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S-NUMBERS AND INTERPOLATION*

BY

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Abstract. We establish some estimates for Gelfand and Kolmogorov numbers of the interpolation operator in the multidimensional case. We also analyse the interpolation property of operator ideals defined by Gelfand and Kolmogorov numbers.

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Key words: interpolation spaces, S-numbers, operator ideals.

1. Introduction. The estimates for Gelfand and Kolmogorov numbers of interpolation operator in the case of Banach couple have been pointed out by PIETSCH ([3]) and TEIXEIRA ([5]). These numbers are important examples of the so-called s-numbers (cf. [1], [2] and [3]).

In this paper we establish some estimates for Gelfand numbers $(c_n(T))_{n\geq 1}$ and Kolmogorov numbers $(d_n(T))_{n\geq 1}$ of interpolation operator T in the case of Sparr methods for $N+1$-tuples of Banach spaces. We consider the case when one of the $N+1$-tuples is reduced to a single Banach space. In the case of Gelfand numbers we can extend the result to the general case when in both sides we have $N+1$-tuples $\overline{A}$ and $\overline{B}$ if $\dim\left(\Delta\left(\overline{A}\right)\right)<\infty$.

We also consider the operator ideals generated by Gelfand and Kolmogorov numbers and analyse the interpolation property for these ideals.

2. Notation and preliminaries. If A and B are Banach spaces, we denote by $\mathcal{L}(A,B)$ the space of all bounded linear operators between A and B with the usual norm. For every $T\in\mathcal{L}(A,B)$

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a) The n-th Gelfand number $c_n(T)$ of T is defined to be

$$c_n(T) = \inf\left\{\left\|T\ J_M^A\right\| : \operatorname{codim}(M) < n\right\},$$

where M is a closed linear subspace of A and the map $J_M^A : M \to A$ is the canonical injection.

b) The n-th Kolmogorov number $d_n(T)$ of T is defined to be

$$d_n(T) = \inf\left\{\left\|Q_M^B\ T\right\| : \dim M < n\right\},$$

where M is a linear subspace of B and $Q_M^B : B \to B/M$ stands for the canonical surjection.

It is easily seen that

$$\|T\| = c_1(T) \geq c_2(T) \geq ... \geq 0$$

and

$$\|T\| = d_1(T) \geq d_2(T) \geq ... \geq 0.$$

Let $\mathcal{L}$ denote the class of all linear and bounded operators between arbitrary Banach spaces and let the classical ideals l_p, $0 < p < \infty$. The class of operators T belonging to $\mathcal{L}$ for which $(c_n(T))_n \in l_p$ or $(d_n(T))_n \in l_p$ form an operator ideal (see [3]) and we shall denote it by G_p or K_p respectively.

In literature there are many interpolation methods for $N+1$ - tuples of Banach spaces. Here we shall analyze Sparrs's methods [4]. We begin by reviewing Sparr constructions. If $\overline{t} = (t_1, ..., t_N)$ and $\overline{s} = (s_1, ..., s_N)$ are N - tuple of positive numbers, we set

$$\overline{t}\overline{s} = (t_1 s_1, ..., t_N s_N);\ \left|\overline{t}\right| = t_1 ... t_n,$$

$$\left|\overline{t}\right|^{\overline{s}} = t_1^{s_1} ... t_n^{s_N};\ \frac{d\overline{t}}{\overline{t}} = \frac{dt_1}{t_1} ... \frac{dt_N}{t_N}.$$

Let $\overline{A} = (A_0, ..., A_N)$ be a Banach $N+1$ -tuple, that is to say, a family of $N+1$ Banach spaces $A_i, i = 0, ..., N$ all of them continuously embedded in a common linear Hausdorff topological vector space. For $\overline{t} = (t_1, ..., t_N) > 0$ we put:

$$K(\overline{t}, a; \overline{A}) = \inf\{\|a_0\|_{A_0} + \sum_{i=1}^{N} t_i \|a_i\|_{A_i} : a = \sum_{i=0}^{N} a_i,\ a_i \in A_i\}$$

for each $a \in A_0 + ... + A_N := \sum(\overline{A})$ and

$$J(\overline{t}, a; \overline{A}) = \max_{1 \le i \le N} (\|a\|_{A_0}, t_i \|a_i\|_{A_i})$$

for all $a \in A_0 \cap ... \cap A_N := \Delta(\overline{A})$.

