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A VIABILITY RESULT FOR SEMILINEAR REACTION-DIFFUSION SYSTEMS*

BY

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Abstract. We consider a reaction-diffusion system of the form

$$\begin{cases} u'(t) = Au(t) + F(u(t), v(t)), & t \geq 0 \\ v'(t) = Bv(t) + G(u(t), v(t)), & t \geq 0 \\ u(0) = \xi, \quad v(0) = \eta, \end{cases}$$

where X and Y are real Banach spaces, $\mathcal{K}$ is a nonempty and locally closed subset in $X \times Y$, $A : D(A) \subseteq X \to X$, $B : D(B) \subseteq Y \to Y$ are the generators of two C_0-semigroups, $\{S_A(t) : X \to X;\, t \geq 0\}$ and $\{S_B(t) : Y \to Y;\, t \geq 0\}$ respectively, $F : \mathcal{K} \to X$, $G : \mathcal{K} \to Y$ are continuous such that $A + F$ and/or $B + G$ are locally of β-compact type. We prove some necessary and sufficient conditions in order that for each $(\xi, \eta) \in \mathcal{K}$, problem above has at least one mild solution $(u, v) : [\,0, T\,] \to \mathcal{K}$.

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1. Introduction. In this paper we are concerned with some viability results referring to a class of semilinear reaction-diffusion systems. Namely, let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be real Banach spaces, $A : D(A) \subseteq X \to X$ and $B : D(B) \subseteq Y \to Y$ the infinitesimal generators of two C_0-semigroups, $\{S_A(t) : X \to X;\, t \geq 0\}$ and $\{S_B(t) : Y \to Y;\, t \geq 0\}$ respectively, $\mathcal{K}$ a nonempty and locally closed subset in $X \times Y$, and $F : \mathcal{K} \to X$, $G : \mathcal{K} \to Y$

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two given functions. We consider a reaction-diffusion system of the form:

$$\begin{cases} u'(t) = Au(t) + F(u(t), v(t)), & t \geq 0 \\ v'(t) = Bv(t) + G(u(t), v(t)), & t \geq 0 \\ u(0) = \xi, \quad v(0) = \eta \end{cases} \tag{1.1}$$

where $\xi \in X,\ \eta \in Y$. We are interested in finding necessary and sufficient conditions in order that $\mathcal{K}$ be a viable set for (1.1), i.e. for each $(\xi, \eta) \in \mathcal{K}$ there exists $T > 0$ such that (1.1) has at least one mild solution $(u, v) : [\,0, T\,] \to \mathcal{K}$ i.e.

$$\begin{cases} u(t) = S_A(t)\xi + \displaystyle\int_0^t S_A(t-s)F(u(s), v(s))\, ds \\ v(t) = S_B(t)\eta + \displaystyle\int_0^t S_B(t-s)G(u(s), v(s))\, ds \end{cases} \tag{1.2}$$

for each $t \in [\,0, T\,]$.

2. Preliminaries. We assume that the reader is familiar with the basic concepts and results concerning infinitesimal generators of C_0-semigroups and mild solutions for Cauchy problems in Banach spaces and we refer to Vrabie [11] for details.

For easy reference we begin by recalling some basic concepts and results concerning the *Hausdorff measure of necompactness* and *uniqueness functions* . In the sequel $(X, \|\cdot\|)$ will always be a Banach space. For $\xi \in X$ and $\rho > 0$, $D(\xi, \rho)$ denotes the closed ball in X of radius ρ centered in ξ and $\operatorname{dist}(\xi, K)$ denotes the distance from $\xi \in X$ to the set $K \subseteq X$. Let $\mathcal{B}(Y)$ be the family of all bounded subsets of $Y \subseteq X$.

Definition 2.1. The function $\beta : \mathcal{B}(X) \to \mathbb{R}_+$, defined by

$$\beta(B) = \inf \left\{ \varepsilon > 0; \exists x_1, x_2, ...x_{n(\varepsilon)} \in X,\ B \subseteq \bigcup_{i=1}^{n(\varepsilon)} D(x_i, \varepsilon) \right\}$$

is called the *Hausdorff measure of necompactness on* X.

Remark 2.1. We have $\beta(B) = 0$ if and only if B is relatively compact set. So, if X is finite dimensional $\beta \equiv 0$, because in this case the class of relatively compact subsets of X coincides with $\mathcal{B}(X)$.

Lemma 2.1. *Let Z be a subspace in X, let $B \in \mathcal{B}(X)$ and let*

$$\beta_Z(B) = \inf \left\{ \varepsilon > 0; \exists x_1, x_2, ...x_{n(\varepsilon)} \in Z,\ B \subseteq \bigcup_{i=1}^{n(\varepsilon)} D(x_i, \varepsilon) \right\}.$$

Then for each $B \in \mathcal{B}(Z)$ we have

$$\beta(B) \leq \beta_Z(B) \leq 2\beta(B).$$

See Akhmerov *et al.* [1], Theorem 1.1.7, p. 4 and Mönch [8].

Lemma 2.2. *Let X be a separable Banach space and $\{F_m; m \in \mathbb{N}\}$ a subset in $L^1(\tau, T; X)$ for which there exists $l \in L^1(\tau, T; \mathbb{R}^+)$ such that*

$$\|F_m(s)\| \leq l(s)$$

for each $m \in \mathbb{N}$ and a.e. for $s \in [\tau, T]$. Then the mapping

$$s \mapsto \beta(\{F_m(s); m \in \mathbb{N}\})$$

is integrable on $[\tau, T]$, and, for each $t \in [\tau, T]$, we have

$$\beta\left(\left\{\int_\tau^t F_m(s)\, ds; m \in \mathbb{N}\right\}\right) \leq \int_\tau^t \beta(\{F_m(s); m \in \mathbb{N}\})\, ds.$$

See Mönch [8].

