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SOME PROPERTIES OF THE HOLSZTYŃSKI SHAPE FUNCTOR

BY

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Abstract. In this Note we consider some functors between pairs and shape theories in the sense of Holsztyński, and the natural transformations between such functors are discussed. An application in the case of Mardešić-Segal shape theory is given.

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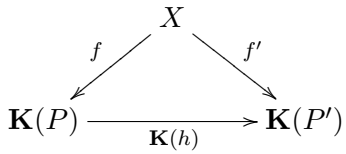
Key words: K-continuous functor, shape theory (functor, category), Holsztyński pair (shape theory), dense subcategory, Mardešić-Segal shape theory.

1. K-continuous functors. In this section we recall from the book of CORDIER and PORTER [1] the notion of **K**-continuous functor, and we add some examples and some remarks.

For a category **A** we will denote by $|\mathbf{A}|$ the class of objects and by $\mathbf{A}(X,Y)$ the set of morphisms from the object X to the object Y in **A**.

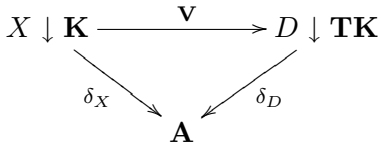
Let **A** and **B** be two categories and $\mathbf{K} : \mathbf{A} \to \mathbf{B}$ a covariant functor. For an object $X \in |\mathbf{B}|$, the *comma category* of **K**-objects under X,denoted $X \downarrow \mathbf{K}$, is the category whose objects are the pairs (f, P), where $P \in |\mathbf{A}|$ and $f \in \mathbf{B}(X, \mathbf{K}(P))$ and where the morphisms $h : (f, P) \to (f', P')$ are the morphisms $h \in \mathbf{A}(P, P')$ which satisfy $\mathbf{K}(h)f = f'$.



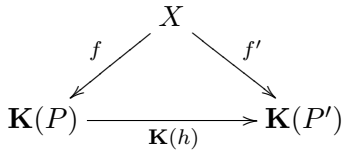
It is denoted by $\delta_X : X \downarrow \mathbf{K} \to \mathbf{A}$ the codomain functor from $X \downarrow \mathbf{K}$ to **A** defined by $\delta_X(f, P) = P$ and $\delta_X(h) = h$.

Let $\mathbf{K} : \mathbf{A} \to \mathbf{B}$ and $\mathbf{T} : \mathbf{B} \to \mathbf{C}$ be two covariant functors.

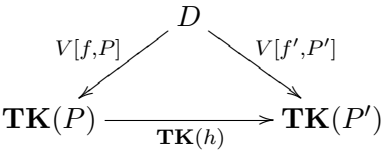
For $X \in |\mathbf{B}|$ and $D \in |\mathbf{C}|$, denote by $Funct_*(X \downarrow \mathbf{K}, D \downarrow T\mathbf{K})$ the set of covariant functors $\mathbf{V} : X \downarrow \mathbf{K} \to D \downarrow \mathbf{TK}$, such that $\delta_D \mathbf{V} = \delta_X$.



This condition implies that for an object (f, P) from $X \downarrow \mathbf{K}$ its image by $\mathbf{V}$ is of the shape $(\mathbf{V}[f, P], P)$, with $\mathbf{V}[f, P] \in \mathbf{C}(D, \mathbf{TK}(P))$, and for a morphism $h : P \to P'$ in $X \downarrow \mathbf{K}$, with



commutes the following diagram

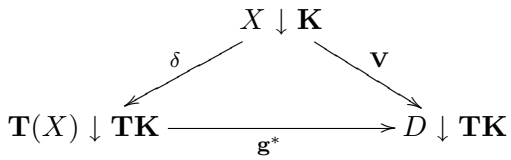


Then we can consider the functor $\delta : X \downarrow \mathbf{K} \to \mathbf{T}X \downarrow \mathbf{TK}$ defined by

$$\delta(f, P) = (\mathbf{T}f, P), \text{ and } \delta(h) = h.$$

because the relation $\mathbf{K}h \circ f = f'$ implies $\mathbf{TK}h \circ \mathbf{T}f = \mathbf{T}f'$.

Definition 1.1. ([1], p. 29) One says that the functor $\mathbf{T}$ is $\mathbf{K}$ - *continuous in* $X \in |\mathbf{B}|$ if, for all $D \in |\mathbf{C}|$ and for all $\mathbf{V} \in Funct_*(X \downarrow \mathbf{K}, D \downarrow \mathbf{TK})$, there is a unique $g \in \mathbf{C}(D, \mathbf{T}(X))$ such that the diagram

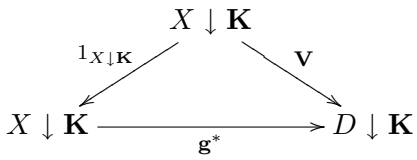


commutes, where $\mathbf{g}^* : \mathbf{T}(X) \downarrow \mathbf{TK} \to D \downarrow \mathbf{TK}$ is the functor given by

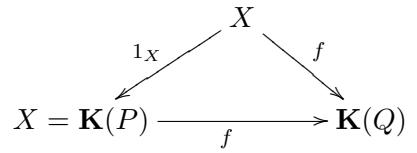
$$\mathbf{g}^*(h,P) = (hg,P) \text{ and } \mathbf{g}^*(k) = k.$$

The functor $\mathbf{T}$ is said to be $\mathbf{K}$-*continuous* if $\mathbf{T}$ is $\mathbf{K}$-continuous in X for all $X \in |\mathbf{B}|$.

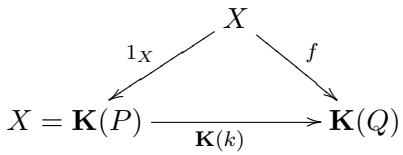
For a useful example, consider a covariant functor $\mathbf{K} : \mathbf{A} \to \mathbf{B}$ and let's see what conditions are sufficient for the $\mathbf{K}$-continuity of the identity functor $1_{\mathbf{B}}$. At first we have that the functor δ is the identity functor $1_{X\downarrow\mathbf{K}}$. Let $X \in |\mathbf{B}|$ be a fixed object for which we will study the $\mathbf{K}$-continuity of $1_{\mathbf{B}}$, and let D be an arbitrary object in $|\mathbf{B}|$, and $\mathbf{V} \in Funct_*(X \downarrow \mathbf{K}, D \downarrow \mathbf{K})$. It is necessary to exist a unique $g \in \mathbf{B}(D,X)$ such that $\mathbf{V} = \mathbf{g}^*$



