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EXCHANGE MOVES AND FIEDLER POLYNOMIAL

BY

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To the memory of my friend Gheorghe Ionesei

Abstract. In order to obtain a Markov theorem without stabilization, Birman and Menasco introduced the notion of exchange related braids. In this paper I study the way the Fiedler polinomial distinguishes conjugacy classes of some particular braided knots. I introduce the Kauffman bracket in the solid torus. Its Taylor expansion give finite type invariants similar to the Fiedler polynomial. I investigate how these invariants distinguish exchange related braids.

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A link is an embedding of $\amalg_{i=1}^{n}\mathbb{S}^1$ in $\mathbb{S}^3$ or in $\mathbb{R}^3$. A famous problem is the classification of links up to isotopy. This problem has an algebraic solution through braids. A braid is an embedding of $\amalg_{i=1}^{n} I$ into the cylinder $\mathbb{D}^2 \times I$, such that the intervals are considered to strictly decrease from the top to the bottom disc. The end points are considered to be fixed.

The braids form a group, the multiplication being given by putting the cylinders in top of one another, such that the end points correspond. The identity element is the braid formed by all the strings going straight from the point $i \times 1$ to $i \times 0$. The braid group B_n is finitely presented given the number of strings also called the braid index. The standard set of generators is $\sigma_1, \ldots, \sigma_{n-1}$, σ_i being represented in the picture below. Moreover the braid σ_i^{-1} is the one in which the crossing of the i-th string is in front of the the $i+1$-th.

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*This is part of my Ph.D. Thesis I defended at Columbia University in the spring of 2001

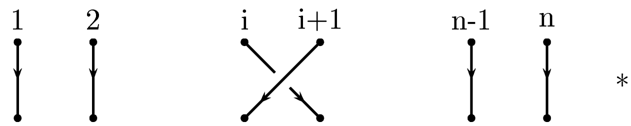

Figure 1: Generator σ_i

Adding a trivial string one obtains a braid in B_{n+1}. This give us an embedding $\mathrm{B}_n \hookrightarrow \mathrm{B}_{n+1}$.

A braid can be closed to a link connecting every point $i \times 0$ to $i \times 1$ by an arc which go around an axis perpendicular to the cross-section of the cylinder. The axis is shown in the above picture by the star. A link obtained in this manner is also called a closed braid.

A theorem of Alexander says that any link in $\mathbb{S}^3$ (or $\mathbb{R}^3$) can be isotoped into a closed braid. Markov theorem says that the closures of two braids α, β are isotopic if and only if the braids are related by a finite sequence of the following two moves and their inverses.

(1) $\beta = \gamma \cdot \alpha \cdot \gamma^{-1}, \qquad \alpha, \beta, \gamma \in \mathrm{B}_n$
(2) $\beta = \alpha \cdot \sigma_n^{\pm 1}$, changing the braid index, $n \leftrightharpoons n+1$

The trouble in the classification problem mentioned above is made by the mixture of the two moves. To be more precise for each isotopy class of links there are infinitely many braids which close to a link in such a class. But there is a braid representative of minimum braid index representing the link. The problem is that given a braid with a larger that minimum braid index, representing a link (up to isotopy), it might be the case that in order to simplify it (meaning to decrease the braid index) one should go up in the tower of braid groups, and then go to a lower index than the one at the beginning.

Examples of this kind were given by Rudolph in [11] for the unlink with two components, refined by Morton in [9] to an example for the unknot.

To overcome the difficulty Birman and Menasco in the paper [4] described a new move which doesn't change the braid index and may be accomplished only by crossing the braid axis as in Markov move (2), above. This new move is given in the following:

Definition 1. Two braids β_1, β_2 are said to be related by an *exchange move*, or called *exchange related*, if $\beta_1 = X\sigma_n^{-1}Y\sigma_n$ *and* $\beta_2 = X\sigma_n Y\sigma_n^{-1}$, where X and $Y \in \mathrm{B}_n$.

The move changes the conjugacy class of generic braids for $n \geq 4$. Our goal is to discover a simple and definitive test for verifying whether two braids which are related by an exchange move are conjugate or not. In [2] the authors describe an algorithmic solution for the conjugacy problem in B_n. This gives, of course, a complete answer to our question, however the algorithm is complicated and one might hope for a simpler solution in our special situation. The matter turns out, however, to be subtle. All the easy invariants take an identical value on exchange-equivalent conjugacy classes.

Let me describe briefly the content of the rest of the paper. In Section 1 I give the definition of the Fiedler invariant, which is a class invariant. I state its properties in Theorem 4 and give a sketch of the proof for the case of braided knots. Although implicit in Morton's work [10], I prove in Proposition 6 that Fiedler's invariant is a type one invariant. Then in Section 2 I use Fiedler's polynomial to detect exchange related braids. Necessary and sufficient conditions for this are given in Proposition 9. At the end of the section I give explicit examples (see Examples 12—14).

1. Fiedler's invariant. FIEDLER in [6] describes in a general setting an isotopy invariant for knots. He considers knots K embedded in an orientable 3-manifold E which is the total space of a real line bundle over a surface. I will consider in what follows only braided knots. The knots are embedded in E= $\mathbb{R}^3 \setminus \{z - axis\}$ and the surface is S= $\mathbb{R}^2 \setminus \{0\}$.

For this particular case we get in fact an invariant of closed braids up to conjugacy, as described in Fiedler's paper. Applying this invariant to $\widehat{\beta_1}$ and $\widehat{\beta_2}$, exchange related braided knots, we see that there are a lot of cases in which it can distinguish when they are not conjugate.

In the sequel I will consider only braids $\gamma = \prod_{r=1}^{k} \sigma_{i_r}^{\epsilon_r}$ such that their closures $\hat{\gamma}$ are knots. By smoothing I will understand the replacement shown in the picture below.

Definition 2. The ascending string of a crossing in a given closed braid $\hat{\gamma}$ (counting all the braid strings from 1 to n at the top of each crossing) is:

- if the sign of the crossing σ_{i_r} is $\epsilon_r = 1$ then the *ascending string* is the i_r-th string after smoothing the crossing;

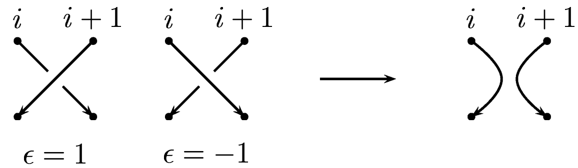

Figure 2: Smoothing

- if the sign of the crossing σ_{i_r} is $\epsilon_r = -1$ then the *ascending string* is the i_{r+1}-th string after smoothing the crossing.

Definition 3. The *Fiedler polynomial* for any $\gamma \in \mathrm{B}_n$ is given by:

$$\mathbf{F}_{\hat{\gamma}}(x) = \sum_{r=1}^{k} \epsilon_r x^{2m(r)-n}$$

where m(r) is the winding number of the ascending string around the braid axis, after smoothing the r-th crossing of the braid.

Let $\pi : \mathrm{B}_n \longrightarrow \Sigma_n$ be the homomorphism which assigns to each braid its associated permutation. Let s_i be the transposition $(i, i+1)$. I will denote with π_X the image of the braid X through π, so, $\pi(\sigma_i) = \pi_{\sigma_i} = s_i$.

Theorem 4 (Fiedler). $\mathbf{F}_{\hat{\gamma}}(x)$ *is a conjugacy class invariant of* $\gamma \in \mathrm{B}_n$ *where* $\hat{\gamma}$ *is a knot, and has the following properties:*

1. $\mathbf{F}_{\hat{\gamma}}(x)$ *is a symmetric Laurent polynomial in x.*
2. *maxdeg of any monomial in* $\mathbf{F}_{\hat{\gamma}}(x)$ *is* $\leq (n-2)$.
3. *If* γ *is conjugate to a positive braid then maxdeg*$(\mathbf{F}_{\hat{\gamma}}(x))$ *is* $n-2$ *and all coefficients are positive*
4. $\mathbf{F}_{\hat{\gamma}}(1) = w(\gamma)$ *(where w(*γ*) is the writhe of* γ*).*