Remark 2.2. If X is not separable, we may proceed as follows. Let $\{F_m; m \in \mathbb{N}\}$ be a subset in $L^1(\tau, T; X)$ for which there exists $l \in L^1(\tau, T; \mathbb{R}_+)$ such that $\|F_m(s)\| \leq l(s)$, for each $m \in \mathbb{N}$ and a.e. for $s \in [\tau, T]$. In view of Theorem 1.1.3, p. 3 and Remark 1.1.2, p. 4 in Vrabie [11], there exists a separable and closed subspace Z of X such that $F_m \in L^1(\tau, T; Z)$ for $m = 1, 2, \ldots$. Let us observe that the restriction of the mapping β_Z to $\mathcal{B}(Z)$ coincides with the Hausdorff measure of noncompactness on Z. Then, from Lemma 2.2, it follows that the function $s \mapsto \beta_Z(\{F_m(s); m \in \mathbb{N}\})$ is integrable on $[\tau, T]$ and we have

$$\beta_Z\left(\left\{\int_\tau^t F_m(s)\, ds; m \in \mathbb{N}\right\}\right) \leq \int_\tau^t \beta_Z(\{F_m(s); m \in \mathbb{N}\})\, ds,$$

for each $t \in [\tau, T]$.

Remark 2.3. Let $(u_n)_n$ be a bounded sequence in X such that

$$\lim_k \beta(\{u_n; n \geq k\}) = 0.$$

Then $\{u_n; n \in \mathbb{N}\}$ is relatively compact.

See CÂRJĂ-NECULA-VRABIE [4].

Definition 2.2. A function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ which is continuous, nondecreasing and the only solution of the Cauchy problem

$$\begin{cases} x'(t) = \omega(x(t)) \\ x(0) = 0. \end{cases}$$

is $x \equiv 0$ is called a *uniqueness function*.

Remark 2.4. If $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is a uniqueness function, then, for each $k > 0$, $k\omega$ is a uniqueness function too.

Lemma 2.3. *Let $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ be a uniqueness function and let $(\gamma_k)_k$ be strictly decreasing to 0. Let $(x_k)_k$ be a bounded sequence of measurable functions, from $[0, T_1]$ to $\mathbb{R}_+$, such that*

$$x_k(t) \leq \gamma_k + \int_0^t \omega(x_k(s))\, ds,$$

for $k = 1, 2, ...$ and for each $t \in [0, T_1]$. Then there exists $T \in (0, T_1]$ such that $\lim_k x_k(t) = 0$ uniformly for $t \in [0, T]$.

For details see CÂRJĂ-NECULA-VRABIE [4].

In the following we recall some viability results concerning the Cauchy problem:

$$\begin{cases} u'(t) = Au(t) + f(u(t)), \quad t \geq 0 \\ u(0) = \xi, \end{cases} \tag{2.1}$$

where $A : D(A) \subseteq X \to X$ is the infinitesimal generator of C_0-semigroup $\{S(t) : X \to X; t \geq 0\}$, K is a nonempty subset in X and $f : K \to X$ is a given function. We recall that the subset $K \subseteq X$ is *locally closed* if for each $\xi \in K$ there exists $\rho > 0$ such that $D(\xi, \rho) \cap K$ is closed. Each subset in X which is either open or closed is locally closed. Moreover, each subset K in X which is closed relative to some open subset D, i.e. for which there exists a closed subset $C \subset X$ such that $K = C \cap D$, is locally closed in X.

Definition 2.3. The set $K \subseteq X$ is *viable* with respect to $A + f$ if for each $\xi \in K$, there exists $T > 0$ such that the Cauchy problem (2.1) has at least one mild solution $u : [0, T] \to K$.

Theorem 2.1. *Let X be a Banach space, $A : D(A) \subseteq X \to X$ the infinitesimal generator of a C_0-semigroup, $\{S(t) : X \to X;\ t \geq 0\}$, K a nonempty subset in X and $f : K \to X$ a continuous function. If K is viable with respect to $A + f$ then, for each $\xi \in K$, we have:*

$$\lim_{h \downarrow 0} \frac{1}{h} \operatorname{dist}(S(h)\xi + hf(\xi); K) = 0. \tag{2.2}$$

For the proof see CÂRJĂ-NECULA-VRABIE [4].

Remark 2.5. Under the general hypotheses of Theorem 2.1, if K is viable with respect to $A + f$ then, for each $\xi \in K$, we have

$$f(\xi) \in \mathcal{T}_K^A(\xi), \tag{2.3}$$

where

$$\mathcal{T}_K^A(\xi) = \{\eta \in X;\ \liminf_{h \downarrow 0} \frac{1}{h} \operatorname{dist}(S(h)\xi + h\eta; K) = 0\}$$

denotes the set of all *A-tangent elements to K in ξ*. We notice that if $A \equiv 0$, then $\mathcal{T}_K^A(\xi)$ is the *contingent cone* at $\xi \in K$ in the sense BOULIGAND [2]. Except for the case when either $A \equiv 0$ or ξ is an interior point of K, $\mathcal{T}_K^A(\xi)$ is not a cone.

