(0) = x\right\}.
\end{aligned}$$

Obviously, for each $\alpha \in \Gamma_0$, we have

$$
\begin{aligned}
&R^s(\mu)[(\mu-\lambda)R^s(\lambda)g+g](x)\\
&\geq \inf_{z\in N_0}\left\{\int_0^{+\infty} e^{-\mu t}[g(x(t))+h(x(t),\alpha[z](t),z(t))]\right.\\
&+(\mu-\lambda)\inf_{\tilde{z}\in N_0}\left\{\int_0^{+\infty} e^{-\lambda s}[g(\tilde{x}(s))+h(\tilde{x}(s),\alpha[\tilde{z}](s),\tilde{z}(s))]ds;\right.\\
&\left.\dot{\tilde{x}}(s)=A\tilde{x}(s)+f(\alpha[\tilde{z}](s),\tilde{z}(s)),\tilde{x}(0)=x(t)\right\}dt;\\
&\left.\dot{x}(t)=Ax(t)+f(\alpha[z](t),z(t)),x(0)=x\right\}.
\end{aligned}
$$

We shall prove that for each $\alpha \in \Gamma_0$ we have

$$
\begin{aligned}
&\inf_{z\in N_0}\left\{\int_0^{+\infty} e^{-\mu t}[g(x(t))+h(x(t),\alpha[z](t),z(t))]\right.\\
&+(\mu-\lambda)\inf_{\tilde{z}\in N_0}\left\{\int_0^{+\infty} e^{-\lambda s}[g(\tilde{x}(s))+h(\tilde{x}(s),\alpha[\tilde{z}](s),\tilde{z}(s))]ds;\right.\\
&\left.\dot{\tilde{x}}(s)=A\tilde{x}(s)+f(\alpha[\tilde{z}](s),\tilde{z}(s)),\tilde{x}(0)=x(t)]\right\}dt;\\
&\left.\dot{x}(t)=Ax(t)+f(\alpha[z](t),z(t)),x(0)=x\right\}\\
&\geq \inf_{z\in N_0}\left\{\int_0^{+\infty} e^{-\mu t}[g(x(t))+h(x(t),\alpha[z](t),z(t))]dt;\right.\\
&\left.\dot{x}(t)=Ax(t)+f(\alpha[z](t),z(t)),x(0)=x\right\}.
\end{aligned}
$$

Let us consider the inequality

(2.18)

$$
\begin{aligned}
&\inf_{z\in N_0}\left\{\int_0^{+\infty} e^{-\lambda t}[g(x(t))+h(x(t),\alpha[z](t),z(t))]dt;\right.\\
&\left.\dot{x}(t)=Ax(t)+f(\alpha[z](t),z(t)),x(0)=x\right\}\\
&\leq \int_0^t e^{-\lambda s}[g(x^*(s))+h(x^*(s),\alpha[z^*](s),z^*(s))]ds\\
&+\int_t^{+\infty} e^{-\lambda s}[g(\tilde{x}^*(s-t))+h(\tilde{x}^*(s-t),\alpha[\tilde{z}^*](s-t),\tilde{z}^*(s-t))]ds,
\end{aligned}
$$

where (x^*, z^*), $(\tilde{x}_t^*, \tilde{z}_t^*)$ satisfy (2.14) with initial conditions $x^*(0) = x$, $\tilde{x}_t^*(0) = x^*(t)$ and they are chosen such that

$$
\begin{aligned}
&\inf_{z\in N_0}\Big\{\int_0^{+\infty} e^{-\mu t}[g(x(t)) + h(x(t), \alpha[z](t), z(t))] \\
&+(\mu-\lambda)\inf_{\tilde{z}\in N_0}\Big\{\int_0^{+\infty} e^{-\lambda s}[g(\tilde{x}(s)) + h(\tilde{x}(s), \alpha[\tilde{z}](s), \tilde{z}(s))]ds; \\
&\dot{\tilde{x}}(s) = A\tilde{x}(s) + f(\alpha[\tilde{z}](s), \tilde{z}(s)), \tilde{x}(0) = x(t)\Big\}\Big]dt; \\
&\dot{x}(t) = Ax(t) + f(\alpha[z](t), z(t)), x(0) = x\Big\} \\
&\geq \int_0^{+\infty} e^{-\mu t}[g(x^*(t)) + h(x^*(t), \alpha[z^*](t), z^*(t))]dt \\
&+(\mu-\lambda)\int_0^{+\infty} e^{-\lambda s}[g(\tilde{x}_t^*(s)) + h(\tilde{x}_t^*(s), \alpha[\tilde{z}_t^*](s), \tilde{z}_t^*(s))]dsdt - \varepsilon,
\end{aligned}
\tag{2.19}
$$

where $\varepsilon > 0$ is arbitrarily chosen.
Now, we multiply inequality (2.18) by $e^{-(\mu-\lambda)t}$ and then we integrate it on $[0; +\infty]$. In this way we obtain

$$
\begin{aligned}
&\frac{1}{\mu-\lambda}\inf_{z\in N_0}\Big\{\int_0^{+\infty} e^{-\lambda t}[g(x(t)) + h(x(t), \alpha[z](t), z(t))]dt; \\
&\qquad \dot{x}(t) = Ax(t) + f(\alpha[z](t), z(t)), x(0) = x\Big\} \\
&\leq \frac{1}{\mu-\lambda}\int_0^{+\infty} e^{-\mu t}[g(x^*(t)) + h(x^*(t), \alpha[z^*](t), z^*(t))]dt \\
&\qquad + \int_t^{\infty} e^{-(\mu-\lambda)t}\int_t^{\infty} e^{-\lambda s}[g(\tilde{x}^*(s-t)) \\
&\qquad +h(\tilde{x}^*(s-t), \alpha[\tilde{z}^*](s-t), \tilde{z}^*(s-t))]dsdt.
\end{aligned}
\tag{2.20}
$$

From (2.20) using (2.19) one can easily obtain

$$
\begin{aligned}
&\inf_{z\in N_0}\Big\{\int_0^{+\infty} e^{-\lambda t}[g(x(t)) + h(x(t), \alpha[z](t), z(t))]dt; \\
&\dot{x}(t) = Ax(t) + f(\alpha[z](t), z(t)), x(0) = x\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \inf_{z\in N_0}\Big\{\int_0^{+\infty} e^{-\mu t}[g(x(t))+h(x(t),\alpha[z](t),z(t)))\\
&+(\mu-\lambda)\inf_{\tilde{z}\in N_0}\Big\{\int_0^{+\infty} e^{-\lambda s}[g(\tilde{x}(s))+h(\tilde{x}(s),\alpha[\tilde{z}](s),\tilde{z}(s))]ds;\\
&\dot{\tilde{x}}(s)=A\tilde{x}(s)+f(\alpha[\tilde{z}](s),\tilde{z}(s)),\tilde{x}(0)=x(t)\Big\}]dt;\\
&\dot{x}(t)=Ax(t)+f(\alpha[z](t),z(t)),x(0)=x\Big\}+\varepsilon
\end{aligned}
\tag{2.21}
$$

Taking supremum over all $\alpha\in\Gamma_0$ it results

$$(R^s(\lambda)g)(x)\leq R^s(\mu)[(\mu-\lambda)R^s(\lambda)g+g](x).$$

Now let us demonstrate the opposite inequality.
Obviously, for each $\beta\in\Delta_0$ we have

$$
\begin{aligned}
&R^d(\mu)[(\mu-\lambda)R^d(\lambda)g+g](x)\\
&=\inf_{\beta\in\Delta_0}\sup_{y\in M_0}\Big\{\int_0^{+\infty} e^{-\mu t}[g(x(t))+h(x(t),y(t),\beta[y](t))\\
&+(\mu-\lambda)\inf_{\tilde{\beta}\in\Delta_0}\sup_{\tilde{y}\in M_0}\Big\{\int_0^{+\infty} e^{-\lambda s}[g(\tilde{x}(s))+h(\tilde{x}(s),\tilde{y}(s),\tilde{\beta}[\tilde{y}](s))]ds;\\
&\dot{\tilde{x}}(s)=A\tilde{x}(s)+f(\tilde{y}(s),\tilde{\beta}[\tilde{y}](s)),\tilde{x}(0)=x(t)\Big\}]dt;\\
&\dot{x}(t)=Ax(t)+f(y(t),\beta[y](t)),x(0)=x\Big\}\\
&\leq\sup_{y\in M_0}\Big\{\int_0^{+\infty} e^{-\mu t}[g(x(t))+h(x(t),y(t),\beta[y](t))\\
&+(\mu-\lambda)\sup_{\tilde{y}\in M_0}\Big\{\int_0^{+\infty} e^{-\lambda s}[g(\tilde{x}(s))+h(\tilde{x}(s),\tilde{y}(s),\beta[\tilde{y}](s))]ds;\\
&\dot{\tilde{x}}(s)=A\tilde{x}(s)+f(\tilde{y}(s),\beta[\tilde{y}](s)),\tilde{x}(0)=x(t)\}]dt;\\
&\dot{x}(t)=Ax(t)+f(y(t),\beta[y](t)),x(0)=x\Big\}
\end{aligned}
$$

$$= -\inf_{y\in M_0}\Big\{\int_0^{+\infty} e^{-\lambda s}[(-g)(x(t))+(-h)(x(t),y(t),\beta[y](t))$$
$$+(\mu-\lambda)\inf_{\tilde{y}\in M_0}\Big\{\int_0^{+\infty} e^{-\lambda s}[(-g)(\tilde{x}(s))+(-h)(\tilde{x}(s),\tilde{y}(s),\beta[\tilde{y}](s))]ds;$$
$$\dot{\tilde{x}}(s)=A\tilde{x}(s)+f(\tilde{y}(s),\beta[\tilde{y}](s)),\tilde{x}(0)=x(t)\Big\}]dt;$$
$$\dot{x}(t)=Ax(t)+f(y(t),\beta[y](t)),x(0)=x\Big\}.$$