Proof. The proof for the case of braided knots is easier than the general case described by Fiedler. One has to see that $\hat{\beta} \longmapsto \mathbf{F}_{\hat{\beta}}(x)$ is a well defined map. More precisely I need to show firstly that

$$\mathbf{F}_{\widehat{\alpha\beta}}(x) = \mathbf{F}_{\widehat{\alpha\sigma_j\sigma_j^{-1}\beta}}(x)$$
$$\mathbf{F}_{\widehat{\alpha\sigma_j\sigma_{j+1}\sigma_j\beta}}(x) = \mathbf{F}_{\widehat{\alpha\sigma_{j+1}\sigma_j\sigma_{j+1}\beta}}(x)$$
$$\mathbf{F}_{\widehat{\alpha\sigma_j\sigma_i\beta}}(x) = \mathbf{F}_{\widehat{\alpha\sigma_i\sigma_j\beta}}(x) \text{ with } |i-j| \geq 2$$

The last two equalities have to be proved because the relations in B_n are $\sigma_j\sigma_{j+1}\sigma_j = \sigma_{j+1}\sigma_j\sigma_{j+1}$ for $j = \overline{1, n-2}$ and $\sigma_j\sigma_i = \sigma_i\sigma_j$ for $|i-j| \geq 2$. After smoothing the r-th crossing of $\widehat{\beta}$, π_β will break into a product of two disjoint cycles. Then m(r) is the length of the disjoint cycle which contains i_r, the index of the r-th ascending string. I will denote by $\mathbf{P}_\theta(x)$ all the monomials in the Fiedler polynomial corresponding to the letters in θ, and also $m(\sigma_{i_r}) = m(i_r)$.

$$\mathbf{F}_{\widehat{\alpha\sigma_j\sigma_j^{-1}\beta}}(x) = \mathbf{P}_\alpha(x) + x^{2m(\sigma_j)-n} - x^{2m(\sigma_j^{-1})-n} + \mathbf{P}_\beta(x).$$

I have to show that $m(\sigma_j) = m(\sigma_j^{-1})$. When I am smoothing any of these two crossings the associated permutations are the same: $\pi_\alpha s_j \pi_\beta$. So the decompositions into disjoint cycles, when smoothing either σ_j or σ_j^{-1} are the same. For σ_j the ascending string is the i_j string and for σ_j^{-1} is also i_j viewed at the top of σ_j. So $m(\sigma_j) = m(\sigma_j^{-1})$ and from here I obtain that:

$$\mathbf{F}_{\widehat{\alpha\sigma_j\sigma_j^{-1}\beta}}(x) = \mathbf{P}_\alpha(x) + \mathbf{P}_\beta(x) = \mathbf{F}_{\widehat{\alpha\beta}}(x).$$

In the same way one can prove the last two equalities, as well as the invariance under conjugation. Everything else in the theorem can be proved using a "skein relation". The assertion is that the difference of the respective values of the Fiedler polynomial on the braids $\alpha\sigma_j\beta$ and $\alpha\sigma_j^{-1}\beta$ is a symmetric polynomial (see (1)). One observation is that the ascending string when smoothing σ_j^{-1} is the descending string for the case when smoothing σ_j, and so $m(\sigma_j^{-1}) = n - m(\sigma_j)$.

$$\begin{aligned}
&\mathbf{F}_{\widehat{\alpha\sigma_j\beta}}(x) - \mathbf{F}_{\widehat{\alpha\sigma_j^{-1}\beta}}(x) \\
&= \mathbf{P}_\alpha(x) + x^{2m(\sigma_j)-n} + \mathbf{P}_\beta(x) - [\mathbf{P}_\alpha(x) - x^{2m(\sigma_j^{-1})-n} + \mathbf{P}_\beta(x)] \\
&= x^{2m(\sigma_j)-n} + x^{n-2m(\sigma_j)}.
\end{aligned} \tag{1}$$

The rest of the proof is exactly as in Fiedler's paper. □

Remark 5. From now on we can think of β as being the conjugacy class of $\beta \in \mathrm{B}_n$. The closure of any braid in β is the same braided knot $\widehat{\beta}$.

Using only the Definition 3 one can prove the following:

Proposition 6. *The Fiedler polynomial is an order 1 invariant for braided knots in* $\mathbb{R}^3 \setminus \{z - axis\}$.

Proof. This observation is implicit in MORTON'S work [10], using the well known theorem of BIRMAN and LIN [3]. A direct proof can be given by a direct calculation.

I use the symbol S_i for the "singular" generator of singular braid monoid (see [1]). That is an elementary braid in which the strings i and $i+1$ intersect transversally at a point. Now I need to see that if I consider a singular closed braid with two singularities S_i and S_j then the Fiedler polynomial vanishes. Solving each singularity in the two possible ways one gets:

$$\begin{aligned}\mathbf{F}_{\widehat{AS_iBS_jC}}(x) &= \mathbf{F}_{\widehat{A\sigma_iB\sigma_jC}}(x) \\ &- \mathbf{F}_{\widehat{A\sigma_iB\sigma_j^{-1}C}}(x) - \mathbf{F}_{\widehat{A\sigma_i^{-1}B\sigma_jC}}(x) + \mathbf{F}_{\widehat{A\sigma_i^{-1}B\sigma_j^{-1}C}}(x)].\end{aligned}$$

Now using the "skein relation" mentioned in the proof of the Theorem 4 I will obtain:

$$\begin{aligned}\mathbf{P}_A(x) \quad &+x^{2m(i)-n} + \mathbf{P}_B(x) + x^{2m(j)-n} + \mathbf{P}_C(x) \\ &-[\mathbf{P}_A(x) + x^{2m(i)-n} + \mathbf{P}_B(x) - x^{n-2m(j)} + \mathbf{P}_C(x)] \\ &-[\mathbf{P}_A(x) - x^{n-2m(i)} + \mathbf{P}_B(x) + x^{2m(j)-n} + \mathbf{P}_C(x)] \\ &+\mathbf{P}_A(x) - x^{n-2m(i)} + \mathbf{P}_B(x) - x^{n-2m(j)} + \mathbf{P}_C(x) = 0.\end{aligned}$$

□

2. Exchange moves and Fiedler polynomial. In this section all braids considered are representatives of knots. I will apply Fiedler's polynomial to exchange related closed braids, $\widehat{\beta_1}$ and $\widehat{\beta_2}$. Recall from Definition **1**, that such braids look like this: $\beta_1 = X\sigma_n^{-1}Y\sigma_n$ and $\beta_2 = X\sigma_nY\sigma_n^{-1}$.

I want to discuss in some more details the braids X and Y which form the two exchange related braids. If X is a word only in letters $\sigma_1, \cdots, \sigma_{n-2}$ then X commutes with $\sigma_n^{\pm 1}$ and $\beta_1 = \sigma_n^{-1}XY\sigma_n$, $\beta_2 = \sigma_nXY\sigma_n^{-1}$ which are conjugate braids. Similarly if Y is a word in letters $\sigma_1, \cdots, \sigma_{n-2}$ then Y commutes with $\sigma_n^{\pm 1}$ and both β_i's are conjugate to XY. I am not interested in either of these two cases. So I will assume that X, Y contain the letter $\sigma_{n-1}^{\pm 1}$ in their expression.

Suppose now $X = X_1\sigma_{n-1}^{2k}X_2$ with no other appearance of σ_{n-1}, which means that X_1, X_2 are words in letters $\sigma_1, \ldots, \sigma_{n-2}$. Then $\pi_X = \pi_{X_1}\pi_{X_2}$ and both π_{X_1} and π_{X_2} fix the letter n. $\pi_{\beta_2} = \pi_{X_1}\pi_{X_2}s_n\pi_Y s_n = s_n\pi_{X_1}\pi_{X_2}\pi_Y s_n$ (because the multiplication of disjoint cycles is commutative).
The last expression is conjugate to $\pi_{X_1}\pi_{X_2}\pi_Y$ which cannot be an $(n+1)$-cycle because it fixes the letter $n+1$. Also $X = X_1\sigma_{n-1}^{2k_1}X_2\sigma_{n-1}^{2k_2}\cdots\sigma_{n-1}^{2k_l}X_l$, with $X_i \in \mathrm{B}_{n-1}$, with no other appearances of σ_{n-1} wouldn't give us by closure a knot, because the n-string of the braid would close itself to a component of the link. The same is true for Y. So at least one of the appearances of σ_{n-1} in X or Y, should be at an odd power.