This concept, for $A \equiv 0$, was introduced by NAGUMO [9]. For the semilinear equation we mention the works of PAVEL [10], CÂRJĂ-VRABIE [6] and the references therein. To handle into a unitary frame, the sufficient conditions in order that the set K is viable with respect $A+f$, we introduce:

Definition 2.4. Let $A : D(A) \subseteq X \to X$ be the infinitesimal generator of a C_0-semigroup $\{S(t) : X \to X;\ t \geq 0\}$ and $f : K \to X$ be a given continuous function. We say that $A + f$ is *locally of β-compact type* if, for each $\xi \in K$, there exist $\rho > 0$, a continuous function $\ell : \mathbb{R}_+ \to \mathbb{R}_+$ and a uniqueness function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$, such that

$$\beta(S(t)f(C)) \leq \ell(t)\omega(\beta(C))$$

for each $t > 0$ and $C \subseteq D(\xi, \rho) \cap K$.

Definition 2.5. Let X be a Banach space and let $K \subseteq X$. A function $f : K \to X$ is called *locally β-compact* if it is continuous and, for each $\xi \in K$, there exist $\rho > 0$, and a uniqueness function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\beta(f(C)) \leq \omega(\beta(C))$$

for each $C \subseteq D(\xi, \rho) \cap K$. A function $f : K \to X$ is called *β-compact* if it is continuous and the condition from above is satisfied for each $C \in \mathcal{B}(K)$.

Remark 2.6. If f is locally β-compact and A generates a C_0-semigroup (case in which there exist $M \geq 1$ and $a \geq 0$ such that $\|S(t)\|_{\mathcal{L}(X)} \leq Me^{at}$ for each $t \geq 0$) then $A+f$ is locally of β-compact type. Indeed, in Definition 2.4 we may take $\ell(t) = Me^{at}$ for $t \geq 0$, while ω is given by Definition 2.5. If f is β-compact and A generates a C_0-semigroup then $A + f$ is locally of β-compact type. We recall that a function $f : K \to X$ is called *locally compact* if it is continuous and, for each $\xi \in K$, there exists $\rho > 0$, such that $f(D(\xi, \rho) \cap K)$ is relatively compact in X. Each locally compact function is locally β-compact.

Definition 2.6. Let X be a Banach space and let $K \subseteq X$. A function $f : K \to X$ is called *locally Lipschitz* if for each $\xi \in K$, there exists $\rho > 0$ and $L = L_\xi > 0$ such that

$$\|f(u) - f(v)\| \leq L\|u - v\|$$

for each $u, v \in D(\xi, \rho) \cap K$.

Remark 2.7. If f is locally Lipschitz function, then $A + f$ is locally of β-compact type. Since $\beta(C) = 0$ if and only if C is relatively compact, then, if K is locally compact, $A + f$ is locally of β-compact type, too. Finally, it follows that whenever f is merely continuous and A generates a compact semigroup, $A + f$ is locally of β-compact type.

We present now some necessary and sufficient conditions in order that the set K be viable with respect to $A + f$. These conditions improve the main result in Pavel [10].

Theorem 2.2. *Let X be a Banach space, $A : D(A) \subseteq X \to X$ the infinitesimal generator of a C_0-semigroup $\{S(t) : X \to X; t \geq 0\}$, K a nonempty and locally closed subset in X and $f : K \to X$ such that $A + f$*

is locally of β-compact type. Then K is viable with respect to $A+f$ if and only if for each $\xi \in K$, the next tangency condition is satisfied:

$$\liminf_{h\downarrow 0} \frac{1}{h}\operatorname{dist}\left(S(h)\xi + hf(\xi); K\right) = 0. \tag{2.4}$$

For proof see Burlică-Roşu [3]. The necessity of Theorem 2.2 is a consequence of Theorem 2.1, while the sufficiency is essentially based on construction of an *ε-approximate* mild solution for the Cauchy problem (2.1), i.e. a 4-uple (σ, θ, g, u) given by lemma below.

Lemma 2.4. *Let X be a Banach space, $A : D(A) \subseteq X \to X$ the infinitesimal generator of a C_0-semigroup, $\{S(t) : X \to X; t \geq 0\}$, K a nonempty and locally closed subset in X and $f : K \to X$ a continuous function satisfying the condition* (2.4). *Let $\xi \in K$ be arbitrary and let $r > 0$ be such that $D(\xi, r) \cap K$ is closed. Then, there exist $\rho \in (0, r\,]$ and $T > 0$ such that, for each $\varepsilon \in (0,1)$, there exist $\sigma : [\,0, T\,] \to [\,0, T\,]$ nondecreasing, $\theta : \{\,(t,s); 0 \leq s < t \leq T\} \to [\,0, T\,]$ measurable, $g : [\,0,\, T\,] \to X$ Riemann integrable and $u : [\,0, T\,] \to X$ continuous such that:*

(i) *$s - \varepsilon \leq \sigma(s) \leq s$ for each $s \in [\,0, T\,]$;*

(ii) *$u(\sigma(s)) \in D(\xi, \rho) \cap K$ for each $s \in [\,0, T\,]$ and $u(T) \in D(\xi, \rho) \cap K$;*

(iii) *$\|g(s)\| \leq \varepsilon$ for each $s \in [\,0, T\,]$;*

(iv) *$\theta(t, s) \leq t$ for each $0 \leq s < t \leq T$ and $t \mapsto \theta(t, s)$ is nonexpansive on $(\,s, T\,]$;*

(v) *$u(t) = S(t)\xi + \int_0^t S(t-s) f(u(\sigma(s)))\, ds + \int_0^t S(\theta(t,s)) g(s)\, ds$ for each $t \in [\,0, T\,]$;*

(vi) *$\|u(t) - u(\sigma(t))\,\| \leq \varepsilon$ for each $t \in [\,0, T\,]$.*

See Lemma 9.3.1 in Cârjă-Necula-Vrabie [4].