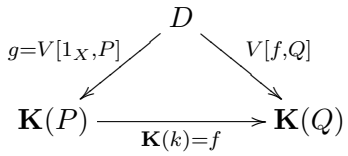
Suppose that $\mathbf{K}$ is surjective on objects. Then we can write the identity morphism 1_X as $1_X : X \to \mathbf{K}(P)$ and we can consider the pair $(1_X, P)$ as an object in the category $X \downarrow \mathbf{K}$. To this object corresponds by the functor $\mathbf{V}$ an object in $D \downarrow \mathbf{K}$ having the shape $(V[1_X, P], P)$, with $V[1_X, P] : D \to \mathbf{K}(P)$. Denote the morphism $V[1_X, P]$ by g , and we can write $g : D \to X$ since $\mathbf{K}(P) = X$. We can verify that for this morphism $\mathbf{V} = \mathbf{g}^*$. At first, we need to prove that, if (f, Q) is an arbitrary object in the category $X \downarrow \mathbf{K}$, then $\mathbf{V}(f,Q) = \mathbf{g}^*(f,Q)$, i.e., $(\mathbf{V}[f,Q],Q) = (fg,Q)$, and therefore that $\mathbf{V}[f,Q] = fg$. To verify this relation, we observe that we have the following commutative diagram



Now if we suppose that the functor $\mathbf{K}$ is full ([6], p. 47), such that the map $\mathbf{K}(P,Q) : \mathbf{A}(P,Q) \to \mathbf{B}(\mathbf{K}(P), \mathbf{K}(Q))$ is surjective, so that we can write $f = \mathbf{K}(k)$, for a morphism $k \in \mathbf{A}(P,Q)$. By this we have the following commutative diagram



and therefore $k : P \to Q$ is a morphism in $X \downarrow \mathbf{K}$. This implies that the following diagram



is commutative. Therefore we have proved $V[f, Q] = fg$, as we have proposed . Then, if we take an arbitrary morphism $l : (f, Q) \to (f', Q')$ in $X \downarrow \mathbf{K}$, then from the definition of $\mathbf{g}^*$ and from the properties of $\mathbf{V}$, we have $\mathbf{g}^*(l) = \mathbf{V}(l) = l$. Thus the functors $\mathbf{g}^*$ and $\mathbf{V}$ coincide. Finally, we have to impose the uniqueness of the morphism g. If $g' : D \to X$ is another morphism for which g'^* coincides to V, we need to have $fg = fg'$, for any morphism $f : X \to \mathbf{K}(Q)$. But by our hypothesis we can write $X = \mathbf{K}(P)$, so we can take $f = 1_X$, which implies $g' = g$. So, we have proved the following proposition.

Proposition 1.2. *If a covariant functor $\boldsymbol{K} : \boldsymbol{A} \to \boldsymbol{B}$ is a full functor, then the identity functor $1_{\boldsymbol{B}}$ is $\boldsymbol{K}$-continuous in every object $X = \boldsymbol{K}(P)$, with $P \in |\boldsymbol{A}|$. In particular, if the functor $\boldsymbol{K}$ is full and surjective on objects, then $1_{\boldsymbol{B}}$ is $\boldsymbol{K}$-continuous.*

This example can be generalized by choosing for the functor $\mathbf{T}$ an arbitrary category isomorphism $\mathbf{T} : \mathbf{B} \to \mathbf{C}$. Then it is easy to see that for each object $X \in |\mathbf{B}|$ the functor $\delta : X \downarrow \mathbf{K} \to \mathbf{T}(X) \downarrow \mathbf{TK}$ is an equivalence of categories. Namely, if $\mathbf{T}^{-1} : \mathbf{C} \to \mathbf{B}$ is the inverse functor of $\mathbf{T}$, then $\delta^{-1} : \mathbf{T}(X) \downarrow \mathbf{TK} \to X \downarrow \mathbf{K}$ is defined by $\delta^{-1}(\overline{f}, P) = (T^{-1}(\overline{f}), P)$ and $\delta^{-1}(k) = k$. Then it is sufficient to prove that, if $\mathbf{V} \in Funct_*(X \downarrow \mathbf{K}, D \downarrow \mathbf{TK})$, with $D \in |\mathbf{C}|$, then there is a unique $g \in \mathbf{C}(D, \mathbf{T}(X))$, such that $\mathbf{g}^* = \mathbf{V}\delta^{-1}$. We will continue the conditions from Proposition 1.2, namely that functor $\mathbf{K}$ is full and surjective on objects. As in the proof of above proposition, we can consider $X = K(P)$

and consider the pair $(1_X, P)$ like an object in $X \downarrow \mathbf{K}$. Let it there be considered $\mathbf{V}(1_X, P) = (g, P)$ with $g : D \to \mathbf{TK}(P) = \mathbf{T}(X)$.If $(\overline{f}, Q)$ is an arbitrary object in $\mathbf{T}(X) \downarrow \mathbf{TK}$, then $(\mathbf{T}^{-1}(\overline{f}), Q)$ is an object in $X \downarrow \mathbf{K}$, and further on we repeat the steps from the proof of Proposition 1.2, to obtain the following generalization.

Proposition 1.3. *If $\boldsymbol{T} : \boldsymbol{B} \to \boldsymbol{C}$ is an isomorphism of categories and $\boldsymbol{K} : \boldsymbol{A} \to \boldsymbol{B}$ is full and surjective on objects functor, then $\boldsymbol{T}$ is a functor $\boldsymbol{K}$-continuous.*

Now we will prove a result which shows the 'natural' character of the notion of $\mathbf{K}$-continuity.

Proposition 1.4. *Let there be given the functors $\boldsymbol{K}, \boldsymbol{K'} : \boldsymbol{A} \to \boldsymbol{B}$ and $\boldsymbol{T}, \boldsymbol{T'} : \boldsymbol{B} \to \boldsymbol{C}$ such that there exist some natural isomorphisms $\alpha : \boldsymbol{K} \to \boldsymbol{K'}$ and $\beta : \boldsymbol{T} \to \boldsymbol{T'}$. If $\boldsymbol{T}$ is $\boldsymbol{K}$-continuous in $X \in |\boldsymbol{B}|$, then it results that $\boldsymbol{T'}$ is $\boldsymbol{K'}$-continuous in X.*

Proof. Let be consider the functors $\delta : X \downarrow \mathbf{K} \to \mathbf{T}(X) \downarrow \mathbf{TK}$ and $\delta' : X \downarrow \mathbf{K'} \to \mathbf{T'}(X) \downarrow \mathbf{T'K'}$ from Definition 1.1 for the pair of functors $(\mathbf{K}, \mathbf{T})$ and $(\mathbf{K'}, \mathbf{T'})$, respectively. We can show that there exist two category isomorphisms $\downarrow \alpha : X \downarrow \mathbf{K} \to X \downarrow \mathbf{K'}$ and $\mathbf{J} : \mathbf{T}(X) \downarrow \mathbf{TK} \to \mathbf{T'}(X) \downarrow \mathbf{T'K'}$, such that $\delta'(\downarrow \alpha) = \mathbf{J}\delta$.

$$\begin{array}{ccc} X \downarrow \mathbf{K} & \xrightarrow{\downarrow\alpha} & X \downarrow \mathbf{K'} \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\delta'} \\ \mathbf{T}(X) \downarrow \mathbf{TK} & \xrightarrow[\mathbf{J}]{} & \mathbf{T'}(X) \downarrow \mathbf{T'K'} \end{array} ,$$

with $(\downarrow \alpha)(f, P) = (\alpha(P)f, P)$,$(\downarrow \alpha)(h) = h$, and

$$\mathbf{J}(f, P) = (\beta(\mathbf{K'}(P))\mathbf{T}(\alpha(P))f\beta(X)^{-1}, P),$$

$\mathbf{J}(u) = u$.