Consider also the braid $\beta = X\sigma_n Y\sigma_n$. The braids β, β_1, β_2 are simultaneously knots, that means their associated permutations are $\pi_\beta = \pi_{\beta_1} = \pi_{\beta_2}$ $= \pi_X s_n \pi_Y s_n$ are $(n+1)$–cycles and I have the relation $\mathrm{w}(X) + \mathrm{w}(Y) \equiv n \pmod 2$. I want to see when $\mathbf{F}_{\widehat{\beta}_1}(x) = \mathbf{F}_{\widehat{\beta}_2}(x)$, for β_1, β_2 exchange related. I will denote with $m_1(\sigma_n)$ the winding number of the ascending string of the first σ_n in the expression of β, and with $m_2(\sigma_n)$ the winding number of the ascending string for the second one.

$$\mathbf{F}_{\hat{\beta}}(x) - \mathbf{F}_{\hat{\beta}_1}(x) = x^{2m_1(\sigma_n)-(n+1)} + x^{(n+1)-2m_1(\sigma_n)},$$
$$\mathbf{F}_{\hat{\beta}}(x) - \mathbf{F}_{\hat{\beta}_2}(x) = x^{2m_2(\sigma_n)-(n+1)} + x^{(n+1)-2m_2(\sigma_n)}$$

$$\begin{aligned}(2)\quad &\mathbf{F}_{\widehat{\beta}_1}(x) - \mathbf{F}_{\widehat{\beta}_2}(x)\\ &= x^{2m_2(\sigma_n)-(n+1)} + x^{(n+1)-2m_2(\sigma_n)} - x^{2m_1(\sigma_n)-(n+1)} - x^{(n+1)-2m_1(\sigma_n)}.\end{aligned}$$

Because $\pi_\beta = \pi_X s_n \pi_Y s_n$ is a $(n+1)$–cycle whenever I set $s_n = 1$ one of its appearances, (smooth one of the σ_n in the braid) I will get the resulting permutation to be a product of 2 disjoint cycles. So one of the cycles of $\pi_X\pi_Y s_n$ has length $m_1(\sigma_n)$, and one of the cycles of $\pi_X s_n \pi_Y$ has length $m_2(\sigma_n)$.

The above difference is a symmetric polynomial and depends only of the permutations $\pi_X\pi_Y s_n$ and $\pi_X s_n\pi_Y$. Let $A = \pi_X$ and $B = s_n\pi_Y s_n$ then $\pi_X\pi_Y s_n = As_nB, \pi_X s_n\pi_Y = ABs_n$.

Remark 7. I will restrict my attention to the case when the above permutations A and B are both cycles, and I will study the case when they multiply to give us a full cycle which is π_β the associated permutation a braided knot $\hat{\beta}$.

Definition 8. Let l be the length of one of the cycles of $\pi_X\pi_Y s_n$.

So is either l is $m_1(\sigma_n)$ or $(n+1) - m_1(\sigma_n)$.

Proposition 9. *Let $n \geq 4$. If n+1 is odd and $l = \frac{n+2}{2}$ then $\mathbf{F}_{\widehat{\beta_1}} = \mathbf{F}_{\widehat{\beta_2}}$ for all pairs of braids β_1 and β_2 related by an exchange move; conversely, if $\mathbf{F}_{\widehat{\beta_1}} = \mathbf{F}_{\widehat{\beta_2}}$ for all pairs of braids β_1 and β_2 related by an exchange move then n+1 and $l = \frac{n+2}{2}$.*

I mentioned the condition $n \geq 4$, (so the exchange related braids will live in B_5 at least), because for braid index 3, the exchange move does not change the conjugacy class of the braids and from the conclusion of the proposition we see that the values of the Fiedler polynomial are different for two exchange related braids living in B_4.

To prove the above result I will study how the permutations A and B multiplied give a full cycle. Let $A = (i_1 \cdots i_a)$, $B = (j_1, \cdots, j_b)$ be both cycles. Consider $A \cup B = \{1, 2, \cdots, n+1\}$, and $A \cap B = \{t_1, \cdots, t_m\}$ to be the union and respectively the intersection of the sets of digits appearing in A and B. Consider $t_1, \cdots, t_m$ to be written in an increasing order. In both A and B these digits will appear in different order. Denote by ν_A the permutation obtained by considering the digits $t_1, \cdots, t_m$ in the order they appear in A, and similarly ν_B. Denote also by $\nu_{A,B} = \nu_A^{-1} \cdot \nu_B$. Let's see what these permutations look like calculating a few examples (computations are done with [7]).

1. (1 7 3 9 5 4 2 8 6)(1 2)=(1 7 3 9 5 4)(2 8 6),
$A \cap B = (1\ 2)$, $\nu_A = \nu_B = id$, $\nu_{A,B} = id$

2. (1 7 3 9 5 4 2 8 6)(1 2 3)=(1 7)(2 8 6)(3 9 5 4)
$A \cap B = (1\ 2\ 3)$, $\nu_A = (2\ 3)$, $\nu_B = id$, $\nu_{A,B} = (2\ 3)$

3. (1 7 3 9 5 4 2 8 6)(1 2 3 5 6)=(1 7 5 4 3 9 6 2 8)
$A \cap B = (1\ 2\ 3\ 5\ 6)$, $\nu_A = (2\ 3\ 5)$, $\nu_B = id$, $\nu_{A,B} = (2\ 5\ 3)$

4. (1 3 4 7 10 5 8 11)(6 7 5 3 2 9 8)=(1 2 9 8 11)(3 4 5 6 7 10)
$A \cap B = (3\ 5\ 7\ 8)$, $\nu_A = (5\ 7)$, $\nu_B = (3\ 7)$, $\nu_{A,B} = (3\ 7\ 5)$

5. (1 3 4 7 10 5 8 11)(6 7 5 3 2 9)=(1 2 9 6 7 10 3 4 5 8 11)
$A \cap B = (3\ 5\ 7)$, $\nu_A = (5\ 7)$, $\nu_B = (3\ 7)$, $\nu_{A,B} = (3\ 7\ 5)$

6. (1 3 4 7 10 5 8 11)(6 7 2 3 9 5)=(1 9 5 8 11)(2 3 4)(6 7 10)
$A \cap B = (3\ 5\ 7)$, $\nu_A = (5\ 7)$, $\nu_B = (3\ 7\ 5)$, $\nu_{A,B} = (3\ 7)$

These computations lead to the following:

Lemma 10. *Let A, B cycles, as above. Then $A \cdot B$ is a full cycle if and only if $|A \cap B|$ is odd and $\nu_{A,B}$ has an even number of inversions.*

Proof. For the direct implication suppose first $A \cap B = \{t_1, \cdots, t_{2m}\}$ and $\nu_{A,B}$ is identity. We have $i_{k_p} = t_p = j_{k_p}$.

$$\begin{aligned}
&A \cdot B = (i_1, \cdots, i_{k_1}, \cdots, i_{k_{2m}}, \cdots, i_a) \cdot (j_1, \cdots, j_{k_1}, \cdots, j_{k_{2m}}, \cdots, j_b)\\
&= (i_1, \cdots, i_{k_1-1}, j_{k_1+1} \cdots, j_{k_2}, i_{k_2+1}, \cdots, j_{k_{2m}}, i_{k_{2m}+1}, \cdots, i_a)\\
&\cdot(j_1, \cdots, j_{k_1-1}, j_{k_1}, i_{k_1+1} \cdots, i_{k_2-1}, j_{k_2+1}, \cdots, i_{k_{2m}-1}, j_{k_{2m}+1}, \cdots, j_b).
\end{aligned}$$

If we are changing $\nu_{A,B}$ with an even permutation, first we will encounter i_a and this will give us a product of 2 cycles as above which is a contradiction.

Now suppose $A \cap B = \{t_1, \cdots, t_{2m+1}\}$ and $\operatorname{sign}(\nu_{A,B}) \equiv 1 \pmod 2$. I will consider the case $\nu_{A,B} = (1\ 2)$. So in this case we have $i_{k_1} = t_1 = j_{k_2}$ and $i_{k_2} = t_2 = j_{k_1}$ and for the rest of p's $i_{k_p} = t_p = j_{k_p}$.

$$\begin{aligned}
A \cdot B\ &= (i_1, \cdots, i_{k_1}, \cdots, i_{k_{2m+1}}, \cdots, i_a) \cdot (j_1, \cdots, j_{k_1}, \cdots, j_{k_{2m+1}}, \cdots, j_b)\\
&= (i_1, \cdots, i_{k_1-1}, j_{k_2+1} \cdots, j_{k_3}, i_{k_3+1}, \cdots, j_{k_{2m}}, i_{k_{2m}+1}, \cdots, i_a)\\
&\cdot(i_{k_1}, \cdots, i_{k_2-1}, j_{k_1+1} \cdots, j_{k_2-1})\\
&\cdot(i_{k_2}, \cdots, i_{k_3-1}, j_{k_3+1}, \cdots, i_{k_{2m+1}-1}, j_{k_{2m}+1}, \cdots, j_b, j_1, \cdots, j_{k_1-1}).
\end{aligned}$$

Again if we are changing $\nu_{A,B}$ with an even permutation we will get the same kind of decomposition in disjoint cycles for $A \cdot B$ as above.