Several important instances where the tangency condition (2.4) is necessary and sufficient for viability are indicated in the remark below.

Remark 2.8. Using Theorem 2.2 and Remark 2.6, we get that K is viable with respect to $A + f$ if and only if for each $\xi \in K$, the condition (2.4) is satisfied in the next situations:

(i) $A : D(A) \subseteq X \to X$ is the infinitesimal generator of a C_0-semigroup, $\{S(t) : X \to X;\ t \geq 0\}$, K is a nonempty and locally closed subset in X and $f : K \to X$ is a locally β-compact function;

(ii) $A : D(A) \subseteq X \to X$ is the infinitesimal generator of a C_0-semigroup, $\{S(t) : X \to X;\ t \geq 0\}$, K is a nonempty and locally closed subset in X and $f : K \to X$ is a locally compact function;

(iii) $A : D(A) \subseteq X \to X$ is the infinitesimal generator of a compact C_0-semigroup, $\{S(t) : X \to X;\ t \geq 0\}$, K is a nonempty and locally closed subset in X and $f : K \to X$ is a continuous function;

(iv) $A : D(A) \subseteq X \to X$ is the infinitesimal generator of a C_0-semigroup, $\{S(t) : X \to X;\ t \geq 0\}$, K is a nonempty and locally compact subset in X and $f : K \to X$ is a continuous function;

(v) $A : D(A) \subseteq X \to X$ is the infinitesimal generator of a C_0-semigroup, $\{S(t) : X \to X;\ t \geq 0\}$, K is a nonempty and locally closed subset in X and $f : K \to X$ is a locally Lipschitz function.

3. The main results. In this section we state the main necessary and sufficient conditions for viability in the case of reaction diffusion systems of the form (1.1). Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ two real Banach spaces, $\mathcal{K} \subseteq X \times Y$, $F : \mathcal{K} \to X$, $G : \mathcal{K} \to Y$, and $\xi \in X$, $\eta \in Y$. We assume that the operators $A : D(A) \subseteq X \to X$ and $B : D(B) \subseteq Y \to Y$ are the generators of two C_0-semigroups.

Definition 3.1. The set $\mathcal{K}$ is *viable* with respect to $(A + F, B + G)$ if for each $(\xi, \eta) \in \mathcal{K}$ there exists $T > 0$ such that (1.1) has at least one mild solution $(u, v) : [\,0, T\,] \to \mathcal{K}$ i.e. (u, v) satisfy (1.2) for each $t \in [\,0, T\,]$.

Remark 3.1. The system (1.1) can be written as a semilinear equation in a product space. Let $\mathcal{X} = X \times Y$ endowed with the norm $\|\cdot\|$, defined by $\|(x, y)\|_{\mathcal{X}} = \|x\|_X + \|y\|_Y$, for each $(x, y) \in \mathcal{X}$. Let $\mathcal{A} = (A, B) : D(\mathcal{A}) \subseteq \mathcal{X} \to \mathcal{X}$ be defined by $\mathcal{A}(x, y) = (\,Ax, By\,)$ for each $(x, y) \in D(\mathcal{A})$ and let $\mathcal{F} : \mathcal{K} \to \mathcal{X}$, $\mathcal{F}(z) = (\,F(z), G(z)\,)$ for each $z = (x, y) \in \mathcal{K}$. Hence, the system (1.1) can be written as

$$\begin{cases} z'(t) = \mathcal{A}z(t) + \mathcal{F}(z(t)) \\ z(0) = \zeta, \end{cases} \tag{3.1}$$

where $\zeta = (\xi, \eta)$. In the hypotheses above, $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup $\{\, \mathcal{S}(t) : \mathcal{X} \to \mathcal{X}; t \geq 0\}$, given by

$$\mathcal{S}(t)(\xi, \eta) = (S_A(t)\xi, S_B(t)\eta)$$

for each $t \geq 0$ and $(\xi, \eta) \in \mathcal{X}$. It readily follows that $\mathcal{K}$ is viable with respect to $(A + F, B + G)$ in sense of Definition 3.1 if and only if $\mathcal{K}$ is viable with respect to $\mathcal{A}+\mathcal{F}$ in sense of Definition 2.3, which means that for each $\zeta \in \mathcal{K}$, there exists $T > 0$ such that the problem (3.1) has at least one mild solution $z : [\, 0, T\,] \to \mathcal{K}$.

Definition 3.2. We say that $(A + F, B + G)$ is *locally of β-compact type* if F, G are continuous functions and for each $\zeta = (\xi, \eta) \in \mathcal{K}$, there exist $\rho > 0$, a continuous function $l : \mathbb{R}_+ \to \mathbb{R}_+$ and a uniqueness function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ such that for each set $C \subseteq D_{X\times Y}(\zeta, \rho) \cap \mathcal{K}$ and for each $t \geq 0$, we have

$$\beta_{X\times Y}(\, (S_A(t)F(C), S_B(t)G(C)\,)\,) \leq l(t)\omega(\beta_{X\times Y}(C)\,).$$

Remark 3.2. It is obvious that $(A + F, B + G)$ is locally of $\beta-$ compact type if and only if $\mathcal{A} + \mathcal{F}$ is locally of $\beta-$ compact type in sense of Definition 2.4.