Now, taking into account the relation (2.21) with $(-g)$ and $(-h)$ instead of g and h, it results

$$\sup_{y\in M_0}\Big\{\int_0^{+\infty} e^{-\mu t}[g(x(t))+h(x(t),y(t),\beta[y](t))$$
$$+(\mu-\lambda)\sup_{\tilde{y}\in M_0}\Big\{\int_0^{+\infty} e^{-\lambda s}[g(\tilde{x}(s))+h(\tilde{x}(s),\tilde{y}(s),\beta[\tilde{y}](s))]ds;$$
$$\dot{\tilde{x}}(s)=A\tilde{x}(s)+f(\tilde{y}(s),\beta[\tilde{y}](s)),\tilde{x}(0)=x(t)\Big\}]dt;$$
$$\dot{x}(t)=Ax(t)+f(y(s),\beta[y](s)),x(0)=x\Big\}$$
$$\leq \sup_{y\in M_0}\Big\{\int_0^{+\infty} e^{-\mu t}[g(x(t))+h(x(t),y(t),\beta[y](t))]dt;$$
$$\dot{x}(t)=Ax(t)+f(y(t),\beta[y](t)),x(0)=x\Big\}+\varepsilon,$$

for ε arbitrarily chosen.

Taking infimum over all $\beta\in\Delta_0$ we obtain

$$R^d(\mu)[(\mu-\lambda)R^d(\lambda)g+g](x)\leq(R^d(\lambda)g)(x),\ \text{for every } x\in H.$$

We make the remark that if $R^s(\lambda)g=R^d(\lambda)g$ for every $g\in BUC(H)$, then the proof is finished.

So, we need of the following lemma:

Lemma 2.1. *In the hypotheses* (1.2), (1.3) *we have*

$$R^s(\lambda)g=R^d(\lambda)g,\quad \textit{for every } g\in BUC(H) \textit{ and } \lambda>0.$$

Proof. Let $A_n x = \dfrac{e^{\frac{1}{n}A}x - x}{\frac{1}{n}}$, $x \in H$.

We shall denote by $x_n(\cdot)$ the solution of (2.1) where we consider A_n instead of A and we shall consider the same differential game with the state $x_n(\cdot)$. First of all we shall prove that $R_n^s(\lambda)g$ and $R_n^d(\lambda)g$ are viscosity solutions for a certain Hamilton–Jacobi equation. Then using a uniqueness result of M.G. Crandall and P.L. Lions we shall obtain that $R_n^s(\lambda)g = R_n^d(\lambda)g$. Making $n \to +\infty$ it will result $R^s(\lambda)g = R^d(\lambda)g$.

Let us consider the following Hamilton–Jacobi equation

$$\begin{aligned}&\lambda V(x) - \max_{y\in Y}\min_{z\in Z}\{(f(y,z), V_x(x)) + (A_n x, V_x(x))\\ &+h(x,y,z) + g(x)\} = 0,\ x \in H; \lambda > 0.\end{aligned} \tag{2.22}$$

Let

$$\begin{aligned}&(R_n^s(\lambda)g)(x) = \sup_{\alpha\in\Gamma_0}\inf_{z\in N_0}\left\{\int_0^{+\infty} e^{-\lambda t}[g(x_n(t)) + h(x_n(t), \alpha[z](t),\right.\\ &\left. z(t))]dt; \dot{x}(t) = A_n x(t) + f(\alpha[z](t), z(t)), x(0) = x\right\}.\end{aligned} \tag{2.23}$$

We shall prove only that $R_n^s(\lambda)g$ is a viscosity solution for (2.22). To prove that $R_n^d(\lambda)g$ is also a viscosity solution for (2.22) is quite similar.
Let $\varphi \in C^1(H)$ and let $x_0 \in H$ be a local maximum point for $(R_n^s(\lambda)g - \varphi)$. We shall prove that

$$\begin{aligned}&\lambda(R_n^s(\lambda)g)(x_0) - \max_{y\in Y}\min_{z\in Z}\{(f(y,z), \varphi_x(x_0)) + (A_n(x_0), \varphi_x(x_0))\\ &+h(x_0,y,z) + g(x_0)\} \le 0.\end{aligned}$$

If this is not true, then there exists some $\theta > 0$ such that

$$\begin{aligned}&\lambda(R_n^s(\lambda)g)(x_0) - \max_{y\in Y}\min_{z\in Z}\{(f(y,z), \varphi_x(x_0)) + (A_n(x_0), \varphi_x(x_0))\\ &+h(x_0,y,z) + g(x_0)\} \ge \theta > 0.\end{aligned}$$

Equivalently

$$\begin{aligned}&-\lambda(R_n^s(\lambda)g)(x_0) + \max_{y\in Y}\min_{z\in Z}\{(f(y,z), \varphi_x(x_0)) + (A_n(x_0), \varphi_x(x_0))\\ &+h(x_0,y,z) + g(x_0)\} \le -\theta < 0.\end{aligned} \tag{2.24}$$

Let

$$\begin{aligned}\Lambda(x,y,z) &= -\lambda(R_n^s(\lambda)g)(x) + \{(f(y,z),\varphi_x(x)) \\ &\quad +(A_n(x),\varphi_x(x)) + h(x,y,z) + g(x)\}.\end{aligned}$$

Using this notation and max-min condition (1.3) we may write (2.24) as

$$\max_{y\in Y}\min_{z\in Z}\Lambda(x_0,y,z) = \min_{z\in Z}\max_{y\in Y}\Lambda(x_0,y,z) \le -\theta < 0.$$

Therefore there exists $z^* \in Z$ such that

$$\max_{y\in Y}\Lambda(x_0,y,z^*) \le -\theta.$$

Since Λ is uniformly continuous we have

$$\max_{y\in Y}\Lambda(x_n(t),y,z^*) \le -\frac{\theta}{2} < 0$$

for $0 \le t \le \sigma$, where $\sigma > 0$ is small enough and $x_n(\cdot)$ is the solution of (2.14) with A_n instead of A and $x_n(0) = x_0$. Hence, for $z(\cdot) \equiv z^*$ and any $\alpha \in \Gamma_0$ we have

$$\begin{aligned}&-\lambda(R_n^s(\lambda)g)(x_n(t)) + (A_n x_n(t),\varphi_x(x_n(t)) + (f(\alpha[z](t),z(t)),\\ &\varphi_x(x_n(t))) + h(x_n(t),\alpha[z](t),z(t)) + g(x_n(t)) \le -\frac{\theta}{2}.\end{aligned} \tag{2.25}$$

We multiply (2.25) by $e^{-\lambda t}$ and integrate from 0 to σ to obtain

$$-\lambda\int_0^\sigma e^{-\lambda t}(R_n^s(\lambda)g)(x_n(t))dt + \int_0^\sigma e^{-\lambda t}[(A_n x_n(t),\varphi_x(x_n(t))$$

$$+(f(\alpha[z](t),z(t)),\varphi_x(x_n(t)))]dt + \int_0^\sigma e^{-\lambda t}h(x_n(t),\alpha[z](t),z(t))dt$$

$$+\int_0^\sigma e^{-\lambda t}g(x_n(t))dt \le -\frac{\theta}{2\lambda}(1-e^{-\lambda\sigma}).$$

Taking into account (2.14) it results

$$-\lambda\int_0^\sigma e^{-\lambda t}(R_n^s(\lambda)g)(x_n(t))dt + \lambda\int_0^\sigma e^{-\lambda t}\varphi(x_n(t))dt - \lambda\int_0^\sigma \varphi(x_n(t))dt$$

$$+\int_0^\sigma e^{-\lambda t}\frac{d}{dt}\varphi(x_n(t))dt+\int_0^\sigma e^{-\lambda t}h(x_n(t),\alpha[z](t),z(t))dt$$
$$+\int_0^\sigma e^{-\lambda t}g(x_n(t))dt\le-\frac{\theta}{2\lambda}(1-e^{-\lambda\sigma}).$$