For the other implication I will consider $A \cap B = \{t_1, \cdots, t_{2m+1}\}$ and $\nu_{A,B}$ is identity.

$$\begin{aligned}
A \cdot B &= (i_1, \cdots, i_{k_1}, \cdots, i_{k_{2m+1}}, \cdots, i_a) \cdot (j_1, \cdots, j_{k_1}, \cdots, j_{k_{2m+1}}, \cdots, j_b)\\
&= (i_1, \cdots, i_{k_1-1}, j_{k_1+1} \cdots, j_{k_2}, i_{k_2+1}, \cdots, i_{k_{2m+1}-1}, j_{k_{2m+1}+1}, \cdots, j_b,\\
&\qquad j_1, \cdots, j_{k_1}, i_{k_1+1}, \cdots, i_{k_2-1}, j_{k_2+1}, \cdots, i_{k_{2m}-1}, j_{k_{2m}+1}, \cdots, i_a).
\end{aligned}$$

Changing $\nu_{A,B}$ with an even permutation we get again an $(n+1)$-cycle because in the multiplication we are getting first j_b and then at the end i_a. □

With A, B as above, $A \cap B = \{t_1, \cdots, t_{2m+1}\}$, and $n, n+1$ are not in this set. Then $As_nB \cap ABs_n = \{t_1, \cdots, t_{2m+1}, n\}$ so the cardinal of this set is even.

Lemma 11. *Let A, B as above, such that $A \cdot B$ is a full cycle, then $\pi_X\pi_Y s_n$ and $\pi_X s_n \pi_Y$ have disjoint cycle decomposition of lengths $l, n+1-l$ and respectively $l-1, n+2-l$.*

Proof. I will consider as in the previous proof only the permutation ν of the intersection set $\{t_1, \cdots, t_{2m-1}\}$ is the identity and n and $n+1$ are at the end between i_{2m-1} and i_a and respectively between j_{2m-1} and j_b. By cyclic permutations of digits we can arrange to have this order in A. So the above assumption is only for B.

$$A = (i_1, \cdots, i_{k_1}, \cdots, i_{k_{2m-1}}, \cdots, i_u, \cdots, i_a) \quad \text{where} \quad i_u = n.$$
$$B = (j_1, \cdots, j_{k_1}, \cdots, j_{k_{2m-1}}, \cdots, j_v, \cdots, j_b) \quad \text{where} \quad j_v = n+1.$$
$$s_nB = (j_1, \cdots, j_{k_1}, \cdots, j_{k_{2m-1}}, \cdots, j_{v-1}, j_v, n, \cdots, j_b)$$
$$Bs_n = (j_1, \cdots, j_{k_1}, \cdots, j_{k_{2m-1}}, \cdots, j_{v-1}, n, j_v, \cdots, j_b)$$

$$\begin{aligned} As_nB &= (i_1, \cdots, i_{k_1}, \cdots, i_{k_{2m-1}}, \cdots, i_u, \cdots, i_a) \\ &\cdot (j_1, \cdots, j_{k_1}, \cdots, j_{k_{2m-1}}, \cdots, j_{v-1}, j_v, n, \cdots, j_b) \\ &= (i_1, \cdots, i_{k_1-1}, j_{k_1+1} \cdots, j_{k_2}, i_{k_2+1}, \cdots, i_{k_{2m-1}-1}, \\ &\qquad j_{k_{2m-1}+1}, \cdots, j_{v-1}, j_v, n, \cdots, i_a) \\ &\cdot (i_{k_1}, \cdots, i_{k_2-1}, j_{k_2+1} \cdots, j_{k_3}, i_{k_3+1}, \cdots, j_{k_{2m-1}}, \\ &\qquad i_{k_{2m-1}+1}, \cdots, i_{u-1}, j_{v+1}, \cdots, j_b, j_1, \cdots, i_{k_1-1}) \end{aligned}$$

$$\begin{aligned} ABs_n &= (i_1, \cdots, i_{k_1}, \cdots, i_{k_{2m-1}}, \cdots, i_u, \cdots, i_a) \\ &\cdot (j_1, \cdots, j_{k_1}, \cdots, j_{k_{2m-1}}, \cdots, j_{v-1}, n, j_v, \cdots, j_b) \\ &= (i_1, \cdots, i_{k_1-1}, j_{k_1+1} \cdots, j_{k_2}, i_{k_2+1}, \cdots, i_{k_{2m-1}-1}, \\ &\qquad j_{k_{2m-1}+1}, \cdots, j_{v-1}, n, i_{u+1}, \cdots, i_a) \\ &\cdot (i_{k_1}, \cdots, i_{k_2-1}, j_{k_2+1} \cdots, j_{k_3}, i_{k_3+1}, \cdots, j_{k_{2m-1}}, \\ &\qquad i_{k_{2m-1}+1}, \cdots, i_{u-1}, j_v, j_{v+1}, \cdots, j_b, j_1, \cdots, i_{k_1-1}). \end{aligned}$$

Sometimes the length of the first cycle in the decomposition of As_nB will be $l-1$ and the length of the similar cycle in ABs_n will be l. Such an example may be obtained as follows: let $i_u = i_a = n$ and $j_v = j_b = n+1$. Consider also $A = (i_{k_1}, i_{k_2}, \cdots, i_{k_{2m-1}}, \cdots, i_u)$, $B = (j_1, j_{k_1}, j_{k_2}, \cdots, j_{k_{2m-1}}, \cdots, j_{v-1}, j_v)$.

Then $s_nB = (j_{k_1}, j_{k_2}, \cdots, j_{k_{2m-1}}, \cdots, j_{v-1}, j_v, n)$, and $Bs_n = (j_{k_1}, j_{k_2}, \cdots, j_{k_{2m-1}}, \cdots, j_{v-1}, n, j_v)$ where $i_{k_p} = j_{k_p}$.

$$\begin{aligned} As_nB &= (i_{k_1}, i_{k_2}, \cdots, i_{k_{2m-1}}, \cdots, i_u) \cdot (j_{k_1}, j_{k_2}, \cdots, j_{k_{2m-1}}, \cdots, j_{v-1}, j_v, n) \\ &= (i_{k_1}, j_{k_3}, j_{k_5}, \cdots, j_{k_{2m-1}}, \cdots, i_{k_{2m-1}+1}, \cdots, i_{u-1}) \\ &\cdot (i_{k_2}, j_{k_4}, j_{k_6}, \cdots, j_{k_{2m-2}}, i_{k_{2m-2}+1}, \cdots, i_{k_{2m-1}-1}, \\ &\qquad j_{k_{2m-1}+1}, \cdots, j_{v-1}, j_v, n). \end{aligned}$$

$$\begin{aligned}ABs_n &= (i_{k_1}, i_{k_2}, \cdots, i_{k_{2m-1}}, \cdots, i_u) \cdot (j_{k_1}, j_{k_2}, \cdots, j_{k_{2m-1}}, \cdots, j_{v-1}, n, j_v) \\ &= (i_{k_1}, j_{k_3}, j_{k_5}, \cdots, j_{k_{2m-1}}, \cdots, i_{k_{2m-1}+1}, \cdots, i_{u-1}, j_v) \\ &\cdot (i_{k_2}, j_{k_4}, j_{k_6}, \cdots, j_{k_{2m-2}}, i_{k_{2m-2}+1}, \cdots, i_{k_{2m-1}-1}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad j_{k_{2m-1}+1}, \cdots, j_{v-1}, n).\end{aligned}$$

The same type of decomposition in disjoint cycles will be obtained for any appropriate A and B, meaning the lengths of the disjoint cycles in As_nB and ABs_n will be as in the above cases. □

Proof of Proposition 9: Since l is either $m_1(\sigma_n)$ or $(n+1) - m_1(\sigma_n)$ it follows from lemma 11 that the difference of the two polynomials will vanish if and only if $l = n + 2 - l \Leftrightarrow 2l = n + 2$. So from here we get two things: n must be even and $l = \frac{n+2}{2}$. □

Example 12. An infinite family of pairs of braids in B_5 $(n = 4)$ which are not conjugate and the Fiedler polynomial cannot distinguish them.