We denote by

$$\Pi_X C = \{u \in X;\ \exists v \in Y,\, (u, v) \in C\},$$

$$\Pi_Y C = \{v \in Y;\ \exists u \in X,\, (u, v) \in C\}.$$

Definition 3.3. Let $\mathcal{K} \subseteq X \times Y$ and let A and F as above. We say that

(i) $A + F$ is *locally of β_X-compact type* if F is continuous and, for each $\zeta = (\xi, \eta) \in \mathcal{K}$, there exist $\rho > 0$, a continuous function $l : \mathbb{R}_+ \to \mathbb{R}_+$ and a uniqueness function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for each subset $C \subseteq D_{X\times Y}(\zeta, \rho) \cap \mathcal{K}$ and for each $t > 0$, we have

$$\beta_X(S_A(t)F(C)) \leq l(t)\omega(\beta_X(\Pi_X(C)));$$

(ii) $A + F$ is *$Y-$uniformly locally of β_X-compact type* if F is continuous and, for each $\zeta = (\xi, \eta) \in \mathcal{K}$, there exist $\rho > 0$, a continuous function

$l : \mathbb{R}_+ \to \mathbb{R}_+$ and a uniqueness function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for each subset $C \subseteq D_{X\times Y}(\zeta, \rho) \cap \mathcal{K}$, with $\Pi_Y C$ relatively compact, and for each $t > 0$, we have

$$\beta_X(S_A(t)F(C)) \leq l(t)\omega(\beta_{X\times Y}(C)).$$

Remark 3.3. If $A+F$ is $Y-$uniformly locally of $\beta-$compact type, then for each subset $C \subseteq D_{\mathcal{X}}(\zeta, \rho) \cap \mathcal{K}$, with $\Pi_Y C$ relatively compact, and for each $t > 0$, we have $\beta_X(S_A(t)F(C)) \leq l(t)\omega(\beta_X(\Pi_X C))$. Indeed, for each bounded subset $C \subseteq X \times Y$, we have $\beta_{X\times Y}(C) \leq \beta_X(\Pi_X C) + \beta_Y(\Pi_Y C)$.

Let us introduce the hypotheses which we will be in effect in the sequel.

(H_1) $A : D(A) \subseteq X \to X$, $B : D(B) \subseteq Y \to Y$ are the generators of two C_0-semigroups, $\{S_A(t) : X \to X;\, t \geq 0\}$ and $\{S_B(t) : Y \to Y;\, t \geq 0\}$ respectively;

(H_2) $\mathcal{K} \subseteq X \times Y$ is nonempty and locally closed;

(H_3) $(A + F, B + G)$ is locally of $\beta_{X\times Y}$-compact type;

(H_4) $A + F$ is $Y-$uniformly locally of β_X-compact type;

(H_5) $B + G$ is locally of β_Y- compact type;

(H_6) $\{S_B(t) : Y \to Y, t \geq 0\}$ is compact;

(H_7) $G : \mathcal{K} \to Y$ is continuous;

(H_8) $F : X \times Y \to X$ is continuous and globally Lipschitz with respect to its first argument.

A main viability result is Theorem 3.1, which is an immediate consequence of Theorem 2.2 and we refer to Burlică-Roşu [3].

Theorem 3.1. *Assume that (H_1), (H_2), (H_3) are satisfied. The necessary and sufficient condition in order that $\mathcal{K}$ be viable with respect to $(A + F, B + G)$ is that, for every $(\xi, \eta) \in \mathcal{K}$*

$$\liminf_{h\downarrow 0} \frac{1}{h}\mathrm{dist}((u(h, \xi, \eta), v(h, \xi, \eta)); \mathcal{K}) = 0, \tag{3.2}$$

where

$$\begin{cases} u(h, \xi, \eta) = S_A(h)\xi + hF(\xi, \eta) \\ v(h, \xi, \eta) = S_B(h)\eta + hG(\xi, \eta). \end{cases} \tag{3.3}$$

Another important viability results are Theorem 3.2, Theorem 3.3 and Theorem 3.4, whose proofs will be delivered in Section 4.

Theorem 3.2. *Assume that* (H_1), (H_2), (H_4), (H_6) *and* (H_7) *are satisfied. The necessary and sufficient condition in order that* $\mathcal{K}$ *be viable with respect to* $(A+F, B+G)$*, is that, for all* $(\xi,\eta)\in\mathcal{K}$*, the tangency condition* (3.2) *is satisfied.*

Theorem 3.3. *Assume that* (H_1), (H_2), (H_4) *and* (H_5) *are satisfied. The necessary and sufficient condition in order that* $\mathcal{K}$ *be viable with respect to* $(A+F, B+G)$*, is that, for all* $(\xi,\eta)\in\mathcal{K}$*, the tangency condition* (3.2) *is satisfied.*

Theorem 3.4. *Assume that* (H_1), (H_2), (H_6), (H_7) *and* (H_8) *are satisfied. The necessary and sufficient condition in order that* $\mathcal{K}$ *be viable with respect to* $(A+F, B+G)$*, is that, for every* $(\xi,\eta)\in\mathcal{K}$*, the tangency condition* (3.2) *is satisfied.*

We conclude this section with a nonautonomous variant of Theorem 3.4.

Theorem 3.5. *Assume that* X *and* Y *are Banach spaces and* (H_1), (H_6) *are satisfied. Let* $\mathcal{K}\subseteq\mathbb{R}\times X\times Y$ *be a nonempty and locally closed set,* $F:\mathbb{R}\times X\times Y\to X$ *and* $G:\mathcal{K}\to Y$ *be continuous. Let us assume that* F *is globally Lipschitz with respect to its second argument. Then a necessary and sufficient condition in order that* $\mathcal{K}$ *be viable with respect to* $(A+F, B+G)$ *is that*

(3.4)
$$\liminf_{h\downarrow 0}\frac{1}{h}\mathrm{dist}\left((\tau+h, S_A(h)\xi+hF(\tau,\xi,\eta), S_B(h)\eta+hG(\tau,\xi,\eta));\mathcal{K}\right)=0$$

for each $(\tau,\xi,\eta)\in\mathcal{K}$.