Then, let we consider $\mathbf{V'} \in Funct_*(X \downarrow \mathbf{K'}, D \downarrow \mathbf{T'K'})$. We will search for a functor $\mathbf{V} \in Funct_*(X \downarrow \mathbf{K}, D \downarrow \mathbf{TK})$. For this we consider the following category isomorphisms: $\downarrow \alpha : X \downarrow \mathbf{K} \to X \downarrow \mathbf{K'}$, defined as above, and $\downarrow (\alpha, \beta) : D \downarrow \mathbf{TK} \to D \downarrow \mathbf{T'K'}$, given by $\downarrow (\alpha, \beta)(h, P) = (\mathbf{T'}(\alpha(P))\beta(\mathbf{K}(P))h, P)$, $\downarrow (\alpha, \beta)(u) = u$. Then we consider the functor $\mathbf{V} = (\downarrow (\alpha, \beta))^{-1}\mathbf{V'}(\downarrow \alpha)$.

$$\begin{array}{ccc} X \downarrow \mathbf{K} & \xrightarrow{\downarrow\alpha} & X \downarrow \mathbf{K}' \\ {\scriptstyle \mathbf{V}}\downarrow & & \downarrow{\scriptstyle \mathbf{V}'} \\ D \downarrow \mathbf{TK} & \xrightarrow[\downarrow(\alpha,\beta)]{} & D \downarrow \mathbf{T}'\mathbf{K}' \end{array}$$

For this it can easily be proved that $\mathbf{V} \in Funct_*(X \downarrow \mathbf{K}, D \downarrow \mathbf{TK})$. Then there is a unique $g : D \to T(X)$ such that $\mathbf{V} = \mathbf{g}^*\delta$. Then we can consider $g' \in \mathbf{C}(D, \mathbf{T}'(X))$, defined by $g' = \beta(X)g$. For this morphism we have the following commutative diagram

$$\begin{array}{ccc} \mathbf{T}(X) \downarrow \mathbf{TK} & \xrightarrow{\mathbf{g}^*} & D \downarrow \mathbf{TK} \\ {\scriptstyle \mathbf{J}}\downarrow & & \downarrow{\scriptstyle \downarrow(\alpha,\beta)} \\ \mathbf{T}'(X) \downarrow \mathbf{T}'\mathbf{K}' & \xrightarrow[\mathbf{g}'^*]{} & D \downarrow \mathbf{T}'\mathbf{K}' \end{array}$$

Then $\mathbf{g}'^*\delta' = \mathbf{g}'^*\mathbf{J}\delta(\downarrow \alpha)^{-1} =\downarrow (\alpha, \beta)\mathbf{g}^*\delta(\downarrow \alpha)^{-1} =\downarrow (\alpha, \beta)\mathbf{V}(\downarrow \alpha)^{-1} =\downarrow (\alpha, \beta)(\downarrow (\alpha, \beta))^{-1}\mathbf{V}'(\downarrow \alpha)(\downarrow \alpha)^{-1} = \mathbf{V}'$.

$$\begin{array}{ccc} & X \downarrow \mathbf{K}' & \\ {\scriptstyle \delta'}\swarrow & & \searrow{\scriptstyle \mathbf{V}'} \\ \mathbf{T}'(X) \downarrow \mathbf{T}'\mathbf{K}' & \xrightarrow[\mathbf{g}'^*]{} & D \downarrow \mathbf{T}'\mathbf{K}' \end{array}$$

The uniqueness of g' easily follows from the uniqueness of g. □

2. Categorical shape theory

Definition 2.1. ([1], p. 29) Let $\mathbf{K} : \mathbf{A} \to \mathbf{B}$ be a covariant functor. A shape theory for $\mathbf{K}$ is a pair $(\mathbf{S}, S)$ with $\mathbf{S}$ a category and S a covariant functor $S : \mathbf{B} \to \mathbf{S}$, such that the following axioms are verified:

(S1) $|\mathbf{S}| = |\mathbf{B}|$ and for all X in $|\mathbf{B}|$, $S(X) = X$.

(S2) If $s \in \mathbf{S}(X, \mathbf{K}(Q))$, there is a unique $f \in \mathbf{B}(X, \mathbf{K}(Q))$, such that $S(f) = s$.

(S3) S is $\mathbf{K}$-continuous.

Example 2.2. Let $\mathbf{K} : \mathbf{A} \to \mathbf{B}$ be a functor which is full and surjective on objects. Let $\mathbf{S}$ be a category with the same objects as $\mathbf{B}$ and let $S : \mathbf{B} \to$

$\mathbf{S}$ be an isomorphism of categories keeping the objects. Then the pair $(\mathbf{S}, S)$ is a shape theory for the functor $\mathbf{K}$. Particularly, in the given conditions for $\mathbf{K}$, the pair $(\mathbf{B}, Id_{\mathbf{B}})$ is a shape theory for the functor $\mathbf{K}$.

Regarding the uniqueness of shape theories for a given functor $\mathbf{K}$, the following result is known.

Proposition 2.3. ([1], *p.*32). *If $(\boldsymbol{S}, S)$ and $(\boldsymbol{S'}, S')$ are two shape theories for a functor $\mathbf{K} : \mathbf{A} \to \mathbf{B}$, then there is a unique isomorphism of categories $\mathbf{R} : \mathbf{S} \to \mathbf{S'}$, such that $\boldsymbol{R}S = S'$.*

Using Proposition 1.4 we obtain the following result.

Corollary 2.4. *Let $\mathbf{K}, \mathbf{K'} : \mathbf{A} \to \mathbf{B}$ be two covariant functors between which there is a natural isomorphism $\alpha : \mathbf{K} \to \mathbf{K'}$. Then a pair $(\boldsymbol{S}, S)$ which is a shape theory for $\boldsymbol{K}$ is a shape theory for $\boldsymbol{K'}$ also.*

3. The Holsztyński shape theory [3]. Let $\mathbf{K} : \mathbf{A} \to \mathbf{B}$ be an arbitrary covariant functor. Let $\mathbf{S_K}$ be the category with $|\mathbf{S_K}| = |\mathbf{B}|$, and $\mathbf{S_K}(X, Y) = Funct_*(Y \downarrow \mathbf{K}, X \downarrow \mathbf{K})$. For an object $X \in |\mathbf{S_K}| = |\mathbf{B}|$, the identity morphism, denoted by $\mathbf{T}$, is given by $\mathbf{Id}_{X \downarrow \mathbf{K}}$. The functor $S_{\mathbf{K}} : \mathbf{B} \to \mathbf{S_K}$ is defined by $S_{\mathbf{K}}(X) = X$ on objects, and if $h \in \mathbf{B}(X, Y)$, then $S_{\mathbf{K}}(h) \in \mathbf{S_K}(X, Y) = Funct_*(Y \downarrow \mathbf{K}, X \downarrow \mathbf{K})$ is given by $S_{\mathbf{K}}(h)((g, P)) = (gh, P)$, on objects, and by $S_{\mathbf{K}}(l) = l$ on morphisms. Let us see in what case the pair $(\mathbf{S_K}, S_{\mathbf{K}})$ is a shape theory for the functor $\mathbf{K}$.