Let

$$\beta_{1,k} = \sigma_3\sigma_2\sigma_1\sigma_4^{-1}\sigma_3\sigma_2^{2k+1}\sigma_1\sigma_4 \quad \text{and} \quad \beta_{2,k} = \sigma_3\sigma_2\sigma_1\sigma_4\sigma_3\sigma_2^{2k+1}\sigma_1\sigma_4^{-1}.$$

They are related by exchange moves for each k, and using the conjugacy algorithm (see [2]) we see that they are not conjugate. Their associated permutations are

$$\pi_{\beta_{1,k}} = \pi_{\beta_{2,k}} = (1\ 3\ 4\ 2\ 5); \quad X = \sigma_3\sigma_2\sigma_1, \quad Y = \sigma_3\sigma_2^{2k+1}\sigma_1.$$

$\pi_X = (1\ 2\ 3\ 4)$, $s_4\pi_Y s_4 = (1\ 2\ 3\ 5)$, and from here we get $\pi_X\pi_Y s_4 = (1\ 3)\cdot(2\ 5\ 4)$ and $\pi_X s_4 \pi_Y = (1\ 3\ 5)\cdot(2\ 4)$.

In this example $m_1(\sigma_n) = 5 - m_2(\sigma_n) = 3$. So $l = \frac{n+2}{2} = \frac{6}{2} = 3$, and their Fiedler polynomial are equal. Similar examples can be obtained by multiplying each of the braids with a pure braid. □

Example 13. Two sets of braids in B_5 $(n = 4)$ which can be distinguished using the Fiedler polynomial.

Let

$$\beta_1 = \sigma_3\sigma_2\sigma_1\sigma_4^{-1}\sigma_2\sigma_1\sigma_3\sigma_4 \quad \text{and} \quad \beta_2 = \sigma_3\sigma_2\sigma_1\sigma_4\sigma_2\sigma_1\sigma_3\sigma_4^{-1}.$$

They are related by exchange moves. Their associated permutations are

$$\pi_{\beta_1} = \pi_{\beta_2} = (1\ 5\ 3\ 4\ 2); \quad X = \sigma_3\sigma_2\sigma_1, \quad Y = \sigma_2\sigma_1\sigma_3.$$

$\pi_X = (1\ 2\ 3\ 4)$, $s_4\pi_Y s_4 = (1\ 2\ 3\ 5)$, and from here we get: $\pi_X\pi_Y s_4 = (1\ 2\ 3\ 4)\cdot(1\ 2\ 5\ 4\ 3) = (1\ 5\ 4\ 2)$ and $\pi_X s_4 \pi_Y = (1\ 2\ 3\ 4)\cdot(1\ 2\ 4\ 5\ 3) = (1\ 4\ 2)\cdot(3\ 5)$.
We have $m_1(\sigma_n) = 4; m_2(\sigma_n) = 2$. In this case $l = 4$ or $l = 1 \neq \frac{n+2}{2} = 3$. Moreover the difference $\mathbf{F}_{\widehat{\beta}_1}(x) - \mathbf{F}_{\widehat{\beta}_2}(x) = x + x^{-1} - x^3 - x^{-3} \neq 0$. So we may conclude that they are not conjugate. □

Example 14. Let n be an even number. In this general case there are families of braids related by exchange moves which are distinguished by the Fiedler polynomial (if $l = \frac{n+2}{2}$).

For $i \neq 1$, let

$$\beta_1 = \sigma_{n-1}\cdots\sigma_1\sigma_n^{-1}\sigma_{n-1}\cdots\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\cdots\sigma_{n-1}^{-1}\sigma_n \quad \text{and}$$

$$\beta_2 = \sigma_{n-1}\cdots\sigma_1\sigma_n\sigma_{n-1}\cdots\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\cdots\sigma_{n-1}^{-1}\sigma_n^{-1}$$

They are related by exchange moves for each k, Their associated permutations are:

1. for $i = 1$ $\quad \pi_{\beta_1} = \pi_{\beta_2} = (1\ 2\cdots n-1\ n\ n+1)$
2. for $i \neq 1$ $\quad \pi_{\beta_1} = \pi_{\beta_2} = (1\ 2\cdots i-1\ n+1\ i\cdots n)$

$$X = \sigma_{n-1}\cdots\sigma_1 \text{ and } Y = \sigma_{n-1}\cdots\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\cdots\sigma_{n-1}^{-1}$$

$\pi_X = (1\ 2\cdots n)$, $s_n\pi_Y s_n = (i\ n+1)$, and from here we get:

1. $i = 1$

$$\pi_X\pi_Y s_n = (1\ 2\cdots n)\cdot(1\ n+1\ n) = (1\ 2\cdots n-1)\cdot(n\ n+1),$$
$$\pi_X s_n\pi_Y = (1\ 2\cdots n)\cdot(1\ n\ n+1) = (1\ 2\cdots n-1\ n+1)$$

and

2. $i \neq 1$

$$\begin{aligned} \pi_X s_n \pi_Y &= (1\ 2\cdots n)\cdot(i\ n+1\ n) \\ &= (1\ 2\cdots i-1\ n+1\ n)\cdot(i\ i+1\cdots n-1), \\ \pi_X\pi_Y s_n &= (1\ 2\cdots n)\cdot(i\ n\ n+1) \\ &= (1\ 2\cdots i-1\ n)\cdot(i\ i+1\cdots n-1\ n+1). \end{aligned}$$

Independent of i we have that $m_1(\sigma_n) = i+1$ and $m_2(\sigma_n) = n+1-i$, so $l = i+1$, and the above permutations are conjugate if and only if $\frac{n+2}{2} = i+1 \Leftrightarrow i = \frac{n}{2}$. $\square$

3. Finite type invariants coming from Kauffman bracket. In his paper [8], Kauffman describes a purely combinatorial way of obtaining the Jones polynomial. His main construction is a regular isotopy invariant for links in $\mathbb{R}^3$ which was called later the Kauffman bracket.

I will do the same construction in the solid torus and the object I'll obtain is an invariant of conjugacy classes of braids. Let $\mathrm{R} = \mathbb{C}[a^{\pm 1}, x]$, where a and x are two variables. The TL_n is the Temperley-Lieb algebra of index n which is generated by $1, e_1, \cdots e_{n-1}$ and has the following defining relations:

$$\begin{aligned} e_i^2 &= d \cdot e_i \\ e_i e_{i\pm 1} e_i &= e_i \\ e_i e_j &= e_j e_i \quad \text{if } |i-j| > 1 \end{aligned}$$

Above, the coefficient $d = -a^2 - a^{-2}$.

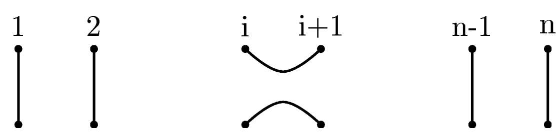

Figure 3: Generator e_i

The elements of the Temperley-Lieb algebra can be viewed as pictures. Consider $2n$ points lying on two parallel horizontal lines in the plane, n on each line. The elements of the Temperley-Lieb algebra, which are also called states can be interpreted as a collection of n disjoint arcs connecting these $2n$ points.

Figure 3 shows a picture of the algebra generator e_i. These generators can be multiplied by putting one in top of the other to give all the elements of the algebra. The identity is the set of segments connecting the i-th point on the top line with the i-th point on the bottom line.

Consider the following map:

$$\Phi : \mathrm{B}_n \hookrightarrow \mathbb{C}\mathrm{B}_n \to TL_n. \tag{3}$$

The map (3) is the representation of the braid group on the Temperley-Lieb algebra, given by:

$$\Phi(\sigma_i) = a^{-1} \cdot e_i + a \cdot 1, \quad \Phi(\sigma_i^{-1}) = a \cdot e_i + a^{-1} \cdot 1.$$

Let me explain the "closure" operation. Think the plane in which the states live as being the sectional plane of a cylinder as in the case of braids. The closure of a state is the exactly the same as the one for braids, meaning connect i-th point on the top line around the axis with the i-th point on the bottom line. Closing a state give us a set of unoriented curves in the solid torus (the one winding around the axis considered), some curves being contractible, some of them winding around the axis.

Let $p = \#$ of contractible components and $q = \#$ of homotopycally non-trivial components in the closure of a state.

TL_n is a vector space over $\mathbb{C}$ with the basis given by the set of states.

Definition 15. Let $f : TL_n \to \mathrm{R} \quad f(w) = d^p x^q$ for w a state and extend by linearity to the entire vector space.

$f \circ \Phi$ associates to each braid a polynomial in $a^{\pm 1}$ and x. One can see that $f(e_i) = d \cdot x^{n-2}$ and $f(1) = x^n$.