Hence

(2.26)
$$-\lambda\int_0^\sigma e^{-\lambda t}[(R_n^s(\lambda)g)(x_n(t))-\varphi(x_n(t))]dt+e^{-\lambda\sigma}\varphi(x_n(\sigma))-\varphi(x_0)$$
$$+\int_0^\sigma e^{-\lambda t}[h(x_n(t),\alpha[z](t),z(t))+g(x_n(t))]dt\le-\frac{\theta}{2\lambda}(1-e^{-\lambda\sigma}).$$

Using the fact that x_0 is a maximum point for $R_n^s(\lambda)g-\varphi$ we obtain

$$(R_n^s(\lambda)g)(x_0)-\varphi(x_0)\ge(R_n^s(\lambda)g)(x_n(t))-\varphi(x_n(t)),\ t\ge0. \tag{2.27}$$

From (2.26) and (2.27) we get

$$-\lambda\int_0^\sigma e^{-\lambda t}[(R_n^s(\lambda)g)(x_0)-\varphi(x_0)]dt+e^{-\lambda\sigma}\varphi(x_n(\sigma))-\varphi(x_0)$$
$$+\int_0^\sigma e^{-\lambda t}[h(x_n(t),\alpha[z](t),z(t))+g(x_n(t))]dt\le-\frac{\theta}{2\lambda}(1-e^{-\lambda\sigma}),$$

i.e.

$$(e^{-\lambda\sigma}-1)(R_n^s(\lambda)g)(x_0)-(e^{-\lambda\sigma}-1)\varphi(x_0)+e^{-\lambda\sigma}(R_n^s(\lambda)g)(x_n(\sigma))$$
$$-e^{-\lambda\sigma}(R_n^s(\lambda)g)(x_0)+e^{-\lambda\sigma}\varphi(x_0)-\varphi(x_0)$$
$$+\int_0^\sigma e^{-\lambda t}[h(x_n(t),\alpha[z](t),z(t))+g(x_n(t))]dt\le-\frac{\theta}{2\lambda}(1-e^{-\lambda\sigma}).$$

Therefore

$$\frac{\theta}{2\lambda}(1-e^{-\lambda\sigma})\le(R_n^s(\lambda)g)(x_0)$$
$$-\left\{\int_0^\sigma e^{-\lambda t}[h(x_n(t),\alpha[z](t),z(t))+g(x_n(t))]dt+e^{-\lambda\sigma}(R_n^s(\lambda)g)(x_n(\sigma))\right\}.$$

Taking now infimum for $z\in N_0$ we obtain

$$0<\frac{\theta}{2\lambda}(1-e^{-\lambda\sigma})\le(R_n^s(\lambda)g)(x_0)$$

$$-\inf_{z\in N_0}\left\{\int_0^\sigma e^{-\lambda t}[h(x_n(t),\alpha[z](t),z(t))+g(x_n(t))]dt+e^{-\lambda\sigma}(R_n^s(\lambda)g)(x_n(\sigma))\right\},$$

for every $\alpha\in\Gamma_0$. Taking now supremum for $\alpha\in\Gamma_0$ and using Theorem 2.1 it results

$$\frac{\theta}{2\lambda}(1-e^{-\lambda\sigma})\leq 0,$$

but $\sigma>0$ and so the assumption (2.24) is false.
We shall prove now that $R^s(\lambda)g$ is a viscosity supersolution.
Let $\varphi\in C^1(H)$ and let $x_0\in H$ be a local minimum for $(R_n^s(\lambda)g-\varphi)$.
If the inequality condition for supersolution is not true in x_0 then there exists $\theta>0$ such that

$$-\lambda(R_n^s(\lambda)g)(x_0)+\max_{y\in Y}\min_{z\in Z}\{(f(y,z),\varphi_x(x_0))$$

$$+(A_nx_0,\varphi_x(x_0))+h(x_0,y,z)+g(x_0)\}\geq\theta>0,$$

i.e.

$$\max_{y\in Y}\min_{z\in Z}\Lambda(x_0,y,z)\geq\theta>0.$$

Using the method from [8], we obtain that for each $z\in Z$, there exists $y(z)\in Y$ such that

$$\Lambda(x_0,y(z),z)\geq\frac{\theta}{2}.$$

Using again the uniform continuity of Λ we get

$$\Lambda(x_0,y(z),u)\geq\frac{\theta}{4},$$

for $u\in V_z$ where V_z is a certain open neighborhood of z .
Because Z is compact we can find some distinct points $z_1,...,z_n\in Z$ to which correspond the points $y_1,...y_n\in Y$ such that

$$Z\subset\bigcup_{i=1}^n V_{z_i}$$

and

$$\Lambda(x_0,y_i,u)\geq\frac{\theta}{4}\ \text{ for } u\in V_{z_i}.$$

Let us define now

$$\varphi:Z\to Y$$

by

$$\varphi(z) = y_i$$

if

$$z \in V_{z_i} \setminus \bigcup_{k=1}^{i-1} V_{z_k}, \quad i = 1, 2, ..., n.$$

Therefore

$$\Lambda(x_0, \varphi(z), z) \geq \frac{\theta}{4} \text{ for all } z \in Z.$$

But Λ is uniformly continuous and so we have for each sufficiently small $\sigma > 0$

$$\Lambda(x_n(s), \varphi(z), z) \geq \frac{\theta}{8} \tag{2.28}$$

for all $z \in Z$, $0 \leq s \leq \sigma$, where $x_n(\cdot)$ is the solution on (2.14) with $y(\cdot), z(\cdot)$ arbitrarily chosen and $x_n(0) = x_0$.
Now we define $\alpha \in \Gamma_0$ by

$$\alpha[z](s) = \varphi(z(s))$$

for each $z \in N_0$, $s \geq 0$.
Using (2.28) we obtain

$$\Lambda(x_n(s), \alpha[z](s), z(s)) \geq \frac{\theta}{8}, \quad 0 \leq s \leq \sigma, \tag{2.29}$$

for every $z \in N_0$.
We multiply (2.27) by $e^{-\lambda t}$ and integrate from 0 to σ to obtain

$$-\lambda \int_0^\sigma e^{-\lambda t}[(R_n^s(\lambda)g)(x_n(t))dt + \int_0^\sigma e^{-\lambda t}[(A_n x_n(t), \varphi_x(x_n(t)))$$

$$+(f(\alpha[z](t), z(t)), \varphi_x(x_n(t)))]dt + \int_0^\sigma e^{-\lambda t} h(x_n(t), \alpha[z](t), z(t))dt$$

$$+ \int_0^\sigma e^{-\lambda t} g(x_n(t))dt \geq \frac{\theta}{2\lambda}(1 - e^{-\lambda\sigma}).$$

Using again (2.14) we get

$$-\lambda \int_0^\sigma e^{-\lambda t}(R_n^s(\lambda)g)(x_n(t))dt + \lambda \int_0^\sigma e^{-\lambda t}\varphi(x_n(t))dt - \lambda \int_0^\sigma e^{-\lambda t}\varphi(x_n(t))dt$$

$$+\int_0^\sigma e^{-\lambda t}\frac{d}{dt}\varphi(x_n(t))dt+\int_0^\sigma e^{-\lambda t}h(x_n(t),\alpha[z](t),z(t))dt$$
$$+\int_0^\sigma e^{-\lambda t}g(x_n(t))dt\geq\frac{\theta}{8\lambda}(1-e^{-\lambda\sigma}).$$

Therefore

$$-\lambda\int_0^\sigma e^{-\lambda t}[(R_n^s(\lambda)g)(x_n(t))-\varphi(x_n(t))]dt$$
$$+e^{-\lambda\sigma}\varphi(x_n(\sigma))-\varphi(x_0)+\int_0^\sigma e^{-\lambda t}[h(x_n(t),\alpha[z](t),z(t)) \tag{2.30}$$
$$+g(x_n(t))]dt\geq\frac{\theta}{8\lambda}(1-e^{-\lambda\sigma}).$$

Taking into account that x_0 is a minimum point for $(R^s(\lambda)g-\varphi)$ and using (2.30) it results after some calculations

$$(e^{-\lambda\sigma}-1)(R_n^s(\lambda)g)(x_0)-(e^{-\lambda\sigma}-1)\varphi(x_0)+e^{-\lambda\sigma}(R_n^s(\lambda)g)(x_n(\sigma))$$
$$-e^{-\lambda\sigma}(R_n^s(\lambda)g)(x_0)+e^{-\lambda\sigma}\varphi(x_0)-\varphi(x_0)$$
$$+\int_0^\sigma e^{-\lambda\sigma}[h(x_n(t),\alpha[z](t),z(t))+g(x_n(t))]dt\geq\frac{\theta}{8\lambda}(1-e^{-\lambda\sigma}),$$

i.e.