Given two Banach $N+1$-tuples $\overline{A} = (A_0, ...A_N)$, $\overline{B} = (B_0, ..., B_N)$ we write $T \in \mathcal{L}\left(\overline{A}, \overline{B}\right)$ or $T : \overline{A} \to \overline{B}$, meaning that T is a linear operator from $\sum\left(\overline{A}\right)$ into $\sum\left(\overline{B}\right)$ whose restriction to each A_i defines a bounded operator from A_i into B_i $(i = 0, ..., N)$. For each $T \in \mathcal{L}\left(\overline{A}, \overline{B}\right)$ we consider the norm:

$$\|T\|_{\overline{A},\overline{B}} = \max_{0 \le i \le N} \left(\|T\|_{A_i, B_i}\right).$$

If one of the $N+1$- tuples $\overline{A}$ or $\overline{B}$ reduces to a single Banach space, i.e. if $A_0 = A_1 = ... = A_N = A$ or $B_0 = B_1 = ... = B_N = B$, then we write $T \in \mathcal{L}\left(A, \overline{B}\right)$ or, respectively, $T \in \mathcal{L}\left(\overline{A}, B\right)$.

Let A be an intermediate Banach space with respect to $\overline{A} = (A_0, ...A_N)$, that is, $\Delta\left(\overline{A}\right) \hookrightarrow A \hookrightarrow \sum\left(\overline{A}\right)$ (with continous embeddings). We say that A is of class $\mathcal{K}\left(\overline{\theta}, \overline{A}\right)$, where $\overline{\theta} = (\theta_1, ..., \theta_N) \in (0,1)^N$, $\sum_{i=1}^N \theta_i < 1$ if there is a constant C such that

(2.1)

$$\|T\|_{A,B} \le C \|T : A_0 \to B\|^{1-\sum_{i=1}^N \theta_i} \|T : A_1 \to B\|^{\theta_1} ... \|T : A_N \to B\|^{\theta_N}$$

for every operator $T \in \mathcal{L}\left(\overline{A}, B\right)$ and all Banach spaces B.

The space A is said to be of class $J\left(\overline{\theta}, \overline{A}\right)$ if there is a constant C such that

(2.2)

$$\|T\|_{B,A} \le C \|T : B \to A_0\|^{1-\sum_{i=1}^N \theta_i} \|T : B \to A_1\|^{\theta_1} ... \|T : B \to A_N\|^{\theta_N}$$

for every operator $T \in \mathcal{L}\left(B, \overline{A}\right)$ and all Banach spaces B.

An important example of space of class $\mathcal{K}\left(\overline{\theta}, \overline{A}\right)$ is the real interpolation space $\overline{A}_{\overline{\theta},p,\mathcal{K}}$ (or Sparr's $\mathcal{K}$- space). We remind that for $1 \le p \le \infty$ and

$$\overline{\theta} = (\theta_1, ..., \theta_N) \in (0,1)^N, \ \sum_{I=1}^N \theta_i < 1$$

the space $\overline{A}_{\overline{\theta},p,\mathcal{K}}$ consists of all $a \in \sum\left(\overline{A}\right)$ which have a finite norm:

$$\|a\|_{\overline{A}_{\overline{\theta},p,\mathcal{K}}} = \left(\int_{\mathbb{R}_+^n} \left(\left|\overline{t}\right|^{-\overline{\theta}} \mathcal{K}\left(\overline{t}, a; \overline{A}\right)\right)^p \frac{d\overline{t}}{\overline{t}}\right)^{\frac{1}{p}}.$$

On the other hand, the real interpolation space $\overline{A}_{\overline{\theta},p,J}$ (or Sparr's J-space) is an important example of space of class $J\left(\overline{\theta},\overline{A}\right)$. We remind that for $1\leq p\leq\infty$ and

$$\overline{\theta}=(\theta_1,...,\theta_N)\in(0,1)^N,\ \sum_{I=1}^{N}\theta_i<1,$$

the space $\overline{A}_{\overline{\theta},p,J}$ consists of all $a\in\sum\left(\overline{A}\right)$ which can represented in the form:

$$a=\int_{\mathbb{R}_+^n}u\left(\overline{t}\right)\frac{d\overline{t}}{\overline{t}}\ \text{(convergence in }\sum\left(\overline{A}\right)\text{)} \tag{2.3}$$

where $u:\mathbb{R}_+^n\to\Delta\left(\overline{A}\right)$ (strongly measurable) and satisfies

$$\left(\int_{\mathbb{R}_+^n}\left(\left|\overline{t}\right|^{-\overline{\theta}}J\left(\overline{t},u\left(\overline{t}\right);\overline{A}\right)\right)^p\frac{d\overline{t}}{\overline{t}}\right)^{\frac{1}{p}}<\infty. \tag{2.4}$$