Lemma 16. *$f : TL_n \to \mathrm{R}$ is a trace function, in other words, it has the following three properties for $(\forall)\ v,\ w \in TL_n$ and for $(\forall)\ \alpha \in \mathbb{C}[a^{\pm 1}]$.*

1. $f(v + w) = f(v) + f(w)$;
2. $f(\alpha \cdot v) = \alpha \cdot f(v)$;
3. $f(v \cdot w) = f(w \cdot v)$.

Proof. The first two properties are true because we have defined the function f as a linear extension. For last property let's look at two generators of the Temperley-Lieb algebra, e_i, e_j. Consider the closure of e_ie_j, we can isotope e_j through the closure arcs in the top of e_i. So the two closures are isotopic in the solid torus, so they have the same number of components, which means that $f(e_ie_j) = f(e_je_i)$. From here, by the same argument we find that f has the third property for every element in the basis of the Temperley-Lieb algebra. Now since each word is a sum of states with some coefficients, by linearity we get the property in general. □

For father computations it will be useful to have some formulas for $\Phi(\sigma_i^k)$.

Lemma 17. *$\Phi(\sigma_i^k) = p_k(a)e_i + a^k$, where*

$$p_k(a) = \begin{cases} \sum\limits_{l=0}^{k-1}(-1)^l a^{k-2-4l} & if \quad k>0 \\ 1 & if \quad k=0 \\ \sum\limits_{l=0}^{-k-1}(-1)^l a^{k+2+4l} & if \quad k<0. \end{cases}$$

Proof. Consider $k>0$. The proof will be by induction. For $k=1$ we have $p_1(a) = a^{-1}$ which is exactly what the formula give us. Assume the formula is true for k. For $k+1$ we have to prove that

$$p_{k+1}(a) = \sum_{l=0}^{k}(-1)^l a^{k+1-2-4l}$$

$$\begin{aligned}\Phi(\sigma_i^{k+1}) &= (a^{-1}e_i + a)\cdot(p_k(a)e_i + a^k) = (a^{-1}p_k(a)d + a^{k-1} + ap_k(a)) \\ &= (-a^{-3}p_k(a) + a^{k-1})e_i + a^{k+1}\end{aligned}$$

The coefficient of e_i is $p_{k+1}(a)$

$$= (-a^{-3})\sum_{l=0}^{k-1}(-1)^l a^{k-2-4l} + a^{k-1} = \sum_{l=0}^{k-1}(-1)^{l+1}a^{k-3-2-4l} + a^{k+1-2}$$

$$= \sum_{l=0}^{k-1}(-1)^{l+1}a^{k+1-2-4(l+1)} + a^{k+1-2}$$

$$= \sum_{l=1}^{k}(-1)^l a^{k+1-2-4l} + a^{k+1-2} = \sum_{l=0}^{k}(-1)^l a^{k+1-2-4l}.$$

The case $k<0$ is similar. □

Proposition 18. *Let $a = e^t$ in $\Phi(\beta)$. Then the coefficient of t^k in $\Phi(\beta)$ is a finite order invariant of order k.*

Proof. Consider a singular braid with $k+1$ transverse double points, and consider its image in the Temperley-Lieb algebra, through the canonical extension of Φ to singular braids.

I will show that all the coefficients of t^k up to power k+1 obtained after we expand $a = e^t$, will be equal to zero. Let $\beta = X_1\tau_{i_1}X_2\tau_{i_2}\cdots X_{k+1}\tau_{i_{k+1}}X_{k+2}$, where $X_i's$ are braids in B_n and τ_i is the standard singular i-th braid generator (the strings i and i+1 cross transversally in one point).

$$\Phi(X_1\tau_{i_1}X_2\tau_{i_2}\cdots X_{k+1}\tau_{i_{k+1}}X_{k+2}) =$$

$$\begin{aligned}
&\sum_{\epsilon_{i_1},\cdots,\epsilon_{i_{k+1}};\epsilon_{i_l}\in\{\pm1\}} \epsilon_{i_1}\ldots\epsilon_{i_{k+1}}\Phi(X_1\sigma_{i_1}^{\epsilon_{i_1}}X_2\sigma_{i_2}^{\epsilon_{i_2}}\cdots X_{k+1}\sigma_{i_{k+1}}^{\epsilon_{i_{k+1}}}X_{k+2}) = \\
&= \sum_{\epsilon_{i_1},\cdots,\epsilon_{i_{k+1}};\epsilon_{i_l}\in\{\pm1\}} \epsilon_{i_1}\ldots\epsilon_{i_{k+1}}\Phi(X_1)(a^{-\epsilon_{i_1}}e_{i_1}+a^{\epsilon_{i_1}}) \\
&\qquad \cdots\Phi(X_{k+1})(a^{-\epsilon_{i_{k+1}}}e_{i_{k+1}}+a^{\epsilon_{i_{k+1}}})\Phi(X_{k+2}).
\end{aligned} \tag{4}$$

In the sum (4) we have 2^{k+1} terms of the form

$$\epsilon_{i_1}\ldots\epsilon_{i_{k+1}}\Phi(X_1)(a^{-\epsilon_{i_1}}e_{i_1}+a^{\epsilon_{i_1}})\cdots\Phi(X_{k+1})(a^{-\epsilon_{i_{k+1}}}e_{i_{k+1}}+a^{\epsilon_{i_{k+1}}})\Phi(X_{k+2})$$

and expanding each parenthesis we will get for each of them a sum of 2^{k+1} terms, each one of them being like $\Phi(X_1)\cdot e_{i_1}\cdots\Phi(X_{k+1})\cdot e_{i_{k+1}}\cdot\Phi(X_{k+2})$, or with some e_j missing. I am interested in the coefficient of each such term. In fact all the coefficients of any of the terms are equal up to a sign. Such term is realized 2^{k+1} times. The sign in front of each appearance will be $(-1)^l$ where l is the number of -1's in the $k+1$-tuple $\epsilon_{i_1},\ldots,\epsilon_{i_{k+1}}$. For a given such l we have $\binom{k+1}{l}$ ways of choosing l -1's out of $k+1$ numbers. As for the a factor for a given l will be of the form $a^l\cdot(a^{-1})^{k+1-l}$, because $\Phi(\sigma_j^{-1}) = (ae_i + a^{-1})$. Putting everything together we get that the coefficient in front of $\Phi(X_1)\cdot e_{i_1}\cdots\Phi(X_{k+1})\cdot e_{i_{k+1}}\cdot\Phi(X_{k+2})$ is

$$\sum_{l=0}^{k+1}\binom{k+1}{l}(-1)^l a^l\cdot(a^{-1})^{k+1-l} = (a^{-1}-a)^{k+1}. \tag{5}$$

For $a = e^t$, we have $a^{-1}-a = (1-t+\cdots)-(1+t+\cdots) = -2t+\cdots$, and we will have $(-a+a^{-1})^{k+1} = (-2t)^{k+1}+\cdots \equiv_k 0$, so the coefficients of $t^j = 0$ for all $1 \leq j \leq k$. Here $\equiv_k$ means truncating the terms of degree $\geq k+1$. Since the coefficient of all states which appear in the decomposition (4) is up to sign equal with (5) we get the conclusion of the proposition. $\square$

Definition 19. Consider $f \circ \Phi(\beta)$, and let $a = e^t$. The coefficient of t^k, a polynomial in x, is denoted by $\mathbf{Q}_{\hat{\beta},k}(x)$.

Corollary 20. *$\mathbf{Q}_{\hat{\beta},k}(x)$ is a $k-th$ order invariant.*

Proof. Consider the image of the word in (4) after replacing a by e^t, through f, We have that $\mathbf{Q}_{\hat{\beta},k}(x) = 0$, because all the coefficients up to order $k+1$ are zero. $\square$

Proposition 21. $\mathbf{Q}_{\hat{\beta},1}(x) \neq \mathbf{F}_{\hat{\beta}}(x)$

Proof. Consider $\sigma_1^3 \in \mathrm{B}_2$. Its closure is the right hand trefoil. $\mathbf{F}_{\widehat{\sigma_1^3}}(x) = 3$. Consider $\Phi(\sigma_1^3) = (a^{-1}e_1+a)^3 = p_3(a)e_1+a^3$ where $p_3(a) = a-a^{-3}+a^{-7}$. Then $f\circ\Phi(\sigma_1^3) = p_3(a)\cdot f\circ\Phi(e_1)+a^3\cdot f\circ\Phi(1) = p_3(a)\cdot d+a^3\cdot x^2$. Let $a = e^t$. Looking only to the terms up to order 2 we will get $d \equiv_2 -(1+2t)-(1-2t) = -2$, $p_3(a) \equiv_2 1+t-(1-3t)+(1-7t) = 1-3t$, $a^3 = 1+3t$. As a result $f \circ \Phi(\sigma_1^3) \equiv_2 (1-3t)\cdot(-2)+(1+3t)x^2 = -2+x^2+3(2+x^2)t$, and $\mathbf{Q}_{\widehat{\sigma_1^3},1}(x) = 3(2+x^2)$. $\square$

We see that the $\mathbf{Q}_{\hat{\beta},k}$ exists for any k, and moreover can be defined for any braids, not only for braided knots.