$$(R_n^s(\lambda)g)(x_0)-\left\{\int_0^\sigma e^{-\lambda\sigma}[h(x_n(t),\alpha[z](t),z(t))+g(x_n(t))]dt\right.$$
$$\left.+e^{-\lambda\sigma}(R_n^s(\lambda)g)(x_n(\sigma))\right\}\leq\frac{\theta}{8\lambda}(1-e^{-\lambda\sigma}),$$

for each $z\in N_0$.
Therefore

$$(R_n^s(\lambda)g)(x_0)-\inf_{z\in N_0}\left\{\int_0^\sigma e^{-\lambda\sigma}[h(x_n(t),\alpha[z](t),z(t))+g(x_n(t))]dt\right.$$
$$\left.+e^{-\lambda\sigma}(R_n^s(\lambda)g)(x_n(\sigma))\right\}\leq-\frac{\theta}{8\lambda}(1-e^{-\lambda\sigma}).$$

Taking now supremum over all $\alpha\in\Gamma_0$ and using again Theorem 2.1 we obtain

$$(R_n^s(\lambda)g)(x_0)-(R_n^s(\lambda)g)(x_0)\leq-\frac{\theta}{8\lambda}(1-e^{-\lambda\sigma})$$

which is a contradiction. Hence $(R_n^s(\lambda)g)(\cdot)$ and $(R_n^d(\lambda)g)(\cdot)$ are viscosity solutions of (2.22).
Calling now Theorem 1 [5] we get

$$R_n^s(\lambda)g = R_n^d(\lambda)g \text{ for every } n \in N.$$

We shall prove now that

$$\lim_{n\to+\infty}(R_n^s(\lambda)g)(x) = (R^s(\lambda)g)(x), \text{ for } x \in H, \tag{2.31}$$

$$\lim_{n\to+\infty}(R_n^d(\lambda)g)(x) = (R^d(\lambda)g)(x), \text{ for } x \in H, \tag{2.32}$$

We prove only (2.31) because (2.32) is proved similarly.
Using the hypotheses on h, g, f and A we have

$$\begin{aligned}
&|(R^s(\lambda)g)(x) - (R_n^s(\lambda)g)(x)| \\
&= \Big|\sup_{\alpha\in\Gamma_0}\inf_{z\in N_0}\int_0^{+\infty} e^{-\lambda t}[h(x(t),\alpha[z](t),z(t)) + g(x(t))]dt \\
&\quad - \sup_{\alpha\in\Gamma_0}\inf_{z\in N_0}\int_0^{+\infty} e^{-\lambda t}[h(x_n(t),\alpha[z](t),z(t)) + g(x_n(t))]dt\Big| \\
&\le \int_0^{+\infty} e^{-\lambda t}[\omega_h|x(t)-x_n(t)| + \omega_g(|x(t)-x_n(t)|)]dt \\
&\le \frac{1}{\lambda}\left(\omega_h\left(\sup_{\substack{t\in[0;T]\\ \alpha\in\Gamma_0, z\in N_0}}|x(t)-x_n(t)|\right)\right. \\
&\quad \left.+\omega_g\left(\sup_{\substack{t\in[0;T]\\ \alpha\in\Gamma_0, z\in N_0}}|x(t)-x_n(t)|\right)\right) + c(T),
\end{aligned} \tag{2.33}$$

where $c(T) \ge 0$, $\lim_{T\to+\infty} c(T) = 0$, and

$$x(t) = e^{At}x + \int_0^t e^{A(t-s)}f(\alpha[z](s), z(s))ds, \ t \in [0;T],$$

$$x_n(t) = e^{A_n t}x + \int_0^t e^{A_n(t-s)}f(\alpha[z](s), z(s))ds, \ t \in [0;T].$$

Therefore

$$|x(t)-x_n(t)| \le |e^{At}x - e^{A_n t}x|$$

$$+\int_0^t |[e^{A(t-s)}f(\alpha[z](s),z(s)) - e^{A_n(t-s)}f(\alpha[z](s),z(s))]|ds,\ t\in[0;T].$$

We shall prove that

$$\lim_{n\to+\infty}\int_0^t |(e^{As}-e^{A_n s})f(y,z)|ds = 0 \tag{2.34}$$

uniformly with respect to $(y,z)\in Y\times Z$, $t\in[0,T]$.
Taking into account that Af is continuous we obtain

$$\lim_{n\to+\infty}|(e^{As}-e^{A_n s})f(y,z)| = 0$$

uniformly for $s\in[0,T]$ and $(y,z)\in Y\times Z$ and therefore (2.34).
Indeed, following [11, Theorem 8.1] we have

$$|(e^{sA}-e^{sA_n})f(y,z)| \le TM^2|Af(y,z)-A_nf(y,z)|,\ s\in[0,T].$$

Let us suppose now that there exists $\varepsilon>0$ such that there exists $\{(y_k,z_k)\}_{k\in N}\subseteq Y\times Z$ such that

$$|Af(y_k,z_k)-A_kf(y_k,z_k)|\ge\varepsilon. \tag{2.35}$$

Because Af is continuous and $Y\times Z$ is compact we may suppose that $(y_k,z_k)\to(\overline{y},\overline{z})$, $Af(y_k,z_k)\to Af(\overline{y},\overline{z})$.
Moreover,

$$k(e^{\frac{1}{k}A}f(y_k,z_k)-f(y_k,z_k)) = k\int_0^{\frac{1}{k}} e^{sA}f(y_k,z_k)ds$$

$$= e^{\xi_k A}Af(y_k,z_k),$$

where $0<\xi_k<\dfrac{1}{k}$. Therefore making $k\to+\infty$ in (2.35) we obtain a contradiction.
In this way it results

$$|x(t)-x_n(t)|\le\ell_n(x),\ t\in[0,T], \tag{2.36}$$

where $\ell_n(x)\underset{n\to+\infty}{\longrightarrow}0$ and $\ell_n(x)$ is independent of $(y,z)\in Y\times Z$.

Now, using (2.33), (2.36) and the fact that $c(T)$ is independent of n, one can easily obtain

$$\lim_{n\to+\infty} |(R^s(\lambda)g)(x) - (R^s_n(\lambda)g)(x)| \leq c(T), \text{ for } T > 0,$$

i.e.

$$\lim_{n\to+\infty} (R^s_n(\lambda)g)(x) = (R^s(\lambda)g)(x), \text{ for every } x \in H.$$

So, the proof of Lemma 2.1 is finished.
In the following denote by $R(\lambda)g$ the function $R^s(\lambda)g = R^d(\lambda)g$.

3. The semigroup approach of Eq. (1.1) Let $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset BUC(H) \to BUC(H)$ be the operator defined by (see [2])

$$\mathcal{A}R(1)g = R(1)g - g \text{ for each } g \in BUC(H) \tag{3.1}$$

with the domain

$$\mathcal{D}(\mathcal{A}) = \{\varphi = R(1)g;\ g \in BUC(H)\}.$$

Proposition 3.1. *The operator $\mathcal{A}$ is $m-$dissipative on $BUC(H) \times BUC(H)$ and*

$$(\lambda I - \mathcal{A})^{-1} = R(\lambda) \text{ for } \lambda > 0.$$

Proof. From Proposition 2.2 (ii), for $\mu = 0$, $0 < \lambda \leq 1$ it results

$$R(\lambda)g = R(1)[(1-\lambda)R(\lambda)g + g],\ 0 < \lambda \leq 1,\ g \in BUC(H). \tag{3.2}$$

We have to prove that

$$(\lambda I - \mathcal{A})R(\lambda) = I. \tag{3.3}$$

But (3.3) is equivalent with

$$\lambda R(\lambda)g - g = \mathcal{A}R(\lambda)g,\ g \in BUC(H)$$

and from (3.2) we have

$$\mathcal{A}R(\lambda)g = \mathcal{A}R(1)[(1-\lambda)R(\lambda)g + g]$$

$$= R(1)[(1-\lambda)R(\lambda)g + g] - (1-\lambda)R(\lambda)g - g = \lambda R(\lambda)g - g.$$

Therefore

$$(\lambda I - \mathcal{A})^{-1} = R(\lambda),\ 0 < \lambda \leq 1.$$

On the other hand, from the definition of $R(\lambda)g$, taking into account that

$$\left| \int_0^{+\infty} [g_1(x(s)) + h(x(s), \alpha[z](s), z(s))]ds \right.$$

$$\left. - \int_0^{+\infty} e^{-\lambda s}[g_2(x(s)) + h(x(s), \alpha[z](s), z(s))]ds \right|$$

$$\leq \lambda^{-1}\|g_1 - g_2\|_b,\ \text{for every } \lambda > 0 \text{ and } g_1, g_1 \in BUC(H),$$

it results

$$\|R(\lambda)g_1 - R(\lambda)g_2\|_b \leq \lambda^{-1}\|g_1 - g_2\|_b$$

for every $\lambda > 0$ and $g_1, g_1 \in BUC(H)$.
Therefore $\mathcal{A}$ is $m-$dissipative (see [1]).
Then, from the resolvent equality it results

(3.4) $$\mathcal{A}R(\lambda)g = \mathcal{A}R(\mu)[(\mu - \lambda)R(\lambda)g + g] \text{ with } \lambda \in (0, 1] \text{ and } \mu > 1.$$