The axiom (S1) is obviously satisfied from the definitions of the category $\mathbf{S_K}$ and the functor $S_{\mathbf{K}}$.

Then the following proposition holds.

Proposition 3.1. ([1], *p.*30). *If the functor $S_{\boldsymbol{K}}$ satisfies the axiom $(S2)$, then it is $\boldsymbol{K}$-continuous , so that the axiom $(S3)$ is satisfied, and therefore the pair $(\mathbf{S_K}, S_{\boldsymbol{K}})$ is a shape theory for the functor $\boldsymbol{K}$.*

Corollary 3.2. ([1], *p.*31). *If a functor $\boldsymbol{K} : \boldsymbol{A} \to \boldsymbol{B}$ is full, particularly if $\boldsymbol{K}$ is the insertion of a full subcategory $\boldsymbol{A}$ into $\boldsymbol{B}$, then the pair $(\mathbf{S_K}, S_{\boldsymbol{K}})$ is a shape theory for $\boldsymbol{K}$.*

By Corollary 2.4 we obtain the following result.

Corollary 3.3. *Let $\mathbf{K} : \mathbf{A} \to \mathbf{B}$ be a covariant functor for which the Holsztyński pair $(\mathbf{S_K}, S_{\boldsymbol{K}})$ is a shape theory for $\boldsymbol{K}$. Then, if another pair $(\boldsymbol{S}, S)$ is a shape theory for $\boldsymbol{K}$ also, then there is a category isomorphism $\boldsymbol{R} : S_{\boldsymbol{K}} \to \boldsymbol{S}$, such that $S = \boldsymbol{R}S_{\boldsymbol{K}}$.*

Example 3.4. Consider $\mathcal{T}$ a category and $\mathcal{P}$ a dense subcategory of $\mathcal{T}$. Let $Sh_{(\mathcal{T},\mathcal{P})}$ be a shape theory in the sense of MARDEŠIĆ-SEGAL ([5], Ch.I, §2), with the shape category **Sh** and the shape functor $Sh : \mathcal{T} \to \mathbf{Sh}$. Then the pair $(\mathbf{Sh}, Sh)$ is a shape theory for the inclusion functor $\mathfrak{I} : \mathcal{P} \to \mathcal{T}$. Moreover, the Hosztyński pair $(\mathbf{S}_\mathfrak{I}, S_\mathfrak{I})$ of the functor $\mathfrak{I}$ is a shape theory for the functor $\mathfrak{I}$, so that there exists an isomorphism of categories $\mathbf{R} : \mathbf{S}_\mathfrak{I} \to \mathbf{Sh}$, such that $Sh = \mathbf{R}S_\mathfrak{I}$.

Example 3.5. Other examples obtained by Proposition 3.1, but for which the functor $\mathbf{K}$ is not full ([1], p.32-35).

(a) $\mathbf{K} = \mathbf{U} : \mathbf{Groups} \to \mathbf{Sets}$, the forgetful functor. Consider the Holsztyński pair $(\mathbf{S_U}, S_\mathbf{U})$. For a set X denote by $\mathbf{F}(X)$ the free group on X. Then it is proved that the pair $(\mathbf{S_K}, S_\mathbf{K})$ is isomorphic to $(\mathbf{Free\ Groups}, \mathbf{F})$ and using the fact that $\mathbf{F}$ is a left adjoint of $\mathbf{U}$, can be verified that $S_\mathbf{U} = \mathbf{F}$ satisfies the axiom (S2).

(b) The previous example can be generalized considering an arbitrary functor $\mathbf{K} : \mathbf{A} \to \mathbf{B}$ with a left adjoint. First, we recall the notion of monad ([4], Ch.VI,§1). A *monad* $\mathbf{T} =< \mathbf{T}, \eta, \mu >$ in a category $\mathbf{B}$ consists of a functor $\mathbf{T} : \mathbf{B} \to \mathbf{B}$ and two natural transformations $\eta : 1_\mathbf{B} \to \mathbf{T}$ and $\mu : \mathbf{T^2} \to \mathbf{T}$ for which the following relations hold:

$$\mu \circ (\mu T) = \mu \circ (T\mu) : T^3 \to T \ , \ \mu \circ (\eta\mathbf{T}) = 1_T, \ \mu \circ (\mathbf{T}\eta) = 1_T,$$

where $\eta\mathbf{T} : \mathbf{T} \to \mathbf{T}^2$ and $\mathbf{T}\eta : \mathbf{T} \to \mathbf{T}^2$ are defined respectively by $(\eta\mathbf{T})_\mathbf{B} = \eta_{\mathbf{T}(\mathbf{B})}$ and $(\mathbf{T}\eta)_\mathbf{B} = \mathbf{T}(\eta_\mathbf{B})$, for all $B \in |\mathbf{B}|$.

For a monad $\mathbf{T} =< \mathbf{T}, \eta, \mu >$, in a category $\mathbf{B}$, the *Kleisly category* $\mathbf{B_T}$, of this monad is defined as follows:

- $|\mathbf{B_T}| = |\mathbf{B}|$;

- if $X, Y \in |\mathbf{B_T}|$, then $\mathbf{B_T}(X, Y) = \mathbf{B}(X, \mathbf{T}(Y))$;

- if $f : X \to \mathbf{T}(Y), g : Y \to \mathbf{T}(Z)$ are morphisms in $\mathbf{B}$, then we consider the composition $g \circ f = \mu\mathbf{T}(g)f : X \to \mathbf{T}(Z)$, and this defines the composition of morphisms in $\mathbf{B_T}$.

Now we can define two functors $\mathbf{F_T} : \mathbf{B} \to \mathbf{B_T}, \mathbf{G_T} : \mathbf{B_T} \to \mathbf{B}$, which are identities on objects and for the morphisms:

- if $k \in \mathbf{B}(X, Y)$, then $\mathbf{F_T}(k) = \eta_Y k : X \to \mathbf{T}Y$;

- if $f \in \mathbf{B_T}(X, Y) = \mathbf{B}(X, \mathbf{T}(Y))$, then $\mathbf{G_T}(f) = \mu_Y\mathbf{T}(f) : T(Y) \to \mathbf{T}^2(Y) \to \mathbf{T}(Y)$.