I will start the study of this new invariant on exchange related knots with a discussion on Morton's braid representative of the unknot appearing in [9]. A conjugate of it is

$$\beta = \sigma_2\sigma_2\sigma_2\sigma_1^{-1}\sigma_2\sigma_3^{-1}\sigma_2^{-1}\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_3$$

It belongs to B_4. To destabilize to the canonical braid representative of the unknot in $e \in \mathrm{B}_1$, one needs either to stabilize first to B_5, as Morton shows in [9], or needs to use exchange moves as in [5]. In fact it is sufficient to use only one exchange move, not two as in [5]. The notations for this example only, are: the standard generators of the braid group, σ_i are replaced with i, and σ_i^{-1} with $\bar{i}$), a twiddle means that the transformation is conjugation in B_n, and an arrow with no specification means that we are using only the braid relations, the rest being clear.

$$
\begin{array}{ccccccc}
222\overline{1}2\overline{322}1\overline{2}3 & \textit{exchange} & 22212\overline{32212}3 & \to & 22212\overline{32121}3 & \to & \\
22212\overline{31211}3 & \to & 22212\overline{132}3\overline{11} & \sim & \overline{11}22212\overline{1}2\overline{32} & \to & \\
\overline{211}22212\overline{1}2\overline{3} & \textit{destabilization} & \overline{211}22212\overline{1}2 & \sim & \overline{11}22212\overline{1} & \to & \\
\overline{11}22121\overline{1} & \to & \overline{11}2212 & \to & \overline{11}2121 & \sim & \\
\overline{1}212 & \to & \overline{1}121 & \sim & 12 & \textit{destabilization} & e
\end{array}
$$

Now let me consider the exchange related braids:

$$
\beta_1 = \sigma_2^{-1}\sigma_3\sigma_2\sigma_2\sigma_2\sigma_1^{-1}\sigma_2\sigma_3^{-1}\sigma_2^{-1}\sigma_2^{-1}\sigma_1,
$$
$$
\beta_2 = \sigma_2^{-1}\sigma_3\sigma_2\sigma_2\sigma_2\sigma_1\sigma_2\sigma_3^{-1}\sigma_2^{-1}\sigma_2^{-1}\sigma_1^{-1},
$$

with $X = \sigma_2^{-1}\sigma_3\sigma_2\sigma_2\sigma_2$ and $Y = \sigma_2\sigma_3^{-1}\sigma_2^{-1}\sigma_2^{-1}$. First let's calculate $\mathbf{Q}_{\widehat{\beta}_1,k}(x)$, and I will start $\Phi(\beta_1)$.

$$
\begin{aligned}
\Phi(\beta_1) &= \Phi(X)(ae_1 + a^{-1})\Phi(Y)\Phi(a^{-1}e_1 + a) \\
&= \Phi(X)e_1\Phi(Y)e_1 + a^2 \cdot \Phi(X)e_1\Phi(Y) \\
&+ a^{-2} \cdot \Phi(X)\Phi(Y)e_1 + \Phi(X)\Phi(Y).
\end{aligned} \tag{6}
$$

The computations involve lemma 17 and we will get:

$$
\begin{aligned}
&\Phi(X)e_1\Phi(Y)e_1 = (-a^{-13} + 2a^{-9} - 3a^{-5} + 2a^{-1} - a^3) \cdot e_3e_2e_1 \\
&+(-2a^{-1} + 3a^3 - 2a^7 + a^{11}) \cdot e_2e_3e_1 + (-2a^{-3} + 2a - a^5) \cdot e_3e_1 \\
&+(-a^{-11} + 3a^{-7} - 5a^{-3} + 4a - 2a^5) \cdot e_2e_1 + (a + 2a^3 - a^7) \cdot e_1
\end{aligned}
$$

$$
\begin{aligned}
&\Phi(X)e_1\Phi(Y) = (a^{-5} - 3a^{-1} + 5a^3 - 3a^7 + a^{11}) \cdot e_2e_3e_1e_2 \\
&+(a^{-7} - 2a^{-3} + 4a + a^3 - a^5 - a^7 - a^9) \cdot e_1e_3e_2 \\
&+(a^{-11} - a^{-7} + a^{-3}) \cdot e_3e_2e_1 + (a^{-7} - 2a^{-3} + 3a - a^5) \cdot e_2e_1e_3 \\
&+a \cdot e_1e_2e_3 + (a^{-9} - a^{-5} + 2a^{-1}) \cdot e_1e_3 \\
&+(a^{-9} - 3a^{-5} + 3a^{-1} - a^3) \cdot e_3e_2 + (a^{-9} - 2a^{-5} + 2a^{-1}) \cdot e_2e_3 \\
&+(a^{-9} - 2a^{-5} + 2a^{-1}) \cdot e_2e_1 + (2a^3 - a^7) \cdot e_1e_2 \\
&+(a^{-11} - a^{-7} + a^{-3}) \cdot e_3 + (2a^{-7} - 5a^{-3} + 6a - 2a^5) \cdot e_2 + a \cdot e_1
\end{aligned}
$$

$$
\begin{aligned}
&\Phi(X)\Phi(Y)e_1 = (-a^{-11} + a^{-7} - a^{-3} + 2a + a^3 - a^7 - a^9) \cdot e_3e_2e_1 \\
&+(-a^{-11} + 2a^{-7} - 2a^{-3} + 2a - a^5) \cdot e_2e_3e_1 \\
&+(-a^{-13} + a^{-9} - a^{-5} + a^{-1}) \cdot e_1e_3 \\
&+(-a^{-9} + 2a^{-5} - 2a^{-1} + 3a^3 - 2a^7 + a^{11}) \cdot e_2e_1 + a \cdot e_1
\end{aligned}
$$

$$\Phi(X)\Phi(Y) = (-a^{-11} + a^{-7} - a^{-3} + a) \cdot e_3e_2$$
$$+(-a^{-11} + 2a^{-7} - 2a^{-3} + 2a - a^5) \cdot e_2e_3 + (-a^{-13} + a^{-9} - a^{-5} + a^{-1}) \cdot e_3$$
$$+(-a^{-9} + 2a^{-5} - 2a^{-1} + 3a^3 - 2a^7 + a^{11}) \cdot e_2 + a.$$

The image of (6) through f. Let's start with the images of the states appearing:

$$\begin{array}{l} f(e_2e_3e_1e_2) = f(e_1e_3) = d^2 \\ f(e_1e_3e_2) = f(e_3e_2e_1) = f(e_2e_1e_3) = f(e_1e_2e_3) = d \\ f(e_3e_2) = f(e_2e_3) = f(e_2e_1) = f(e_1e_2) = x^2 \\ f(e_3) = f(e_2) = f(e_1) = d \cdot x^2. \end{array}$$

Replacing all these in (6), we get:

$$\begin{aligned} f \circ \Phi(\beta_1) &= f(\Phi(X)e_1\Phi(Y)e_1) + a^2 \cdot f(\Phi(X)e_1\Phi(Y)) \\ &\quad +a^{-2} \cdot f(\Phi(X)\Phi(Y)e_1) + f(\Phi(X)\Phi(Y)) \\ &= (-a^{-19} + 2a^{-15} - 2a^{-11} - a^{-7} + 3a^{-3} - a^{-1} - 5a - a^3 \\ &\quad +2a^5 + a^7 + 4a^9 + a^{11} - a^{13} + a^{17}) \\ &\quad +(a^{-15} - 4a^{-11} + 9a^{-7} - 15a^{-3} - a^{-1} + 9a - a^3 \\ &\quad -11a^5 + 4a^9 - a^{13}) \cdot x^2 + a \cdot x^4 \end{aligned}$$

and similarly

$$\begin{aligned} f \circ \Phi(\beta_2) &= f(\Phi(X)e_1\Phi(Y)e_1) + a^{-2} \cdot f(\Phi(X)e_1\Phi(Y)) \\ &\quad +a^2 \cdot f(\Phi(X)\Phi(Y)e_1) + f(\Phi(X)\Phi(Y)) \\ &= (-a^{-3} - a^{-1} - a - a^3 + a^5 + a^7 + 2a^9 + a^{11} + a^{13}) \\ &\quad +(-2a^{-3} - a^{-1} - 4a - a^3 - 2a^5) \cdot x^2 + a \cdot x^4. \end{aligned}$$

It is clear, even from the images in the Temperley-Lieb algebra, that the two braids are different, but we can see that also their images through f are different. We can look at $\mathbf{Q}_{\widehat{\beta_i},k}(x)$, where $i = 1, 2$ and $k \geq 0$. For example $\mathbf{Q}_{\widehat{\beta_1},1}(x) = 70 - 14x^2 + x^4$ and $\mathbf{Q}_{\widehat{\beta_2},1}(x) = 54 - 10x^2 + x^4$.