Because $(\lambda I - \mathcal{A})^{-1} = R(\lambda)$, $0 < \lambda \leq 1$, we have

(3.5) $$g = \lambda R(\lambda)g - \mathcal{A}R(\lambda)g,\ \mathcal{D}(\mathcal{A}) = \{R(\lambda)g;\ g \in BUC(H)\}$$

with $0 < \lambda \leq 1$.
From the relations (3.4), (3.5) and using again the resolvent equality, we easily obtain

(3.6) $$\mathcal{A}R(\mu)[(\mu-\lambda)R(\lambda)g+g] = \mu R(\mu)[(\mu-\lambda)R(\lambda)g+g]-(\mu-\lambda)R(\lambda)g-g.$$

But $(\mu - \lambda)R(\lambda) + I : BUC(H) \to BUC(H)$ is surjective because the equation

(3.7) $$(\mu - \lambda)(\lambda I - \mathcal{A})^{-1}g + g = v$$

has a solution $g \in BUC(H)$ for every $v \in BUC(H)$.
To prove this fact we use (3.5) to obtain

$$\mu R(\lambda)g - g - \mathcal{A}R(\lambda)g + g = v$$

or

$$(\mu I - \mathcal{A})R(\lambda)g = v,$$

but $\mathcal{A}$ is $m-$dissipative and so the equation (3.7) has a solution $g \in BUC(H)$. Thus we have obtained that

$$\mathcal{A}R(\lambda)g = \mu R(\mu)g - g, \text{ for every } g \in BUC(H)$$

i.e.

$$R(\mu)g = (\mu I - \mathcal{A})^{-1}, \text{ for each } \mu > 0.$$

Let us define the function

$$(3.8) \quad \begin{aligned}(S(t)\varphi_0)(x) = \varphi(t,x) = \sup_{\alpha\in\Gamma_0}\inf_{z\in N_0}\Big\{\int_0^t [g(x(s)) \\ +h(x(s),\alpha[z](s),z(s))]ds + \varphi_0(x(t))\Big\}\end{aligned}$$

where $x(\cdot)$ is the solution of (2.14).

Remark 3.1. In the relation (3.8) the infimum may be taken over all measurable controls $z : [0;t] \to Z$ and the supremum may be taken over all strategies defined on the set of measurable controls $z : [0;t] \to Z$.

Proposition 3.2. *$S(t)$ maps $BUC(H)$ into itself and*

$$(3.9) \quad S(t)S(s)\varphi_0 = S(t+s)\varphi_0, \text{ for every } t,s,\geq 0, \text{ and } \varphi_0 \in BUC(H)$$

$$(3.10) \quad \|S(t)\varphi_0 - S(t)\tilde{\varphi}_0\|_b \leq \|\varphi_0 - \tilde{\varphi}_0\|_b, \text{ for every } t \geq 0, \varphi_0, \tilde{\varphi}_0 \in BUC(H).$$

Proof. Relation (3.9) is obtained in the same way as (2.5) and (2.6) from Theorem 2.1 (Dynamic Programming Principle) and relation (3.10) results directly from the expression (3.8) of $S(t)$.
In the other words, $S(t)$ is a semigroup of nonlinear contractions on $BUC(H)$.

Proposition 3.3. *Let $\mathcal{E} = \{g \in BUC(H) \cap \mathrm{Lip}(H);\ |g(x(t)) - g(x)| \leq C_g \cdot t$ for $t \in [0;1], x(\cdot)$ being the solution of (2.14) for $\alpha \in \Gamma_0$ and $z \in N_0\}$. Then $\overline{\mathcal{E}} = \overline{\mathcal{D}(\mathcal{A})}$.*

Proof. We shall prove that for $g \in \mathcal{E}$

$$(I - \mu\mathcal{A})^{-1} g \to g \text{ in } BUC(H) \text{ as } \mu \to 0.$$

Indeed, for $g \in \mathcal{E}$ we have

$$((I - \mu\mathcal{A})^{-1} g)(x) = (R(\mu^{-1})(\mu^{-1} g))(x)$$

$$= \sup_{\alpha \in \Gamma_0} \inf_{z \in N_0} \left\{ \int_0^{+\infty} e^{-\mu^{-1} t} \left[\frac{1}{\mu} g(x(t)) + h(x(t), \alpha[z](t), z(t)) \right] dt; \right.$$

$$\left. x(\cdot) \text{ verifies (2.14)} \right\}.$$

Then there exists $\alpha_\mu \in \Gamma_0$ such that

$$((I-\mu\mathcal{A})^{-1} g)(x) \leq \left\{ \int_0^{+\infty} e^{-\mu^{-1} t} \left[\frac{1}{\mu} g(x_\mu^1(t)) + h(x_\mu^1(t), \alpha_\mu[z](t), z(t)) \right] dt; \right.$$

$$\left. x_\mu^1(\cdot) \text{ verifies (2.14) with } \alpha = \alpha_\mu \right\} + \mu, \text{ for every } z \in N_0.$$

In the same manner, there exists $z_\mu \in N_0$ such that

$$((I-\mu\mathcal{A})^{-1} g)(x) \geq \left\{ \int_0^{+\infty} e^{-\mu^{-1} t} \left[\frac{1}{\mu} g(x_\mu^2(t)) + h(x_\mu^2(t), \alpha[z_\mu](t), z_\mu(t)) \right] dt; \right.$$

$$\left. x_\mu^2(\cdot) \text{ verifies (2.14) with } z = z_\mu \right\} - \mu, \text{ for every } \alpha \in \Gamma_0.$$

Therefore, there exists α_μ, z_μ such that

(3.11)

$$\int_0^{+\infty} e^{-\mu^{-1} t} \left[g(x_\mu(t)) + h(x_\mu(t), \alpha_\mu[z_\mu](t), z_\mu(t)) \right] dt - \mu$$

$$\leq ((I - \mu\mathcal{A})^{-1} g)(x) \leq \int_0^{+\infty} \left[\frac{1}{\mu} g(x_\mu(t)) + h(x_\mu(t), \alpha_\mu[z_\mu](t), z_\mu(t)) \right] dt$$

$$+\mu, \text{ where } x_\mu^1(\cdot) \text{ verifies (2.14) with } z = z_\mu \text{ and } \alpha = \alpha_\mu.$$

From (3.11) it results

$$|((I - \mu^{-1}\mathcal{A})^{-1} g)(x) - g(x)|$$

$$\leq \left| \int_0^{+\infty} e^{-\mu^{-1}t} \left[\frac{1}{\mu}(g(x_\mu(t)) - g(x)) + h(x_\mu(t), \alpha_\mu[z_\mu](t), z_\mu(t)) \right] dt \right| + \mu$$

$$\leq \int_0^1 \frac{1}{\mu} e^{-\mu^{-1}t} C_g t dt + \int_1^{+\infty} e^{-\mu^{-1}t} \frac{1}{\mu} 2M_g dt$$

$$+ \int_0^{+\infty} e^{-\mu^{-1}t} M_h dt + \mu = (e^{-\mu^{-1}} - \mu e^{-\mu^{-1}} + \mu) C_g$$

$$+ e^{-\mu^{-1}} 2M_g + \mu M_h + \mu = \gamma(\mu),$$

for every $x \in H$.
So, we have obtained

$$\|((I - \mu\mathcal{A})^{-1} g)(\cdot) - g(\cdot)\|_b \leq \gamma(\mu)$$

with $\gamma(\mu) \to 0$ as $\mu \to 0$.
Using the fact that $(I - \mu^{-1}\mathcal{A})^{-1} g$ belongs to $\mathcal{D}(\mathcal{A})$, it results that $f \in \overline{\mathcal{D}}(\mathcal{A})$.
Hence $\mathcal{E} \subset \overline{\mathcal{D}(\mathcal{A})}$.
Now we shall prove that

$$(I - \mu\mathcal{A})^{-1} g \in \mathcal{E} \text{ for } \mu > 0 \text{ and } g \in BUC(H) \cap \mathrm{Lip}(H).$$

Let $\mu > 0$ and $g \in BUC(H) \cap \mathrm{Lip}(H)$.
From Proposition 2.2 we have that $(I - \mu\mathcal{A})^{-1} g$ belongs to $BUC(H) \cap \mathrm{Lip}(H)$. For every $\varepsilon > 0$ there are $\alpha_\varepsilon \in \Gamma_0$ and $z_\varepsilon \in N_0$ such that

$$\left| (R(\mu^{-1})(\mu^{-1} g))(x) - \int_0^{+\infty} e^{-\mu^{-1}t} [g(x_\varepsilon(t)) \right.$$
$$\left. + h(x_\varepsilon(t), \alpha_\varepsilon[z_\varepsilon](t), z_\varepsilon(t))] dt \right| \leq \varepsilon. \tag{3.12}$$