The norm in $\overline{A}_{\overline{\theta},p,J}$ is

$$\|a\|_{\overline{A}_{\overline{\theta},p,J}}=\inf\left\{\left(\int_{\mathbb{R}_+^n}\left(\left|\overline{t}\right|^{-\overline{\theta}}J\left(\overline{t},u\left(\overline{t}\right);\overline{A}\right)\right)^p\frac{d\overline{t}}{\overline{t}}\right)^{\frac{1}{p}}\right\}$$

where the infimum is taken over all representations $u\left(\overline{t}\right)$ of a satisfying (2.3) and (2.4).

For every $T\in\mathcal{L}\left(\overline{A},\overline{B}\right)$ we have

$$\|T\|_{\overline{A}_{\overline{\theta},p,\mathcal{K}},\overline{B}_{\overline{\theta},p,\mathcal{K}}}\leq\prod_{j=0}^{N}M_j^{\theta_j} \tag{2.5}$$

where $M_j=\|T\|_{A_j,B_j}$ and $\theta_0=1-\sum_{j=1}^{N}\theta_j$.

Similar result we have in the case of J-spaces.

3. Estimates for Gelfand numbers

Theorem 3.1. *Let $\overline{B}=(B_0,...,B_N)$ be a Banach $N+1$-tuple, A a Banach space and suppose that B is a Banach space of class $J\left(\overline{\theta},\overline{B}\right)$*

with constant C, for some $\overline{\theta} = (\theta_1, ..., \theta_N) \in (0,1)^N$, $\sum_{i=1}^N \theta_i < 1$. If $T \in \mathcal{L}\left(A, \overline{B}\right)$ then, for all $n_i \in \mathbb{N}$ $(i = 0, ...N)$ we have

$$c_{n_0+...+n_N-N}\left(T_{A,B}\right) \leq C \;\; c_{n_0}\left(T_0\right)^{\theta_0} ... c_{n_N}\left(T_N\right)^{\theta_N} \tag{3.1}$$

where $T_i = T : A \to B_i$, $i = 0, ...N$ and $\theta_0 = 1 - \sum_{i=1}^N \theta_i$.

Proof. Given $\varepsilon > 0$ we can choose subspaces M_i of A with $\operatorname{codim}\left(M_i\right) < n_i$ such that $\left\|TJ_{M_i}^A : M_i \to B_i\right\| \leq (1+\varepsilon)\, c_{n_i}\left(T_i\right)$, $(i = 0, ..., N)$.

Put $M := M_0 \cap ... \cap M_N$. Then $\operatorname{codim}\left(M\right) < n_0 + ... + n_N - N$. It follows from $\left\|TJ_M^A : M \to B_i\right\| \leq \left\|TJ_{M_i}^A : M_i \to B_i\right\|$ and the estimation (2.2) that

$$\begin{aligned} c_{n_0+...+n_N-N}\left(T_{A,B}\right) &\leq \left\|TJ_M^A : M \to B\right\| \\ &\leq C \left\|TJ_M^A : M \to B_0\right\|^{\theta_0} ... \left\|TJ_M^A : M \to B_N\right\|^{\theta_N} \\ &\leq C \left\|TJ_{M_0}^A : M_0 \to B_0\right\|^{\theta_0} ... \left\|TJ_{M_N}^A : M_N \to B_N\right\|^{\theta_N} \\ &\leq C\,(1+\varepsilon)\, c_{n_0}\left(T_0\right)^{\theta_0} ... c_{n_N}\left(T_N\right)^{\theta_N}. \end{aligned}$$

Since ε is an arbitrary positive number we obtain (3.1). □

Corollary 3.2. *Let $\overline{\theta} = (\theta_1, ..., \theta_N) \in (0,1)^N$, $\sum_{i=1}^N \theta_i < 1$ and $1 \leq p \leq \infty$. For any Banach $N+1$-tuple $\overline{B} = (B_0, ..., B_N)$, any Banach space A and any operator $T \in \mathcal{L}\left(A, \overline{B}\right)$, we have*

$$c_{n_0+...+n_N-N}\left(T : A \to \overline{B}_{\overline{\theta},p,J}\right) \leq C \;\; c_{n_0}\left(T_0\right)^{\theta_0} ... c_{n_N}\left(T_N\right)^{\theta_N}. \tag{3.2}$$