Consider now in general two braids which are exchange related $\beta_1 = X\sigma_n^{-1}Y\sigma_n$ and $\beta_2 = X\sigma_nY\sigma_n^{-1}$. We would be interested to compute the difference $f \circ \Phi(\beta_1) - f \circ \Phi(\beta_2)$. Let's look first to the difference of the images of β_i in the TL_n algebra.

$$\begin{aligned}
\delta &= \Phi(X)\Phi(\sigma_n^{-1})\Phi(Y)\Phi(\sigma_n) - \Phi(X)\Phi(\sigma_n)\Phi(Y)\Phi(\sigma_n^{-1}) \\
&= \Phi(X) \cdot (ae_n + a^{-1})\Phi(Y)\Phi(a^{-1}e_n + a) \\
&\quad -\Phi(X) \cdot (a^{-1}e_n + a)\Phi(Y)\Phi(ae_n + a^{-1}).
\end{aligned}$$

So we can rewrite δ as:

$$\delta = (a^2 - a^{-2}) \cdot [\Phi(X)e_n\Phi(Y) - \Phi(X)\Phi(Y)e_n]. \tag{7}$$

Using the power series expansion $a = e^t$, we see that $a^2 - a^{-2} \equiv_2 4t$. In case we want to look only to $\mathbf{Q}_{\widehat{\beta_1},1}(x) - \mathbf{Q}_{\widehat{\beta_2},1}(x)$, we have then to compute only the free term of $f(\Phi(X)e_n\Phi(Y)) - f(\Phi(X)\Phi(Y)e_n)$.

Let me investigate the braids in example (12) using these invariants.

Example 22 (Example(12) **revisited).** The following two sets of braids in B_5 are not conjugate, as we already know, and they cannot be distinguished either by the Fiedler polynomial, or by $\mathbf{Q}_{\widehat{\beta},1}(x)$, but they can be distinguished by $\mathbf{Q}_{\widehat{\beta},2}(x)$

$$\beta_{1,j} = \sigma_3\sigma_2\sigma_1\sigma_4^{-1}\sigma_3\sigma_2^{2j+1}\sigma_1\sigma_4 \quad \text{and} \quad \beta_{2,j} = \sigma_3\sigma_2\sigma_1\sigma_4\sigma_3\sigma_2^{2j+1}\sigma_1\sigma_4^{-1}.$$

Set $X = \sigma_3\sigma_2\sigma_1$ and $Y = \sigma_3\sigma_2^{2j+1}\sigma_1$.

$$\begin{aligned}
&\Phi(X)e_4\Phi(Y) - \Phi(X)\Phi(Y)e_4 \\
&= \Phi(\sigma_3)\Phi(\sigma_2\sigma_1)e_4\Phi(\sigma_2^{2j+1}\sigma_1) - \Phi(\sigma_3)\Phi(\sigma_2\sigma_1)\Phi(\sigma_2^{2j+1}\sigma_1)e_4 \\
&= a^{-2} \cdot e_3\Phi(\sigma_2\sigma_1)e_4e_3\Phi(\sigma_2^{2j+1}\sigma_1) - a^{-2} \cdot e_3\Phi(\sigma_2\sigma_1)e_3\Phi(\sigma_2^{2j+1}\sigma_1)e_4 \\
&+\Phi(\sigma_2\sigma_1)e_4e_3\Phi(\sigma_2^{2j+1}\sigma_1) - \Phi(\sigma_2\sigma_1)e_3\Phi(\sigma_2^{2j+1}\sigma_1)e_4.
\end{aligned}$$

The last equality is obtained using the expression for $\Phi(\sigma_3)$ and the commutativity relations of e_4 with e_1, e_2. We can reduce it more because we are interested in the images through f which are the same for cyclic permutations of the words. So in the image of f the last difference will cancel out. This lead us to the investigation of:

$$
\begin{aligned}
&a^{-2} \cdot \Phi(\sigma_2\sigma_1)e_3\Phi(\sigma_2^{2j+1}\sigma_1) \cdot [e_3e_4 - e_4e_3] \\
&= a^{-1}p_{2j+1}(a) \cdot (e_2e_1e_3e_4 - e_2e_1e_3e_2e_4e_3) \\
&+ \left(a^{-3}p_{2j+1}(a) + a^{2j-2}d + 2a^{2j}\right) \cdot (de_2e_1e_3e_4 - e_2e_1e_3) \\
&+ap_{2j+1}(a) \cdot (e_1e_3e_4 - e_1e_3e_2e_4e_3) + ap_{2j+1}(a) \cdot (e_3e_1e_4 - e_3e_2e_4e_3e_1) \\
&+ \left(a^{-1}p_{2j+1}(a) + a^{2j}d + 2a^{2j+2}\right) \cdot (de_1e_3e_4 - e_1e_3) \\
&+a^3p_{2j+1}(a) \cdot (e_3e_4 - e_3e_2e_4e_3) + a^{2j+2} \cdot (de_2e_3e_4 - e_2e_3) \\
&+a^{-1}p_{2j+1}(a) \cdot (e_2e_1e_3e_4 - e_2e_1e_4e_3) \\
&+a^{2j+4} \cdot (de_3e_4 - e_3) + ap_{2j+1}(a) \cdot (e_2e_3e_4 - e_2e_4e_3)
\end{aligned}
$$

Let's evaluate the function f for the above states.

$$
\begin{aligned}
&f(e_2e_1e_3e_4) = f(e_2e_1e_3e_2e_4e_3) = x \\
&f(e_1e_3e_2e_4e_3) = f(e_3e_2e_4e_3e_1) = dx \\
&f(e_2e_1e_3) = f(e_3e_1e_4) = f(e_2e_3e_4) = f(e_2e_4e_3) = dx \\
&f(e_1e_3) = f(e_3e_2e_4e_3) = d^2x; f(e_3e_4) = f(e_2e_3) = x^3; f(e_3) = dx^3 \\
&f(e_2e_1e_3e_4) = f(e_2e_1e_4e_3) = x
\end{aligned}
$$

The difference $f(\Phi(X)e_4\Phi(Y)) - f(\Phi(X)\Phi(Y)e_4)$ using the expressions above becomes:

$$a^3p_{2j+1}(a) \cdot (x^3 - d^2x) + a^{2j+2} \cdot (d^2x - x^3),$$

The image of δ through f, will be:

$$
\begin{aligned}
&f(\delta) = (a^2 - a^{-2}) \cdot [f(\Phi(X)e_4\Phi(Y)) - f(\Phi(X)\Phi(Y)e_4)] \\
&= (a^2 - a^{-2}) \cdot [(a^3p_{2j+1}(a) - a^{2j+2}) \cdot x^3 + d^2(a^{2j+2} - a^3p_{2j+1}(a)) \cdot x]
\end{aligned}
$$

and after expanding $a = e^t$, we get up to degree 3 in t:

$$f(\delta) \equiv_3 4\left(-4jx^3 + 16jx\right) \cdot t^2.$$

So we have learned that $\mathbf{Q}_{\widehat{\beta_1,j},k}(x) - \mathbf{Q}_{\widehat{\beta_2,j},k}(x) = 0$ if $k = 0, 1$, and the first non-zero difference is $\mathbf{Q}_{\widehat{\beta_1,j},2}(x) - \mathbf{Q}_{\widehat{\beta_2,j},2}(x) = -16j(x^3 - 4x)$. $\square$

Because of the above example and similar computations I will make the following:

Conjecture 23. $\mathbf{Q}_{\widehat{\beta},1}(x)$ braids will vanish in the same way on exchange related as the Fiedler's polynomial does (see Proposition 9).

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