Let $x(\cdot)$ be a solution of (2.14) for some $\alpha \in \Gamma_0$ and $z \in N_0$.
Using the properties of f we obtain

$$|x(t) - x_\varepsilon(t)| \leq 2M_f \cdot t, \quad \text{for } t \geq 0, \tag{3.13}$$

where $x(\cdot)$ and $x_\varepsilon(\cdot)$ solves (2.14) for (α, z) and $(\alpha_\varepsilon, z_\varepsilon)$, respectively.
Therefore

$$|(R(\mu^{-1})(\mu^{-1} g))(x(t)) - (R(\mu^{-1})(\mu^{-1} g))(x_\varepsilon(t))| \leq ct \text{ for } t \geq 0, \tag{3.14}$$

where c is a positive constant.
Now from Theorem 2.1 we have

$$R(\mu^{-1})(\mu^{-1}g))(x) = \sup_{\alpha\in\Gamma_0}\inf_{z\in N_0}\left\{\int_0^t e^{-\mu^{-1}s}\left[\frac{1}{\mu}g(x(s)) + h(x(s),\alpha[z](s),z(s))\right]ds + e^{-\mu^{-1}t}(R(\mu^{-1})(\mu^{-1}g))(x(t))\right\}. \tag{3.15}$$

From (3.13) and (3.15) it results that for each $\varepsilon > 0$ there exists $(\alpha_\varepsilon, z_\varepsilon)$ such that

$$\left|(R(\mu^{-1})(\mu^{-1}g))(x) - e^{-\mu^{-1}t}(R(\mu^{-1})(\mu^{-1}g))(x_\varepsilon(t)) - \int_0^t e^{-\mu^{-1}s}[\mu^{-1}g(x_\varepsilon(s)) + h(x_\varepsilon(s),\alpha_\varepsilon[z_\varepsilon](s),z_\varepsilon(s))]\right|ds \le \varepsilon,$$

where $x_\varepsilon(\cdot)$ is the solution of (2.14) for $\alpha = \alpha_\varepsilon$ and $z = z_\varepsilon$.
Hence

$$|(R(\mu^{-1})(\mu^{-1}g))(x) - e^{-\mu^{-1}t}(R(\mu^{-1})(\mu^{-1}g))(x_\varepsilon(t))| \le ct + \varepsilon \text{ for } t \ge 0. \tag{3.16}$$

Using the relations (3.15), (3.16) and making the following calculation, we get

$$\begin{aligned}
&|R(\mu^{-1})(\mu^{-1}g)(x) - (R(\mu^{-1})(\mu^{-1}g))(x(t))| \\
&\le |R(\mu^{-1})(\mu^{-1}g)(x) - e^{-\mu^{-1}t}(R(\mu^{-1})(\mu^{-1}g))(x_\varepsilon(t))| \\
&+|e^{-\mu^{-1}t}(R(\mu^{-1})(\mu^{-1}g))(x_\varepsilon(t)) - e^{-\mu^{-1}t}(R(\mu^{-1})(\mu^{-1}g))(x(t))| \\
&+|e^{-\mu^{-1}t}(R(\mu^{-1})(\mu^{-1}g))(x(t)) - (R(\mu^{-1})(\mu^{-1}g))(x(t))| \\
&\le c\cdot t + \varepsilon, \text{ for } t > 0,
\end{aligned}$$

where c is a positive constant independent of ε.
Therefore $(I-\mu\mathcal{A})^{-1}g$ belongs to $\mathcal{E}$ for $g \in BUC(H)\cap \mathrm{Lip}(H)$.
Since $BUC(H)\cap \mathrm{Lip}(H)$ is dense in $BUC(H)$ (see [2, 10]) and $(I-\mu\mathcal{A})^{-1}$ is nonexpansive in $BUC(H)$ we obtain that $(I-\mu\mathcal{A})^{-1}g \in \overline{\mathcal{E}}$ for all $g \in BUC(H)$. It is well known that $\lim_{\mu\to 0}(I-\mu\mathcal{A})^{-1}g = g$ in $BUC(H)$, for all $g \in \overline{\mathcal{D}(\mathcal{A})}$.
Taking into account that $(I-\mu\mathcal{A})^{-1}g \in \mathcal{E}$ for $g \in BUC(H)\cap \mathrm{Lip}(H)$ it results that $\overline{\mathcal{D}(\mathcal{A})} \subset \overline{\mathcal{E}}$.

Using now the Crandall - Ligget theorem we obtain that for each $\varphi_0 \in \overline{\mathcal{D}(\mathcal{A})}$ and $g \in BUC(H)$ the Cauchy problem

$$\begin{cases} \dfrac{d\varphi}{dt} \in \mathcal{A}\varphi + g \text{ in } R_+ \\ \varphi(0) = \varphi_0 \end{cases} \tag{3.17}$$

has a unique weak solution $\varphi : R^+ \to BUC(H)$ defined by the exponential formula

$$\varphi(t) = \lim_{n\to+\infty} \left(I - \frac{t}{n}\mathcal{A}\right)^{-n} \left(\frac{t}{n}g + \varphi_0\right).$$

The map

$$T(t) : \overline{\mathcal{D}(\mathcal{A})} \to \overline{\mathcal{D}(\mathcal{A})}$$

defined by

$$T(t)\varphi_0 = \varphi(t),\ t \geq 0$$

(where $\varphi(\cdot)$ is given by (3.17)) is a continuous semigroup of nonlinear contractions on $\overline{\mathcal{D}(\mathcal{A})}$.

The main result is contained in the following theorem:

Theorem 3.1. *If $h, g \in BUC(H) \cap \mathrm{Lip}(H)$ then*

$$S(t)\varphi_0 = T(t)\varphi_0 \text{ for all } t \geq 0 \text{ and } \varphi_0 \in \overline{\mathcal{D}(\mathcal{A})}.$$

Moreover, the operator $\mathcal{A}$ is single valued and for all $\varphi_0 \in \mathcal{D}(\mathcal{A})$

$$\lim \frac{1}{t}[(S(t)\varphi_0)(x) - \varphi_0(x)] = (\mathcal{A}\varphi_0)(x) + g(x),\ \text{for } x \in H \tag{3.18}$$

and the limit in (3.18) *is in the strong topology of H.*

Proof. First of all we shall prove that $S(t)(\overline{\mathcal{D}(\mathcal{A})}) \subseteq \overline{\mathcal{D}(\mathcal{A})}$.
Indeed, let $\varphi_0 \in \mathcal{E}$. Using Dynamic Programming Principle we have for all $0 \leq s \leq t$

(3.19)

$$(S(t)\varphi_0)(x) = \sup_{\alpha\in\Gamma_0} \inf_{z\in N_0} \left\{ \int_0^s [g(x(\tau)) \right.$$

$$\left. + h(x(\tau), \alpha[z](\tau), z(\tau))]d\tau + (S(t)\varphi_0)(x(t-s)); x(\cdot) \text{ verifies (2.14)} \right\}$$

Let $\varepsilon > 0$. Then there exist $\alpha_\varepsilon \in \Gamma_0, z_\varepsilon \in N_0$ such that

$$\int_0^s [g(x_\varepsilon(\tau)) + h(x_\varepsilon(\tau), \alpha_\varepsilon[z_\varepsilon](\tau), z_\varepsilon(\tau))]d\tau + (S(t-s)\varphi_0)(x_\varepsilon(s)) - \varepsilon$$

$$\leq (S(t)\varphi_0)(x) \leq \int_0^s [g(x_\varepsilon(\tau)) + h(x_\varepsilon(\tau), \alpha_\varepsilon[z_\varepsilon](\tau), z_\varepsilon(\tau))]d\tau$$

$$+(S(t-s)\varphi_0)(x_\varepsilon(s)) + \varepsilon.$$

Therefore

$$\int_0^s [g(x_\varepsilon(\tau)) + h(x_\varepsilon(\tau), \alpha_\varepsilon[z_\varepsilon](\tau), z_\varepsilon(\tau))]d\tau + (S(t-s)\varphi_0)(x_\varepsilon(s))$$

$$-(S(t)\varphi_0)(x_\varepsilon(s)) - \varepsilon \leq (S(t)\varphi_0)(x) - (S(t)\varphi_0)(x_\varepsilon(s))$$

$$\leq \int_0^s [g(x_\varepsilon(\tau)) + h(x_\varepsilon(\tau), \alpha_\varepsilon[z_\varepsilon](\tau), z_\varepsilon(\tau))]d\tau$$

$$+(S(t-s)\varphi_0)(x_\varepsilon(s)) - (S(t)\varphi_0)(x_\varepsilon(s)) + \varepsilon.$$

Using the last relation we obtain

$$|(S(t)\varphi_0)(x) - (S(t)\varphi_0)(x_\varepsilon(s))| \leq cs + \|S(s)\varphi_0 - \varphi_0\|_b + \varepsilon \tag{3.20}$$

for certain positive constant c and $s \in (0; t)$.
Let $x(\cdot)$ be the solution of (2.14). Then we have

$$|(S(t)\varphi_0)(x_\varepsilon(s)) - (S(t)\varphi_0)(x)| \leq c|x_\varepsilon(s) - x(s)| \leq 2M_f cs, \text{ for } s \geq 0. \tag{3.21}$$

Using the fact that $\varphi_0 \in \mathcal{E}$ and the definition of $S(t)\varphi_0$ one can easily obtain

$$\|S(s)\varphi_0 - \varphi_0\| \leq cs, \text{ for } s \geq 0. \tag{3.22}$$

From (3.20), (3.21) and (3.22) it results

$$|S(t)\varphi_0)(x(s)) - (S(t)\varphi_0)(x)| \leq cs \text{ for } 0 \leq s \leq t.$$

Therefore $S(t)\varphi_0 \in \mathcal{E}$.
Using now Proposition 3.3 we obtain the desired result.
Next, we give a nonlinear version of the Chernoff theorem (see [2], [4]).