To extend the result to the general case when in both sides we have $N+1$-tuples $\overline{A}$ and $\overline{B}$ we need that $\dim\left(\Delta\left(\overline{A}\right)\right) < \infty$ and the following result

Lemma 3.3. *Let $\overline{A} = (A_0, ..., A_N)$ be a Banach $N+1$-tuple, and C a finite-dimensional subspace of $\Delta\left(\overline{A}\right)$. Then, there exists in $\sum\left(\overline{A}\right)$ a topological complement B with respect to C such that for every $p \in [1, \infty]$, and every $\overline{\theta} = (\theta_1, ..., \theta_N) \in (0,1)^N$, $\sum_{i=1}^N \theta_i < 1$, we have*

$$(A_0 \cap B, ..., A_N \cap B)_{\overline{\theta},p,J} = \overline{A}_{\overline{\theta},p,J} \cap B. \tag{3.3}$$

Moreover, the norm of the inclusion map

$$I : \overline{A}_{\overline{\theta},p,J} \cap B \to (A_0 \cap B, ..., A_N \cap B)_{\overline{\theta},p,J}$$

satisfies

$$\|I\| \leq n^{\frac{1}{2}} + 1,\ n = \dim(C). \tag{3.4}$$

Proof. On the spaces $A_0 \cap B, ..., A_N \cap B$ and $\overline{A}_{\overline{\theta},p,J} \cap B$ are given the topologies of $A_0, ..., A_N$ and $\overline{A}_{\overline{\theta},p,J}$ respectively. Since C is finite-dimensional, there exists a closed linear subspace B of $\sum\left(\overline{A}\right)$ such that $\sum\left(\overline{A}\right) = B \oplus C$. Since $A_i \cap B \hookrightarrow B$ and $A_i \cap B \hookrightarrow A_i$, $(i = 0, ..., N)$, we obtain by interpolation $(A_0 \cap B, ..., A_N \cap B)_{\overline{\theta},p,J} \hookrightarrow B$ and $(A_0 \cap B, ..., A_N \cap B)_{\overline{\theta},p,J} \hookrightarrow \overline{A}_{\overline{\theta},p,J}$ respectively.

So, it then follows that $(A_0 \cap B, ..., A_N \cap B)_{\overline{\theta},p,J} \hookrightarrow \overline{A}_{\overline{\theta},p,J} \cap B$.

Now, we prove the reverse inclusion. Since C is a finite-dimensional subspace of A_i, then there is a projection $Q_i \in \mathcal{L}(A_i, A_i)$ on C such that $\|Q_i\|_{A_i,A_i} \leq n^{\frac{1}{2}}$, $(i = 0, ..., N)$, $n = \dim(C)$ (see [5] p. 386). Now we consider $Q : \sum\left(\overline{A}\right) \to \sum\left(\overline{A}\right)$ defined by $Qa = Q_0a_0 + ... + Q_Na_N$, where $a_0 + ... + a_N$ is any representation of a in $\sum\left(\overline{A}\right)$. It is not hard to show that, Q is well defined, $Q \in \mathcal{L}\left(\overline{A}, \overline{A}\right)$, Q is a projection on C and satisfies

$$\|Q\|_{A_i,A_i} \leq n^{\frac{1}{2}}\ (i = 0, ..., N),\ \|Q\| \leq n^{\frac{1}{2}},\ n = \dim(C).$$

Now, the map $P = I - Q$ is a projection on B and belongs to

$$\mathcal{L}((A_0, ..., A_N), (A_0 \cap B, ..., A_N \cap B)).$$

By the interpolation property we obtain a map

$$P_{\overline{\theta},p} : \overline{A}_{\overline{\theta},p,J} \to (A_0 \cap B, ..., A_N \cap B)_{\overline{\theta},p,J}$$

such that

$$\left\|P_{\overline{\theta},p}\right\| \leq \|P\|_{A_0,A_0}^{1-\sum_{i=1}^{N}\theta_i} \|P\|_{A_1A_1}^{\theta_1} ... \|P\|_{A_NA_N}^{\theta_N} \leq n^{\frac{1}{2}} + 1. \tag{3.5}$$