If we have two functors $\mathbf{K} : \mathbf{A} \to \mathbf{B}$ and $\mathbf{L} : \mathbf{B} \to \mathbf{A}$, where $\mathbf{L}$ is a left adjoint of $\mathbf{K}$, then we can define a monad $< \mathbf{T}, \eta, \mu >$, with $\mathbf{T} = \mathbf{KL}$,

$\eta : \mathbf{Id_B} \to \mathbf{KL}$, the adjunction unit, and $\mu = \mathbf{K}\varepsilon\mathbf{L}$, with $\varepsilon : \mathbf{LK} \to \mathbf{Id_A}$, the corresponding co-unit. To this monad we can consider its Kleisli category $\mathbf{B_T}$ and the pair of functors $\mathbf{L_T}$ si $\mathbf{K_T}$. In this case the Holsztyński category $\mathbf{S_K}$ is isomorphic to the Kleisly category $\mathbf{B_T}$, and then the functor $S_\mathbf{K} : \mathbf{B} \to \mathbf{S_K}$ can be identified with the functor $\mathbf{F_T} : \mathbf{B} \to \mathbf{B_T}$. Now it can easily be verified for $\mathbf{F_T}$ the axiom (S2) and we can apply Proposition 3.1 to deduce that the Holsztyński pair $(\mathbf{S_K}, S_\mathbf{K})$ is a shape theory for the functor $\mathbf{K}$.

The following property of the Holsztyński shape functor $S_\mathbf{K}$ will be useful.

Proposition 3.6. *Let $\mathbf{K} : \mathbf{A} \to \mathbf{B}$ be a functor with the Holsztyński pair $(\mathbf{S_K}, S_\mathbf{K})$. Then the functor $S_\mathbf{K} : \mathbf{B} \to \mathbf{S_K}$ is an epifunctor i.e., carries an epimorphism in an epimorphism.*

Proof. Let $\varphi \in \mathbf{B}(X, Y)$ be an epimorphism, and let $\Phi := S_\mathbf{K}(\varphi)$ be its image by the shape functor. Then $\Phi \in \mathbf{S_K}(X, Y) = Funct_*(Y \downarrow \mathbf{K}, X \downarrow \mathbf{K})$. Suppose that $\Psi_1, \Psi_2 \in \mathbf{S_K}(Y, Z)$ such that $\Psi_1 \circ \Phi = \Psi_2 \circ \Phi$, in $\mathbf{S_K}$. Write Ψ_1, Ψ_2 as functors $\Psi_1, \Psi_2 : Z \downarrow \mathbf{K} \to Y \downarrow \mathbf{K}$ from $Funct_*(X \downarrow \mathbf{K}, Z \downarrow \mathbf{K})$, such that $\Phi \circ \Psi_1 = \Phi \circ \Psi_2$. We need to prove that $\Psi_1 = \Psi_2$. For an arbitrary object $(h, P) \in |Z \downarrow \mathbf{K}|$, let it be denoted $\Psi(h, P) = (k_1, P)$ and $\Psi_2(h, P) = (k_2, P)$. Then from $(\Phi \circ \Psi_1)(h, P) = (\Phi \circ \Psi_2)(h, P)$ we have $(k_1 \circ \varphi, P) = (k_2 \circ \varphi, P)$, which implies $k_1 \circ \varphi = k_2 \circ \varphi$,and since φ is an epimorphism this implies $k_1 = k_2$. Therefore $\Psi_1(h, P) = \Psi_2(h, P)$ and because from the affiliation to the class $Funct_*(Z \downarrow \mathbf{K}, Y \downarrow \mathbf{K})$ the functors Ψ_1, Ψ_2 coincide on the morphisms, we can conclude that $\Psi_1 = \Psi_2$. □

Example 3.7. If $f : G \to H$ is an epimorphism of groups, then the induced homomorphism between the free groups $\mathbf{F}(f) : \mathbf{F}(G) \to \mathbf{F}(H)$ is an epimorphism.

Proposition 3.8. *Let $\mathbf{K} : \mathbf{A} \to \mathbf{B}$ be a functor with the Holsztyński pair $(\mathbf{S_K}, S_\mathbf{K})$. If a morphism $\varphi \in \boldsymbol{B}(X, Y)$ satisfies the condition that for any object $P \in \boldsymbol{A}$ the map $\boldsymbol{B}(Y, K(P)) \to \boldsymbol{B}(X, \boldsymbol{K}(P))$, given by $g \to g\varphi$, is an onto map, then $S_{\boldsymbol{K}}(\varphi)$ is a monomorphism.*

Proof. Let $\varphi \in \mathbf{B}(X, Y)$ be the given morphism, and let $\Phi := S_\mathbf{K}(\varphi)$ be its image by the shape functor.Then $\Phi \in \mathbf{S_K}(X, Y) = Funct_*(Y \downarrow \mathbf{K}, X \downarrow \mathbf{K})$. Suppose that $\Psi_1, \Psi_2 \in \mathbf{S_K}(Z, X)$ such that $\Phi \circ \Psi_1 = \Phi \circ \Psi_2$, in $\mathbf{S_K}$. Write Ψ_1, Ψ_2 as functors $\Psi_1, \Psi_2 : X \downarrow \mathbf{K} \to Z \downarrow \mathbf{K}$ from $Funct_*(X \downarrow \mathbf{K}, Z \downarrow \mathbf{K})$, such that $\Psi_1 \circ \Phi = \Psi_2 \circ \Phi$, and we need to prove the equality

$\Psi_1 = \Psi_2$. By hypothesis, for an arbitrary object $(f, P) \in |X \downarrow K|$ we can write $(f, P) = (g\varphi, P) = \Phi(g, P)$. Then $\Psi_1(f, P) = (\Psi_1 \circ \Phi)(g, P) = (\Psi_2 \circ \Phi)(g, P) = \Psi_2(f, P)$. This coincidence on objects, together with the affiliation to $\mathbf{S_K}(Z, X)$ of the functors Ψ_1 and Ψ_2, shows that these functor are equal.

Example 3.9. If $f : G \to H$ is a monomorphism of groups, then the induced homomorphism between the free groups $\mathbf{F}(f) : \mathbf{F}(G) \to \mathbf{F}(H)$ is a monomorphism. This means that in fact the functor $S_{\mathbf{U}} : \mathbf{Sets} \to \mathbf{FreeGroups}$ is a monofunctor. □

Remark 3.10. Generally a Holsztyński functor $S_{\mathbf{K}} : \mathbf{B} \to \mathbf{S_K}$ is neither a full nor faithful functor. One can have morphisms in $\mathbf{S_K}$ not of the form $S_{\mathbf{K}}(f)$, and also different morphisms $f, g \in \mathbf{B}(X, Y)$ with $S_{\mathbf{K}}(f) = S_{\mathbf{K}}(g)$ ([5], p. 26).

4. Change of models. Since in considering the shape theory relative to a functor $\mathbf{K} : \mathbf{A} \to \mathbf{B}$, one is considering approximations to objects in $\mathbf{B}$ by objects from $\mathbf{A}$, it is natural to call the objects of $\mathbf{A}$ the 'models' for the shape theory.The simplest situation in which a functor is induced between shape theories is when one changes the models in the following way. The following theorem is a generalization of a theorem which appears in [1], p.112.