Proposition 3.4. *Let C be a closed convex subset of a Banach space Y and let $\mathcal{A}_0$ be a $m-$dissipative subset of $Y \times Y$. Let $\{G(t);\ t \geq 0\}$ be a family of nonexpansive mapping from C into itself such that*

$$\lim_{\rho \searrow 0} \left(I - \lambda \frac{G(\rho) - I}{\rho} \right)^{-1} x = (I - \lambda \mathcal{A}_0)^{-1} x \tag{3.23}$$

for all $x \in \overline{\mathcal{D}(\mathcal{A}_0)} \cap C$ and $\lambda > 0$. Then

$$\lim_{n \to +\infty} \left(G\left(\frac{t}{n} \right) \right)^n = e^{\mathcal{A}_0 t} x, \ \text{for } t \geq 0, \ x \in \overline{\mathcal{D}(\mathcal{A}_0)} \cap C,$$

where $e^{\mathcal{A}_0 t}$ is the semigroup generated by $\mathcal{A}_0$.

We shall apply Proposition 3.4 with $C = \overline{\mathcal{E}}$, $G(\rho) = S(\rho)$ and $\mathcal{A}_{\prime}\varphi = \mathcal{A}\varphi + g$ for $\varphi \in \mathcal{D}(\mathcal{A})$. In this case the relation (3.23) becomes

$$\lim_{\rho \searrow 0} \left(I - \lambda \frac{S(\rho) - I}{\rho} \right)^{-1} \varphi_0 = (I - \lambda \mathcal{A}_0)^{-1} (\varphi_0 + \lambda g) \tag{3.24}$$

for $\varphi_0 \in \mathcal{E}$.

Using the nonexpansivity of $(I - \lambda \mathcal{A})^{-1}$ and $\left(I - \lambda \frac{S(\rho) - I}{\rho} \right)^{-1}$ on $\overline{\mathcal{E}} = \overline{\mathcal{D}(\mathcal{A})}$ we remark that it is sufficient to prove (3.24) for $\varphi_0 \in \mathcal{E}$.

We put

$$\varphi_\rho = \left(I - \lambda \frac{S(\rho) - I}{\rho} \right)^{-1} \varphi_0, \ \varphi = (I - \lambda \mathcal{A})^{-1} (\varphi_0 + \lambda g).$$

With these notations we have

$$\varphi_\rho = \frac{\rho}{\rho + \lambda} \varphi_0 + \frac{\lambda}{\rho + \lambda} S(\rho) \varphi_\rho. \tag{3.25}$$

Taking into account the definition of $S(\rho)$ we may write

$$\begin{aligned} \varphi_\rho(x) = \frac{\rho}{\rho + \lambda} \varphi_0(x) + \frac{\lambda}{\rho + \lambda} \sup_{\alpha \in \Gamma_0} \inf_{z \in N_0} \Big\{ \int_0^\rho [g(x(t)) \\ + h(x(t), \alpha[z](t), z(t))] dt + \varphi_\rho(x(\rho));\ x(\cdot) \text{ verifies (2.14)} \Big\}. \end{aligned} \tag{3.26}$$

Using the definition of $S(\rho)$ and the fact that $\varphi_0 \in \mathcal{E}$ we get

$$\|S(\rho)\varphi_0\|_{\mathrm{Lip}(H)} \leq \rho M \|g\|_{\mathrm{Lip}(H)} + M \|\varphi_0\|_{\mathrm{Lip}(H)} \tag{3.27}$$

and

$$\|S(\rho)\varphi_0\|_b \le \rho(\|g\|_b + c) + \|\varphi_0\|_b, \tag{3.28}$$

for some positive constants c.
From (3.25), (3.27) and (3.28) it results

$$\|\varphi_\rho\|_{\mathrm{Lip}(H)} \le M\|\varphi_0\|_{\mathrm{Lip}(H)} + \lambda M\|g\|_{\mathrm{Lip}(H)}, \tag{3.29}$$

$$\|\varphi_\rho\|_b \le \|\varphi_0\|_b + \lambda(\|g\|_b + c). \tag{3.30}$$

Using now Theorem 2.1 we obtain

$$\varphi(x) = \sup_{\alpha\in\Gamma_0}\inf_{z\in N_0}\left\{\int_0^\rho e^{-\lambda^{-1}t}[g(x(t)) + \lambda^{-1}\varphi_0(x(t))\right.$$

$$\left.+h(x(t),\alpha[z](t),z(t))]dt + e^{-\lambda^{-1}\rho}\varphi(x(\rho));\ x(\cdot)\text{ solves (2.14)}\right\}.$$

Therefore there exist $\alpha_\rho^1 \in \Gamma_0$, $z_\rho^1 \in N_0$ such that

$$\begin{aligned}
&\varphi_\rho(x) - \varphi(x)\\
&\le \frac{\rho}{\rho+\lambda}\varphi_0(x) + \frac{\lambda}{\rho+\lambda}\left\{\int_0^\rho [g(x_\rho(t))\right.\\
&\left.+h(x_\rho(t),\alpha_\rho[z_\rho](t),z_\rho(t))]dt + \varphi_\rho(x_\rho(\rho))\right\}\\
&-\int_0^\rho e^{-\lambda^{-1}t}[g(x_\rho(t)) + \lambda^{-1}\varphi_0(x_\rho(t))\\
&+h(x_\rho(t),\alpha_\rho[z_\rho](t),z_\rho(t))]dt - e^{-\lambda^{-1}\rho}\varphi(x_\rho(\rho)) + 2\rho^2,
\end{aligned} \tag{3.31}$$

where $x_\rho(\cdot)$ solves (2.14) for $\alpha = \alpha_\rho$ and $z = z_\rho$.
In the same manner we get

$$\begin{aligned}
\varphi(x) - \varphi_\rho(x) &\le \frac{\rho}{\rho+\lambda}\varphi_0(x) + \frac{\lambda}{\rho+\lambda}\left\{\int_0^\rho [g(\tilde{x}_\rho(t))\right.\\
&\left.+h(\tilde{x}_\rho(t),\tilde{\alpha}_\rho[\tilde{z}_\rho](t),\tilde{z}_\rho(t))]dt + \varphi_\rho(\tilde{x}(\rho))\right\}\\
&-\int_0^\rho e^{-\lambda^{-1}t}[g(\tilde{x}_\rho(t)) + \lambda^{-1}\varphi_0(\tilde{x}_\rho(t))\\
&+h(\tilde{x}_\rho(t),\tilde{\alpha}_\rho[\tilde{z}_\rho](t),\tilde{z}_\rho(t))]dt - e^{-\lambda^{-1}\rho}\varphi_0(\tilde{x}_\rho(\rho)) + 2\rho^2,
\end{aligned} \tag{3.32}$$

for some $\tilde{\alpha}_\rho \in \Gamma_0$ and $\tilde{z}_\rho \in N_0$.
From (3.31) we obtain

$$\varphi_\rho(x) - \varphi(x) \le \left|\frac{\rho}{\rho+\lambda}\varphi_0(x) - \frac{1}{\lambda}\int_0^\rho e^{-\lambda^{-1}t}\varphi_0(x_\rho(t))dt\right|$$

$$+\left|\frac{\lambda}{\rho+\lambda}\int_0^\rho g(x_\rho(t))dt - \int_0^\rho e^{-\lambda^{-1}t}g(x_\rho(t))dt\right|$$

$$+\left|\frac{\lambda}{\rho+\lambda}\int_0^\rho h(x_\rho(t),\alpha_\rho[z_\rho](t),z_\rho(t))dt\right.$$

$$\left.-\int_0^\rho e^{-\lambda^{-1}t}h(x_\rho(t),\alpha_\rho[z_\rho](t),z_\rho(t))dt\right| + \left|\frac{\lambda}{\rho+\lambda}\varphi_\rho(x_\rho(\rho))\right.$$

$$\left.-e^{-\lambda^{-1}\rho}\varphi(x_\rho(\rho))\right| + 2\rho^2 \le \int_0^\rho \left|\frac{1}{\rho+\lambda}\varphi_0(x) - \frac{1}{\lambda}e^{-\lambda^{-1}t}\varphi_0(x_\rho(t))\right| dt$$