4. Proofs of Theorem 3.2, Theorem 3.3 and Theorem 3.4. In view of Remark 3.1, Theorem 3.2 can be reformulated as:

Theorem 4.1. *Assume that* (H_1), (H_2), (H_4), (H_6) *and* (H_7) *are satisfied. The necessary and sufficient condition in order that* $\mathcal{K}$ *be viable with respect to* $\mathcal{A}+\mathcal{F}$ *is that, for every* $\zeta\in\mathcal{K}$ *the tangency condition*

$$\liminf_{h\downarrow 0}\frac{1}{h}\mathrm{dist}(\mathcal{S}(h)\zeta+h\mathcal{F}(\zeta);\mathcal{K})=0. \tag{4.1}$$

is satisfied.

The necessity follows from Theorem 8.1.1 in CÂRJĂ-NECULA-VRABIE [4].

Proof. *Sufficiency.* We will use Lemma 2.4 to obtain a sequence of ε_n- approximate solutions to the Cauchy problem (3.1). Let $\zeta = (\xi, \eta) \in \mathcal{K}$ and $r > 0$ such that $D_{\mathcal{X}}(\zeta, r) \cap \mathcal{K}$ be closed. Let us choose $\rho \in (0, r]$, $N > 0$, $M \geq 1$ and $a \geq 0$ such that $\|\mathcal{F}(z)\|_{\mathcal{X}} \leq N$, for every $z \in D_{\mathcal{X}}(\zeta, \rho) \cap \mathcal{K}$ and $\|\mathcal{S}(t)\|_{\mathcal{L}(\mathcal{X})} \leq Me^{at}$, for every $t \geq 0$. Since $t \mapsto \mathcal{S}(t)\zeta$ is continuous in $t = 0$ and $\mathcal{S}(0)\zeta = \zeta$, we may find a sufficiently small $T > 0$ such that $\sup_{t\in[0,T]} \|\mathcal{S}(t)\zeta - \zeta\|_{\mathcal{X}} + TMe^{aT}(N + 1) \leq \rho$. As $A + F$ is Y-uniformly locally of β_X-compact type, diminishing $\rho > 0$ and $T > 0$, if necessary, it follows that there exist a continuous function $l : \mathbb{R}_+ \to \mathbb{R}_+$ and a uniqueness function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for each subset $C \subseteq D_{\mathcal{X}}(\zeta, \rho) \cap \mathcal{K}$, with $\Pi_Y C$ relatively compact, and for each $t > 0$, we have

$$\beta_X(S_A(t)F(C)) \leq l(t)\omega(\beta_{X\times Y}(C)). \tag{4.2}$$

Let $(\varepsilon_n)_n \downarrow 0$ be a sequence in $(0, 1)$ and let $((\sigma_n, \theta_n, \mathcal{G}_n, z_n))_n$ be a sequence of ε_n-approximate solution defined on $[0, T]$ whose existence is ensured by Lemma 2.4. From (v)† we have

$$z_n(t) = \mathcal{S}(t)\zeta + \int_0^t \mathcal{S}(t-s)\mathcal{F}(z_n(\sigma_n(s)))ds + \int_0^t \mathcal{S}(\theta_n(t,s))\mathcal{G}_n(s)ds, \tag{4.3}$$

for each $n \in \mathbb{N}$ and $t \in [0, T]$, where

$$\begin{cases} \mathcal{G}_n(s) = (\mathcal{G}_n^X(s), \mathcal{G}_n^Y(s)) \\ \mathcal{S}(\theta_n(t,s))\mathcal{G}_n(s) = (S_A(\theta_n(t,s))\mathcal{G}_n^X(s), S_B(\theta_n(t,s))\mathcal{G}_n^Y(s)), \end{cases}$$

for each $n \in \mathbb{N}$ and $0 \leq s < t \leq T$.

This means that $z_n = (u_n, v_n)$ satisfies $z_n(\sigma_n(t)) \in D_{\mathcal{X}}(\zeta, \rho) \cap \mathcal{K}$, for $n = 1, 2, \ldots$ and each $t \in [0, T]$, where (u_n, v_n) satisfies

$$\begin{cases} u_n(t) = S_A(t)\xi + \int_0^t S_A(t-s)F(z_n(\sigma_n(s)))ds + \int_0^t S_A(\theta_n(t,s))\mathcal{G}_n^X(s)ds \\ v_n(t) = S_B(t)\eta + \int_0^t S_B(t-s)G(z_n(\sigma_n(s)))ds + \int_0^t S_B(\theta_n(t,s))\mathcal{G}_n^Y(s)ds, \end{cases}$$

for each $n \in \mathbb{N}$ and $t \in [0, T]$. Since $\{G(z_n(\sigma_n(\cdot)))); n \in \mathbb{N}\}$ is uniformly bounded on $[0, T]$, $\|\mathcal{G}_n^Y(s)\| \leq \varepsilon_n$ for $n \in \mathbb{N}^*$ and the C_0-semigroup

†Throughout this section, reference to (i)–(vi) are to the items in Lemma 2.4.