Theorem 4.1. *Suppose given a commutative diagram of categories and functors*

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{K} & \mathbf{B} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle G} \\ \mathbf{A}' & \xrightarrow[K']{} & \mathbf{B}' \end{array}$$

and consider the Holsztyński pairs $(\mathbf{S_K}, S_{\mathbf{K}})$, $(\mathbf{S_{K'}}, S_{\mathbf{K'}})$ *for the functors* $\boldsymbol{K}$ *and* $\boldsymbol{K'}$ *respectively. If the functor* $\boldsymbol{G}$ *is an equivalence of categories then there exists a covariant functor* $(\boldsymbol{F}, \boldsymbol{G})^* : \mathbf{S_{K'}} \to \mathbf{S_K}$, *with the following commutative diagram*

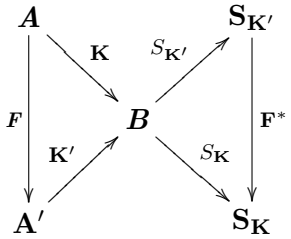
$$\begin{array}{ccc} \boldsymbol{B} & \xrightarrow{S_K} & \mathbf{S_K} \\ {\scriptstyle G}\downarrow & & \uparrow{\scriptstyle (\mathbf{F},\mathbf{G})^*} \\ \mathbf{B}' & \xrightarrow[S_{K'}]{} & \mathbf{S_{K'}} \end{array}$$

Proof. Denote by $\mathbf{G}^{-1} : \mathbf{B}' \to \mathbf{B}$ the 'inverse' functor of $\mathbf{G}$. Then on objects we define $(\mathbf{F}, \mathbf{G})^*(X') = \mathbf{G}^{-1}(X')$ for all $X' \in |\mathbf{B}'|$. If $u' \in \mathbf{S}_{\mathbf{K}'}(X', Y') = Funct_*(Y' \downarrow \mathbf{K}', X' \downarrow \mathbf{K}')$, denote $\mathbf{G}^{-1}(X') = X, \mathbf{G}^{-1}(Y') = Y$, and we need define a morphism $u \in \mathbf{S}_{\mathbf{K}}(X, Y) = Funct_*(Y \downarrow \mathbf{K}, X \downarrow \mathbf{K})$. If $(f, P) \in |Y \downarrow \mathbf{K}|$ with $P \in |\mathbf{A}|$ and $f : Y \to \mathbf{K}(P)$ a morphism in $\mathbf{B}$, then we consider $\mathbf{G}(f) : \mathbf{G}(Y) \to \mathbf{G}(\mathbf{K}(P)) = \mathbf{K}'(\mathbf{F}(P))$. By this the pair $(\mathbf{G}(f), \mathbf{F}(P))$ appears as an object in $\mathbf{G}(Y) \downarrow \mathbf{K}' = Y' \downarrow \mathbf{K}'$. Let $u'(\mathbf{G}(f), \mathbf{F}(P)) = (k, \mathbf{F}(P))$, with $k : X' \to \mathbf{K}'(\mathbf{F}(P)) = \mathbf{G}(\mathbf{K}(P))$ a morphism in $\mathbf{B}'$. Consider $g = \mathbf{G}^{-1}(k) : \mathbf{G}^{-1}(X') = X \to \mathbf{K}(P)$, and define $u(f, P) = (g, P) \in |X \downarrow \mathbf{K}|$. Let there be another object $(f', P') \in |Y \downarrow \mathbf{K}|$, $u(f', P') = (g', P')$, with $u'(\mathbf{G}(f'), \mathbf{F}(P')) = (\mathbf{G}(g'), \mathbf{F}(P')$ and suppose that $h : (f, P) \to (f', P')$ is a morphism in $Y \downarrow \mathbf{K}$. Then $h \in \mathbf{A}(P, P')$ and $f' = \mathbf{K}(h)f$. This implies $\mathbf{G}(f') = \mathbf{G}\mathbf{K}(h)\mathbf{G}(f) = \mathbf{K}'(\mathbf{F}(h))\mathbf{G}(f)$ and this shows that $\mathbf{F}(h)$ is a morphism from $(\mathbf{G}(f), \mathbf{F}(P))$ to $(\mathbf{G}(f'), \mathbf{F}(P'))$ in the category $\mathbf{G}(Y) \downarrow \mathbf{K}' = \mathbf{Y}' \downarrow \mathbf{K}'$. Since $u' \in Funct_*(Y' \downarrow \mathbf{K}', X' \downarrow \mathbf{K}')$, it follows that $\mathbf{F}(h)$ is also a morphism from $(\mathbf{G}(g), \mathbf{F}(P))$ to $(\mathbf{G}(g'), \mathbf{F}(P'))$ in $X' \downarrow \mathbf{K}'$, so that $\mathbf{G}(g') = \mathbf{K}'(\mathbf{F}(h))\mathbf{G}(g) = \mathbf{G}(\mathbf{K}(h))\mathbf{G}(g) \Rightarrow g' = \mathbf{K}(h)g$. By this we see that if we can put $u(h) = h$ to obtain a functor $u \in Funct_*(Y \downarrow \mathbf{K}, X \downarrow \mathbf{K})$, and define $(\mathbf{F}, \mathbf{G})^*(u') = u$. If $u' = 1_{X' \downarrow \mathbf{K}'}$, it is immediate that $u = 1_{X \downarrow \mathbf{K}}$. Then, if $u' \in Funct_*(Y' \downarrow \mathbf{K}', X' \downarrow \mathbf{K}')$ and $v' \in Funct_*(Z' \downarrow \mathbf{K}', Y' \downarrow \mathbf{K}')$, with $(\mathbf{F}, \mathbf{G})^*(u') = u$, $(\mathbf{F}, \mathbf{G})^*(v') = v$ and $(\mathbf{G}, \mathbf{F})^*(u'v') = w$, then if for an object $(f, P) \in |Z \downarrow \mathbf{K}|$ we put $v(f, P) = (f', P) \Rightarrow v'(\mathbf{G}(f), \mathbf{F}(P)) = (\mathbf{G}(f'), \mathbf{F}(P))$ if $u(v(f, P)) = (f'', P) \Rightarrow u'(\mathbf{G}(f'), \mathbf{F}(P)) = (\mathbf{G}(f''), \mathbf{F}(P))$. Thus $(u'v')(\mathbf{G}(f), \mathbf{F}(P)) = (\mathbf{G}(f''), \mathbf{F}(P))$, which implies $w(f, P) = (f'', P) = (uv)(f, P)$. This implies $(\mathbf{F}, \mathbf{G})^*(u'v') = (\mathbf{F}, \mathbf{G})^*(u)(\mathbf{F}, \mathbf{G})^*(v)$, so that $(\mathbf{F}, \mathbf{G})^*$ is a covariant functor. The commutativity of the diagram on objects is obvious. If $\alpha \in \mathbf{B}(X, Y)$, and $(f, P) \in |Y \downarrow \mathbf{K}|$, then on the one hand $S_{\mathbf{K}}(\alpha)(f, P) = (f\alpha, P)$, and on the other hand if $((\mathbf{F}, \mathbf{G})^* S_{\mathbf{K}'} G)(\alpha)(f, P) = (g, P)$, then $(S_{\mathbf{K}'}\mathbf{G}(\alpha))(\mathbf{G}(f), \mathbf{F}(P)) = (\mathbf{G}(g), \mathbf{F}(P)) \Rightarrow (\mathbf{G}(f)\mathbf{G}(\alpha), \mathbf{F}(P)) = (\mathbf{G}(g), \mathbf{F}(P)) \Rightarrow g = f\alpha$. Therefore, $(\mathbf{F}, \mathbf{G})^* S_{\mathbf{K}'}\mathbf{G} = S_{\mathbf{K}}$ which ends the proof. □

If we take $\mathbf{B}' = \mathbf{B}$ and $\mathbf{G} = 1_{\mathbf{B}}$, then we obtain the following result from [1], p. 112 (see also [2]).