$$+\int_0^\rho \left|\frac{\lambda}{\rho+\lambda} - e^{-\lambda^{-1}t}\right| (|g(x_\rho(t))| + |h(x_\rho(t),\alpha_\rho[z_\rho](t),z_\rho(t))|)dt$$

$$+\frac{\lambda}{\rho+\lambda}\|\varphi_\rho - \varphi\|_b + \left|\frac{\lambda}{\rho+\lambda} - e^{-\lambda^{-1}\rho}\right| \|\varphi\|_b + 2\rho^2$$

$$\le \int_0^\rho \left|\frac{1}{\rho+\lambda}\varphi_0(x) - \frac{1}{\rho+\lambda}\varphi_0(x_\rho(t))\right| dt$$

$$+\int_0^\rho \left|\frac{1}{\rho+\lambda} - \frac{e^{-\lambda^{-1}t}}{\lambda}\right| |\varphi_0(x_\rho(t))|dt$$

$$+\int_0^\rho \left|\frac{\lambda}{\rho+\lambda} - e^{-\lambda^{-1}t}\right| (M_g + M_h)dt$$

$$+\frac{\lambda}{\rho+\lambda}\|\varphi_\rho - \varphi\|_b + \left|\frac{\lambda}{\rho+\lambda} - e^{-\lambda^{-1}\rho}\right| \|\varphi\|_b + 2\rho^2$$

$$\le \frac{1}{\rho+\lambda}\int_0^\rho ctdt + \int_0^\rho \left|\frac{1}{\rho+\lambda} - \frac{e^{-\lambda^{-1}t}}{\lambda}\right| \|\varphi_0\|_b dt$$

$$+\int_0^\rho \left|\frac{\lambda}{\rho+\lambda} - e^{-\lambda^{-1}t}\right| (M_g + M_h)dt$$

$$+\frac{\lambda}{\rho+\lambda}\|\varphi_\rho - \varphi\|_b + \left|\frac{\lambda}{\rho+\lambda} - e^{-\lambda^{-1}\rho}\right| \|\varphi_0\|_b + 2\rho^2$$

$$= a(\rho) + \frac{\lambda}{\rho+\lambda}\|\varphi_\rho - \varphi\|_b.$$

Analogously, using (3.32) we obtain

$$\varphi(x) - \varphi_\rho(x) \le a(\rho) + \frac{\lambda}{\rho+\lambda}\|\varphi_\rho - \varphi\|_b,\ \forall x \in H.$$

Therefore

$$\frac{\rho}{\rho+\lambda}\|\varphi - \varphi_\rho\|_b \le a(\rho),\ \rho > 0.$$

Since $\frac{a(\rho)}{\rho} \underset{\rho\to 0}{\longrightarrow} 0$, we get

$$\lim_{\rho\to 0}\|\varphi - \varphi_\rho\|_b = 0.$$

Applying Proposition 3.4 we have

$$T(t)\varphi_0 = \lim_{n\to+\infty}\left(S\left(\frac{t}{n}\right)\right)^n \varphi_0 = S(t)\varphi_0,$$

for every $\varphi_0 \in \overline{\mathcal{E}} = \overline{\mathcal{D}(\mathcal{A})}$, $t \ge 0$.
Now we shall prove (3.18).
Let $\varepsilon > 0$. Then there exists $\alpha_\varepsilon \in \Gamma_0$, such that

$$(3.33)\qquad (S(t)\varphi_0)(x) - \varphi_0(x) \le \int_0^t [g(x_\varepsilon^1(\tau)) + h(x_\varepsilon^1(\tau), \alpha_\varepsilon[z](\tau), z(\tau))]d\tau + \varphi_0(x_\varepsilon^1(t)) - \varphi_0(x) + \varepsilon,$$

for every $z \in N_0$ and $x_\varepsilon^1(\cdot)$ solves (2.14) for $\alpha = \alpha_\varepsilon$ and $z(\cdot)$.
Using now Proposition 2.2 and since $\varphi_0 \in \mathcal{D}(\mathcal{A})$ there exists $f \in BUC(H)$ such that

$$\varphi_0(x) = \sup_{\alpha\in\Gamma_0}\inf_{z\in N_0}\left\{\int_0^t [f(x(\tau)) + h(x(\tau), \alpha[z](\tau), z(\tau))]d\tau + e^{-t}\varphi_0(x(t))\right\}.$$

Therefore, there exists $z_\varepsilon \in \Gamma_0$ such that

$$(3.34)\qquad \varphi_0(x) \ge \left\{\int_0^t [f(x_\varepsilon^2(\tau)) + h(x_\varepsilon^2(\tau), \alpha[z_\varepsilon](\tau), z_\varepsilon(\tau))]d\tau + e^{-t}\varphi_0(x_\varepsilon^2(t))\right\} - \varepsilon,$$

for every $\alpha \in \Gamma_0$.
Taking $\varepsilon = t^2$ and using (3.33) and (3.34) it results

$$(S(t)\varphi_0)(x) - \varphi_0(x) \leq \int_0^t g(x_{t^2}(\tau))d\tau$$

$$+ \int_0^t (1 - e^{-\tau})h(x_{t^2}(\tau), \alpha_{t^2}[z_{t^2}](\tau), z_{t^2}(\tau))d\tau$$

$$+(1 - e^{-t})\varphi_0(x_{t^2}(t)) - \int_0^t e^{-t} f(x_{t^2}(\tau))d\tau + 2t^2.$$

Dividing now the last relation by t and making $t \to 0$ we obtain

$$\lim_{t\to 0_+} \frac{1}{t}((S(t)\varphi_0)(x) - \varphi_0(x)) \leq g(x) - f(x) + \varphi_0(x).$$

Similarly we get

$$\lim_{t\to 0_+} \frac{1}{t}((S(t)\varphi_0)(x) - \varphi_0(x)) \geq g(x) - f(x) + \varphi_0(x).$$

Therefore

$$\lim_{t\to 0_+} \frac{1}{t}((S(t)\varphi_0)(x) - \varphi_0(x)) = \varphi_0(x) - f(x) + g(x)$$

$$= (R(1)f)(x) - f(x) + g(x) = (\mathcal{A}(R(1)f))(x)$$

$$+g(x) = (\mathcal{A}\varphi_0)(x) + g(x), \text{ for every } x \in H.$$

Remark 3.1. We may consider the operator A nonlinear, but in this case the proof for the resolvent equality would be more difficult. So we should have to work with the Yoshida approximation of A and the evaluations would be more complicated.

We also remark that the function $S(t)\varphi_0$ given by (3.8) is also the viscosity solution of (1.1). Therefore, for the problem (1.1) the variational solution $S(t)\varphi_0$, the semigroup solution $T(t)\varphi_0$ and the viscosity solution coincide.

Acknowledgement. I want to thank to Professor V. Barbu for the helpful discussions.

REFERENCES

1. Barbu, V. – *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff, Leyden, 1976.
2. Barbu, V. – *A Semigroup Approach to Hamilton–Jacobi Equation in Hilbert spaces*, Studia Univ. Babeş–Bolyai, Matematica XXXIII, 1, 1988, 63–78.
3. Barbu, V.; Da Prato, G. – *Hamilton–Jacobi Equations in Hilbert Spaces; Variational and Semigroups Approach*, Annali Matematica Pura ed Appl. vol. CXLII, 1985, 303–349.
4. Brézis, H.; Pazy, A. – *Convergence and Approximation of Semigroups of Nonlinear Operators in Banach Spaces*, J. Funct. Anal. 7, 1972, 63–74.
5. Crandall, M.G.; Lions, P.L. – *Hamilton–Jacobi Equations in Infinite Dimensions. I. Uniqueness of Viscosity solutions*, J. Funct. Anal. 62, 1985, 379–396.
6. Elliott, R.J.; Kalton, N.J. – *The Existence of Value in Differential Games*, Mem. Amer. Math. Soc., 277, 1983, 1–42.
7. Elliott, R.J.; Kalton, N.J. – *Cauchy Problems for Certain Isaacs–Bellman Equations and Games of Survival*, Trans. Amer. Math. Soc. 198, 1974, 45-72.
8. Evans, L.C.; Sougamidis, P.E. – *Differential Games and Representation Formulas for Solutions of Hamilton–Jacobi-Isaacs Equations*, Indiana Univ. Math. J., vol. 33, No. 5, 1984, 773–797.
9. Friedman, A. – *Differential Games*, Wiley, New York, 1971.
10. Lasry, J.M.; Lions, P.L. – *A Remark on Regularization in Hilbert Spaces*, Israel J. Math. 55, 1986, 257–266.
11. Pazy, A. – *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer–Verlag, New York, Berlin, Heidelberg, Tokyo, 1983.

Received: 2.V.2007

Faculty of Mathematics,
"Al.I. Cuza" University,
Bd. Carol I, no 11, Iaşi,
ROMÂNIA
havi@uaic.ro

Romanian Academy,
Institute of Mathematics
"O. Mayer", Bd. Carol I, Iaşi,
ROMÂNIA