$\{S_B(t) : Y \to Y; t \geq 0\}$ is compact, from Theorem 8.4.2 in VRABIE [11], it follows that $\{v_n;\ n = 1, 2, \ldots\}$ is relatively compact in $C(\,[\,0, T\,]; Y\,)$. So, for each $t \in [\,0, T\,]$, $\{v_n(\sigma_n(t));\ n = 1, 2, \ldots\}$ is relatively compact in Y. Let us prove that $\{u_n;\ n = 1, 2, \ldots\}$ is relatively compact in $C(\,[\,0, T\,]; X\,)$.

We consider first the case when X is a separable space. We know that

$$\begin{cases} \Pi_X\{\ (u_n(\sigma_n(t)), v_n(\sigma_n(t)));\, n \geq k\ \} = \{u_n(\sigma_n(t));\, n \geq k\ \} \\ \Pi_Y\{\ (u_n(\sigma_n(t)), v_n(\sigma_n(t)));\, n \geq k\ \} = \{v_n(\sigma_n(t));\, n \geq k\ \} \end{cases}$$

and $A + F$ is $Y-$uniformly locally of $\beta-$compact type. So, from (4.2) and Remark 3.3, we get

$$\begin{aligned} &\beta_X(\{S_A(t-s)F(u_n(\sigma_n(s)), v_n(\sigma_n(s)));\, n \geq k\}) \\ &\leq l(t-s)\omega(\beta_X(\{u_n(\sigma_n(s)\,);\, n \geq k\})), \end{aligned} \tag{4.4}$$

for each $k = 1, 2, \ldots$ and $0 \leq s < t \leq T$.

From (4.4) and Lemma 2.2, we deduce that

$$\begin{aligned} &\beta_X(\{u_n(t);\, n \geq k\}\ ) \leq \beta_X\left(\left\{\int_0^t S_A(t-s)F(z_n(\sigma_n(s)))ds;\, n \geq k\right\}\right) \\ &+\beta_X\left(\left\{\int_0^t S_A(\theta_n(t,s))\mathcal{G}_n^X(s)ds, n \geq k\right\}\right) \\ &\leq \int_0^t \beta_X(\{S_A(t-s)F(z_n(\sigma_n(s)));\, n \geq k\})ds \\ &+\int_0^t \beta_X(\{S_A(\theta_n(t,s))\mathcal{G}_n^X(s);\, n \geq k\})ds \\ &\leq \int_0^t l(t-s)\omega(\beta_X(\{u_n(\sigma_n(s));\, n \geq k\}))ds + MTe^{aT}\varepsilon_k \\ &\leq \int_0^t l(t-s)\omega(\beta_X(\{u_n(s);\, n \geq k\} + \{u_n(\sigma_n(s)) - u_n(s);\, n \geq k\}))ds \\ &+MTe^{aT}\varepsilon_k \\ &\leq \int_0^t m\omega\left(\beta_X(\{u_n(s);\, n \geq k\}) + \beta_X(\{u_n(\sigma_n(s)) - u_n(s);\, n \geq k\})\right)ds \\ &+MTe^{aT}\varepsilon_k, \end{aligned}$$

where $m = \sup_{t\in[\,0,T\,]} l(t)$. Hence

$$\beta_X(\{u_n(t);\, n \geq k\}\ ) \leq \int_0^t m\omega\left(\beta_X(\{u_n(s);\, n \geq k\}) + \varepsilon_k\right)ds + MTe^{aT}\varepsilon_k,$$

for each $k = 1, 2, \ldots$ and $t \in [\,0, T\,]$.

Set $x_k(t) = \beta_X(\{u_n(t); n \geq k\}) + \varepsilon_k$ and $\gamma_k = (MTe^{aT} + 1)\varepsilon_k$, for $k = 1, 2, \ldots$, and $\omega_0 = m\omega$. Thus, we deduce that

$$x_k(t) \leq \gamma_k + \int_0^t \omega_0(x_k(s))ds,$$

for $k = 1, 2, \ldots$ and $t \in [\,0, T\,]$.

From Remark 2.4, ω_0 is a uniqueness function, so, by Lemma 2.3, diminishing T, if necessary, we have $\lim_k x_k(t) = 0$, uniformly for $t \in [\,0, T\,]$, which means that $\lim_k \beta_X(\{u_n(t); n \geq k\}) = 0$, uniformly for $t \in [\,0, T\,]$. From Remark 2.3, it follows that for each $t \in [\,0, T\,]$, $\{u_n(t); n = 1, 2, \ldots\}$ is relatively compact in X. But, for each $t \in [\,0, T\,]$, $\{v_n(t); n = 1, 2, \ldots\}$ is relatively compact in Y, so, for each $t \in [\,0, T\,]$, $\{z_n(t); n = 1, 2, \ldots\}$ is relatively compact in $\mathcal{X}$. Since $(\mathcal{F}(z_n))_n$ is bounded, it is uniformly integrable, so, by Theorem 8.4.1 in Vrabie [11], there exists $z \in C([\,0, T\,]; \mathcal{X})$ such that, on a subsequence at least,

$$\lim_n \left(z_n(t) - \int_0^t \mathcal{S}(\theta_n(t, s)\mathcal{G}_n(s)ds \right) = z(t),$$

uniformly for $t \in [\,0, T\,]$. But, by (iii),

$$\lim_n \int_0^t \mathcal{S}(\theta_n(t, s))\mathcal{G}_n(s)ds = 0,$$

uniformly for $t \in [\,0, T\,]$, so $\lim_n z_n(t) = z(t)$, uniformly for $t \in [\,0, T\,]$. Using (ii) and (vi), we deduce that $z(t) \in \mathcal{K}$ for each $t \in [\,0, T\,]$ and, passing to the limit in (4.3), we get

$$z(t) = \mathcal{S}(t)\zeta + \int_0^t \mathcal{S}(t - s)\mathcal{F}(z(s))ds$$

for each $t \in [\,0, T\,]$. This completes the proof if X is separable.