Corollary 4.2. *Let* $\mathbf{K} : \mathbf{A} \to \mathbf{B}$ *and* $\mathbf{K}' : \mathbf{A}' \to \mathbf{B}$ *be covariant functors with the Holsztyński pairs* $(\mathbf{S}_{\mathbf{K}}, S_{\mathbf{K}})$ *and* $(\mathbf{S}_{\mathbf{K}'}, S_{\mathbf{K}'})$ *respectively. Suppose a functor* $\mathbf{F} : \mathbf{A} \to \mathbf{A}'$ *is given such that* $\mathbf{K}'\mathbf{F} = \mathbf{K}$. *Then there exists a*

functor $\mathbf{F}^* : \mathbf{S}_{\mathbf{K}'} \to \mathbf{S}_{\mathbf{K}}$ *such that* $\mathbf{F}^*\mathbf{S}_{\mathbf{K}'} = \mathbf{S}_{\mathbf{K}}$.



Proposition 4.3. *Suppose that for the functors* $\mathbf{K} : \mathbf{A} \to \mathbf{B}$, $\mathbf{K}' : \mathbf{A}' \to \mathbf{B}$ *and* $\mathbf{K}'' : \mathbf{A}'' \to \mathbf{B}$ *there exist the Holsztyński shape theories. Suppose that* $\mathbf{F} : \mathbf{A} \to \mathbf{A}'$, $\mathbf{G} : \mathbf{A}' \to \mathbf{A}''$ *are two functors. Then the functor* $\mathbf{F}^*$ *between shape theories constructed in Corollary* 4.2 *has the following properties:*

(*i*) $(1_{\mathbf{A}})^* = 1_{\mathbf{S}_{\mathbf{K}}}$;

(*ii*) $(\mathbf{GF})^* = \mathbf{F}^*\mathbf{G}^*$.

Proof. We recall that for $u' \in \mathbf{S}_{\mathbf{K}'}(X, Y) = Funct_*(Y \downarrow \mathbf{K}', X \downarrow \mathbf{K}')$,$(f, P) \in |Y \downarrow \mathbf{K}|$ and h a morphism in $Y \downarrow \mathbf{K}$, we have:

$$u'(f, \mathbf{F}(P)) = (g, \mathbf{F}(P)) \Rightarrow \mathbf{F}^*(u')(f, P) = (g, P), \tag{4.1}$$

and

$$\mathbf{F}^*(u')(h) = h. \tag{4.2}$$

(i) is obvious.

(ii) If $u'' \in Funct_*(Y \downarrow \mathbf{K}'', X \downarrow \mathbf{K}'')$ and h is a morphism in $Y \downarrow K$, then $(\mathbf{GF})^*(u'')(h) = h = (\mathbf{F}^*\mathbf{G}^*)(u'')(h)$. Then, for $(f, P) \in |Y \downarrow \mathbf{K}|$, $(\mathbf{GF})^*(u'')(f, P) = (g, P)$, for $u''(f, \mathbf{GF}(P)) = (g, \mathbf{GF}(P))$. But the last relation implies $\mathbf{G}^*(u'')(f, \mathbf{F}(P)) = (g, \mathbf{F}(P))$, and this implies $F^*(G^*(u''))(f, P) = (g, P)$, so that $(\mathbf{GF})^*(u'')(f, P) = (\mathbf{F}^*\mathbf{G}^*)(u'')(f, P)$. Thus we have obtained the equality

$$(\mathbf{GF})^*(u'') = (\mathbf{F}^*\mathbf{G}^*)(u''),$$

which together with the coincidence of the functors on objects imply (ii).□

Corollary 4.4. *In the condition of Corollary* 4.2*, if* $\mathbf{F}$ *is an isomorphism of categories, then* $\mathbf{F}^*$ *is an isomorphism of categories.*

Example 4.5. Let $\mathbf{K} : \mathbf{A} \to \mathbf{B}$ be a category with the Holsztyński pair $(\mathbf{S_K}, S_\mathbf{K})$. If we consider the pair $(\mathbf{B}, 1_\mathbf{B})$ as the Holsztyński pair for the the identity functor $1_\mathbf{B}$, then by Corollary 4.2 we obtain that $\mathbf{K}^* = S_\mathbf{K}$.

Example 4.6. Let $\mathbf{K} : \mathbf{A} \to \mathbf{B}$ and $\mathbf{F} : \mathbf{A}' \to \mathbf{A}$ be functors and consider the Holsztyński pairs $(\mathbf{S_K}, S_\mathbf{K})$ and $(\mathbf{S_{KF}}, S_\mathbf{KF})$. Then, by Corollary 4.2 we obtain $S_\mathbf{KF} = \mathbf{F}^* S_\mathbf{K}$. This example becomes interesting if we suppose that $\mathbf{K}$ and $\mathbf{F}$ admit some left adjoint functors. Then $\mathbf{KF}$ also admits a left adjoint functor, and the two pairs really are some shape theories.

Example 4.7. If we denote by $\mathbf{U} : \mathbf{Groups} \to \mathbf{Sets}$ and

$$\mathbf{U}' : \mathbf{Fin.Groups} \to \mathbf{Sets}$$

the forgetful functors, and by $\mathbf{K} : \mathbf{Fin.Groups} \to \mathbf{Groups}$ the inclusion functor, then there is a covariant functor

$$\mathbf{K}^* : \mathbf{FreeGroups} \to \mathbf{pro}(\mathbf{Fin.Groups}),$$

satisfying $\mathbf{K}^* S_\mathbf{U} = S_{\mathbf{U}'}$.

Remark 4.8. Proposition 4.3 can be generalized for the case from Theorem 4.1 and then we can deduce that if the functor $\mathbf{F}$ is an isomorphism of categories, then $(\mathbf{F}, \mathbf{G})^*$ is an isomorphism of categories also.

5. Natural transformations

Theorem 5.1. *Let* $\mathbf{K} : \mathbf{A} \to \mathbf{B}$, $\mathbf{K}' : \mathbf{A}' \to \mathbf{B}$ *admitting the Holsztyński shape theories, and* $\mathbf{F}, \mathbf{G} : \mathbf{A} \to \mathbf{A}'$ *functors such that* $\mathbf{K}'\mathbf{F} = \mathbf{K}'\mathbf{G} = \mathbf{K}$. *Then for a natural transformation,* $\varphi : \mathbf{F} \to \mathbf{G}$ *is induced a natural transformation* $\varphi^* : \mathbf{F}^* \to \mathbf{G}^*$.