Since the restriction of $P_{\overline{\theta},p}$ to $\overline{A}_{\overline{\theta},p,J} \cap B$ is the identity map I, we have

$$\overline{A}_{\overline{\theta},p,J} \cap B \hookrightarrow (A_0 \cap B, ..., A_N \cap B)_{\overline{\theta},p,J}$$

and, by (3.5) the norm of I satisfies (3.4). □

Theorem 3.4. *Let $\overline{A} = (A_0, ..., A_N)$ and $\overline{B} = (B_0, ..., B_N)$ be two Banach $N + 1$ -tuples and let $A = \overline{A}_{\overline{\theta},p,J}$, $B = \overline{B}_{\overline{\theta},p,J}$ for some $\overline{\theta} =$*

$(\theta_1, ..., \theta_N) \in (0,1)^N$, $\sum_{i=1}^N \theta_i < 1$ *and some* $p \in [1, \infty)$. *Suppose that* $\Delta(\overline{A})$ *is finite-dimensional. If* $T \in \mathcal{L}(\overline{A}, \overline{B})$ *then we have*

$$c_{n_0+...+n_N-N}(T_{A,B}) \leq [\dim(\Delta(\overline{A}))]^{1/2} c_{n_0}(T_0)^{\theta_0} ... c_N(T_N)^{\theta_N} \tag{3.6}$$

where $T_i = T : A_i \to B_i, i = 0, ..., N$ *and* $\theta_0 = 1 - \sum_{i=1}^N \theta_i$.

Proof. Given $\varepsilon > 0$ we can choose subspaces M_i of A_i with

$$\operatorname{codim}_{A_i}(M_i) < n_i$$

such that

$$\left\| T_i J_{M_i}^{A_i} \right\| \leq c_{n_i}(T_i), (i = 0, ..., N) \tag{3.7}$$

We may assume that $M_i \neq \{0\}$, for, if $M_i = \{0\}, i = 0, ..., N$ then $c_{n_i}(T_i) = \|T_i\|$ and the theorem the follows from the interpolation property. Put $M = M_0 \cap ... \cap M_N$. Denote $M^{(i)}$ the set M, when given the A_i topology and by $M_{\overline{\theta},p}$ the same set when endowed with the $\overline{A}_{\overline{\theta},p,J}$ topology Then, by (3.3) $M_{\overline{\theta},p}$ and $(M^{(0)}, ..., M^{(N)})_{\overline{\theta},p,J}$ are topologically isomorphic with the norm of inclusion map $I, I : M_{\overline{\theta},p} \to (M^{(0)}, ..., M^{(N)})_{\overline{\theta},p,J}$ satisfying:

$$\|I\| \leq (\dim(M))^{1/2} \leq [\dim(\Delta(\overline{A}))]^{1/2} \tag{3.8}$$

Denote by S the restriction of T to $M^{(0)} + ... + M^{(N)}$. Then, by interpolation property, we obtain a bounded linear map $S_{\overline{\theta},p} : (M^{(0)}, ..., M^{(N)})_{\overline{\theta},p,J} \to \overline{B}_{\overline{\theta},p,J}$. Since $M \hookrightarrow M_i (i = 0, ..., N)$, it follows from (3.8) and (3.7) that

(3.9)

$$\left\| T_{A,B} J_{M_{\overline{\theta},p}}^{A} \right\| \leq \left\| S_{\overline{\theta},p} \right\| \|I\| \leq \|I\| \left\| T_{A_0,B_0} J_{M^{(0)}}^{A_0} \right\|^{\theta_0} ... \left\| T_{A_N,B_N} J_{M^{(N)}}^{A_N} \right\|^{\theta_N}$$

$$\leq [\dim(\Delta(\overline{A}))]^{1/2} [(1+\varepsilon) c_{n_0}(T_0)]^{\theta_0} ... [(1+\varepsilon) c_{n_N}(T_N)]^{\theta_N}.$$

But $c_{n_0+...+n_N-N}(T_{A,B}) \leq \left\| T_{A,B} J_M^A \right\|$and $\Delta(\overline{A})$ is dense in $\overline{A}_{\overline{\theta},p,J}$. Then, since ε is an arbitrary positive number, (3.6) follows from (3.9) and the theorem is proved. □

The following result is an easy consequence of Theorem 3.1 and Hölder's inequality

Theorem 3.5. *Let* $\overline{B} = (B_0, ..., B_N)$ *be a Banach* $N+1$ *-tuple,* A *a Banach space and let* $B = \overline{B}_{\overline{\theta},p,J}$ *for some* $\overline{\theta} = (\theta_1, ..., \theta_N) \in (0,1)^N$ $\sum_{i=1}^N \theta_i <$