Proof. For an object $X \in |\mathbf{B}|$ define $\varphi^*_X : \mathbf{F}^*(X) \to \mathbf{G}^*(X)$ a morphism in $\mathbf{S_K}$, i.e., $\varphi^*_X \in Funct_*(X \downarrow \mathbf{K}, X \downarrow \mathbf{K})$ by

$$\varphi^*_X(f, P) = (\mathbf{K}'(\varphi_P) \circ f, P), \tag{5.1}$$

for $(f, P) \in |X \downarrow \mathbf{K}|$, and

$$\varphi^*_X(h) = h \tag{5.2}$$

for a morphism $h : P \to P'$ in $X \downarrow \mathbf{K}$. This is correct since if $\mathbf{K}(h)f = f'$, then from the equality $G(h)\varphi_P = \varphi_{P'}F(h)$ we deduce $\mathbf{K}'(G(h))\mathbf{K}'(\varphi_P) = \mathbf{K}'(\varphi_{P'})\mathbf{K}'(\mathbf{F}(h))$, which implies

$$\mathbf{K}(h)\mathbf{K}'(\varphi_P) = \mathbf{K}'(\varphi_{P'})\mathbf{K}(h) \Rightarrow$$

$$\mathbf{K}(h)\mathbf{K}'(\varphi_P)f = \mathbf{K}'(\varphi_{P'})\mathbf{K}(h)f = \mathbf{K}'(\varphi_{P'})f'.$$

Now if $u' : Y \downarrow \mathbf{K}' \to X \downarrow \mathbf{K}'$ is a morphism in $\mathbf{S}_{\mathbf{K}'}$, then we have $\mathbf{G}^*(u')\varphi_Y^* = \varphi_X^*\mathbf{F}^*(u')$, i.e., the following diagram commutes.

$$\begin{array}{ccc} Y \downarrow \mathbf{K} & \xrightarrow{\varphi_Y^*} & Y \downarrow \mathbf{K} \\ {\scriptstyle \mathbf{F}^*(u')}\downarrow & & \downarrow{\scriptstyle \mathbf{G}^*(u')} \\ X \downarrow \mathbf{K} & \xrightarrow[\varphi_X^*]{} & X \downarrow \mathbf{K} \end{array}$$

If for an object $(f, P) \in |Y \downarrow \mathbf{K}|$, $u'(f, \mathbf{F}(P)) = (g, \mathbf{F}(P))$, then $u'(\mathbf{K}'(\varphi_P)f, \mathbf{G}(P)) = (\mathbf{K}'(\varphi_P)g, \mathbf{G}(P))$.

By this we have $(\varphi_X^*F^*(u'))(f, P) = \varphi_X^*(g, P) = (K'(\varphi_P)g, P)$, and $(G^*(u')\varphi_Y^*)(f, P) = G^*(u')(K'(\varphi_P)f, P) = (K'(\varphi_P)g, P)$. Thus, the diagram is commutative on objects, and since the commutativity on morphism is trivial, we conclude that the diagram commutes in the category $\mathbf{S}_{\mathbf{K}}$. □

Corollary 5.2. *In the conditions of Theorem* 5.1 *the natural transformation $\varphi : \boldsymbol{F} \to \boldsymbol{G}$ is a natural equivalence, then $\varphi^* : \mathbf{F}^* \to \mathbf{G}^*$ is a natural equivalence also.*

6. Application: the case of Mardešić-Segal shape theories

Theorem 6.1. *Let $\mathcal{T}$ be a category with $\mathcal{S}$ a full dense subcategory of $\mathcal{T}$, and $\mathcal{P}$ a full dense subcategory of $\mathcal{S}$, $\mathcal{P} \subset \mathcal{S} \subset \mathcal{T}$. Then $\mathcal{P}$ is a dense subcategory of $\mathcal{T}$, and if for two objects $X, Y \in |\mathcal{T}|$ we have $sh_{(\mathcal{T},\mathcal{S})}(X) = sh_{(\mathcal{T},\mathcal{S})}(Y)$ then $sh_{(\mathcal{T},\mathcal{P})}(X) = sh_{(\mathcal{T},\mathcal{P})}(Y)$.*

Proof. It is known ([7], Corollary 2.5) that a subcategory $\mathcal{K}$ of the category $\mathcal{C}$ is dense if and only if $pro-\mathcal{K}$ is reflective in $pro-\mathcal{C}$, which means that, if $J : \mathcal{K} \to \mathcal{C}$ is the inclusion functor, then the functor $pro-J : pro-\mathcal{K} \to pro-\mathcal{C}$ has a left adjoint. In our case, if $J : \mathcal{P} \hookrightarrow \mathcal{S}$, $J' : \mathcal{S} \hookrightarrow \mathcal{T}$ and $J'' : \mathcal{P} \hookrightarrow \mathcal{T}$ are the inclusion functors, then $pro-J : pro-\mathcal{P} \hookrightarrow pro-\mathcal{S}$

and $pro-J' : pro-\mathcal{S} \hookrightarrow pro-\mathcal{T}$ have some left adjoint functors. Then since $J'' = J'J$ we have $pro-J'' = (pro-J')(pro-J)$. Then since the adjunctions can be composed in a natural fashion, it follows that $pro-J''$ has a left adjoint. This implies that $\mathcal{P}$ is a dense subcategory of $\mathcal{T}$, so that we can also consider the shape $S_{(\mathcal{T},\mathcal{P})} : \mathcal{T} \to Sh_{(\mathcal{T},\mathcal{P})}$. By Corollary 3.2 there exists the functor $J^* : Sh_{(\mathcal{T},\mathcal{S})} \to Sh_{(\mathcal{T},\mathcal{P})}$. Now because the relation $sh_{(\mathcal{T},\mathcal{S})}(X) = sh_{(\mathcal{T},\mathcal{S})}(Y)$ means that there exists a shape isomorphism $F \in Sh_{(\mathcal{T},\mathcal{S})}((X,Y)$, we deduce a shape isomorphism $J^*(F) \in Sh_{(\mathcal{T},\mathcal{P})}(X,Y)$.□

Remark 6.2. A proof of the result from Theorem 6.1 based on inverse systems is complicated enough since it is necessary to obtain from a $\mathcal{S}$-expansion $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ of an object X a $\mathcal{P}$- expansion of X and this needs to use the density of $\mathcal{P}$ in $\mathcal{S}$.

Remark 6.3. In the conditions of Theorem 6.1 we can also consider the Mardešić-Segal shape theory $S_{(\mathcal{S},\mathcal{P})} : \mathcal{S} \to Sh_{(\mathcal{S},\mathcal{P})}$. But the category $Sh_{(\mathcal{S},\mathcal{P})}$ is a subcategory of the category $Sh_{(\mathcal{T},\mathcal{P})}$ and the functor $S_{(\mathcal{S},\mathcal{P})}$ is the restriction of the functor $S_{(\mathcal{S},\mathcal{T})}$ to the category $Sh_{(\mathcal{S},\mathcal{P})}$. We remark that this situation does not integrate in Theorem 4.1 since the inclusion functor J' is not an isomorphism of categories.

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