1 and some $1 \leq p \leq \infty$. If $T \in \mathcal{L}(A, \overline{B})$ is such that $T_{A,B_i} \in G_{p_i}, (i = 0, ..., N)$ with $1 < p_0, ..., p_N < \infty$, then $T_{A,B} \in G_p$ with

$$\frac{1}{p} = \frac{1 - \sum_{i=1}^{N} \theta_i}{p_0} + \frac{\theta_1}{p_1} + ... + \frac{\theta_N}{p_N}.$$

4. Estimates for Kolmogorov numbers

Theorem 4.1. *Let $\overline{A} = (A_0, ..., A_N)$ be a Banach $N+1$-tuple, B a Banach space and suppose that A is a Banach space of class $\mathcal{K}\left(\overline{\theta}, \overline{A}\right)$ with constant C, for some $\overline{\theta} = (\theta_1, ..., \theta_N) \in (0,1)^N$, $\sum_{i=1}^{N} \theta_i < 1$. If $T \in \mathcal{L}\left(\overline{A}, B\right)$ then, for all $n_i \in \mathbb{N}$ $(i = 0, ..., N)$ we have*

$$d_{n_0+...+n_N-N}\left(T_{A,B}\right) \leq C\ d_{n_0}\left(T_0\right)^{\theta_0} ... d_{n_N}\left(T_N\right)^{\theta_N} \tag{4.1}$$

where $T_i = T : A_i \to B,\ i = 0, ...N$ and $\theta_0 = 1 - \sum_{i=1}^{N} \theta_i$.

Proof. Given $\varepsilon > 0$ we can choose subspaces M_i of B with $\dim(M_i) < n_i$ such that

$$\left\|Q^B_{M_i} T_i\right\| \leq (1+\varepsilon)\, d_{n_i}(T_i),\ (i = 0, ..., N). \tag{4.2}$$

Put $M := M_0 + ... + M_N$. Then $\dim(M) < n_0 + ... + n_N - N$.
Since $M_i \hookrightarrow M$ it follows that $\left\|Q^B_M T_i\right\| \leq \left\|Q^B_{M_i} T_i\right\|,\ (i = 0, ..., N)$.
Now, by the estimate (2.1) and (4.2) we obtain

$$\begin{aligned} d_{n_0+...+n_N-N}\left(T_{A,B}\right) &\leq \left\|Q^B_M T\right\| \leq C \left\|Q^B_M T_0\right\|^{\theta_0} ... \left\|Q^B_M T_N\right\|^{\theta_N} \\ &\leq C \left\|Q^B_{M_0} T_0\right\|^{\theta_0} ... \left\|Q^B_{M_N} T_N\right\|^{\theta_N} \\ &\leq C\,(1+\varepsilon)\, d_{n_0}\left(T_0\right)^{\theta_0} ... d_{n_N}\left(T_N\right)^{\theta_N}. \end{aligned}$$

Since ε is an arbitrary positive number we obtain (4.1). □

Corollary 4.2. *Let $\overline{\theta} = (\theta_1, ..., \theta_N) \in (0,1)^N$, $\sum_{i=1}^{N} \theta_i < 1$ and $1 \leq p \leq \infty$. For any Banach $N+1$-tuple $\overline{A} = (A_0, ..., A_N)$, any Banach space B and any operator $T \in \mathcal{L}\left(\overline{A}, B\right)$, we have*

$$d_{n_0+...+n_N-N}\left(T : \overline{A}_{\overline{\theta},p,\mathcal{K}} \to B\right) \leq C\ d_{n_0}\left(T_0\right)^{\theta_0} ... d_{n_N}\left(T_N\right)^{\theta_N}.$$

The following result is an easy consequence of Theorem 4.1 and Hölder's inequality.

Theorem 4.3. *Let $\overline{A} = (A_0, ..., A_N)$ be a Banach $N+1$-tuple, B a Banach space and let $A = \overline{A}_{\overline{\theta},p,K}$ for some $\overline{\theta} = (\theta_1, ..., \theta_N) \in (0,1)^N$, $\sum_{i=1}^N \theta_i < 1$ and some $1 \leq p \leq \infty$. If $T \in \mathcal{L}\left(\overline{A}, B\right)$ is such that $T_{A_i,B} \in K_{p_i}$, $(i = 0, ..., N)$ with $1 < p_0, ..., p_N < \infty$, then $T_{A,B} \in K_p$ with*

$$\frac{1}{p} = \frac{1 - \sum_{i=1}^N \theta_i}{p_0} + \frac{\theta_1}{p_1} + ... + \frac{\theta_N}{p_N}.$$

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