If X is not separable, in view of Remark 2.2, there exists a separable and closed subspace Z of X such that

$$S_A(t)\xi,\ S_A(r)F(u_n(\sigma_n(s)), v_n(\sigma_n(s))),\ S_A(\theta_n(r, s))\mathcal{G}_n^X(s) \in Z$$

for $n = 1, 2, \ldots$ and a.e. for $t, r, s \in [\,0, T\,]$. Using Lemma 2.1 and the monotonicity of ω, we have

$$\begin{aligned} \beta_Z(S_A(t)F(C)) &\leq 2\beta_X(S_A(t)F(C)) \leq 2l(t)\omega(\beta_X(\Pi_X C)) \\ &\leq 2l(t)\omega(\beta_Z(\Pi_X C)), \end{aligned}$$

for each $t > 0$ and for each set $C \subseteq D_{\mathcal{X}}(\zeta, \rho) \cap \mathcal{K} \cap (Z \times Y)$ with $\Pi_Y C$ relatively compact. Since the restriction of β_Z to $\mathcal{B}(Z)$ is the Hausdorff measure of noncompactness on Z, we repeat now the arguments in the separable case with β_X replaced by β_Z and ω replaced by 2ω. This completes the proof. □

In view of Remark 3.1, Theorem 3.3 can be reformulated as

Theorem 4.2. *Assume that* (H_1), (H_2), (H_4) *and* (H_5) *are satisfied. The necessary and sufficient condition in order that* $\mathcal{K}$ *be viable with respect to* $\mathcal{A} + \mathcal{F}$*is that, for every* $\zeta \in \mathcal{K}$ *the tangency condition* (4.1) *is satisfied.*

We focus our attention on the sufficiency because the necessity of this result follows from Theorem 8.1.1 in CÂRJĂ-NECULA-VRABIE [4].

Proof. *Sufficiency.* We will prove that there exists at least one sequence $(\varepsilon_n)_n \downarrow 0$ for which the corresponding sequence $((\sigma_n, \theta_n, \mathcal{G}_n, z_n))_n$ of $(\varepsilon_n)_n-$ approximate solutions, has the property that $(z_n)_n$ is uniformly convergent on $[\,0, T\,]$ to a function $z : [\,0, T\,] \to \mathcal{K}$ which is a mild solution for the Cauchy problem (3.1). Let $\zeta = (\xi, \eta) \in \mathcal{K}$ be arbitrary and let $r > 0$, $\rho \in (0, r\,]$, $N > 0$, $M \geq 1$, $a \geq 0$ and $T > 0$ as in proof of Theorem 4.1. Since $A + F$ is Y-uniformly locally of β_X-compact type and $B + G$ is locally of β_Y-compact type, diminishing $\rho > 0$ and $T > 0$, if necessary, and using Remark 3.3 we conclude that there exist a continuous function $l : \mathbb{R}_+ \to \mathbb{R}_+$ and a uniqueness function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\beta_X(S_A(t)F(C)) \leq l(t)\omega(\beta_X(\Pi_X C)) \tag{4.5}$$

for each subset $C \subseteq D_{\mathcal{X}}(\zeta, \rho) \cap \mathcal{K}$, with $\Pi_Y C$ relatively compact, and for each $t > 0$, and

$$\beta_Y(S_B(t)G(C)) \leq l(t)\omega(\beta_Y(\Pi_Y C)) \tag{4.6}$$

for each subset $C \subseteq D_{\mathcal{X}}(\zeta, \rho) \cap \mathcal{K}$ and for each $t > 0$. Let $(\varepsilon_n)_n \downarrow 0$ be a sequence in $(0, 1)$ and let $((\sigma_n, \theta_n, \mathcal{G}_n, z_n))_n$ of $(\varepsilon_n)_n-$ approximate solutions defined on $[\,0, T\,]$ whose existence is ensured by the Lemma 2.4. From (v), we have

$$z_n(t) = \mathcal{S}(t)\zeta + \int_0^t \mathcal{S}(t-s)\mathcal{F}(z_n(\sigma_n(s)))ds + \int_0^t \mathcal{S}(\theta_n(t,s))\mathcal{G}_n(s)ds, \tag{4.7}$$

for each $n \in \mathbb{N}^*$ and $t \in [\,0, T\,]$, where

$$\begin{cases} \mathcal{G}_n(s) = (\mathcal{G}_n^X(s), \mathcal{G}_n^Y(s)) \\ \mathcal{S}(\theta_n(t,s))\mathcal{G}_n(s) = (S_A(\theta_n(t,s))\mathcal{G}_n^X(s), S_B(\theta_n(t,s))\mathcal{G}_n^Y(s)), \end{cases}$$

for each $n \in \mathbb{N}^*$ and $0 \le s < t \le T$. This means that $z_n = (u_n, v_n)$ satisfies

$$\begin{cases} u_n(t) = S_A(t)\xi + \displaystyle\int_0^t S_A(t-s)F(z_n(\sigma_n(s)))ds + \int_0^t S_A(\theta_n(t,s))\mathcal{G}_n^X(s)ds \\ v_n(t) = S_B(t)\eta + \displaystyle\int_0^t S_B(t-s)G(z_n(\sigma_n(s)))ds + \int_0^t S_B(\theta_n(t,s))\mathcal{G}_n^Y(s)ds, \end{cases}$$

for each $n \in \mathbb{N}^*$ and $t \in [\,0, T\,]$. We prove that the set $\{v_n(t); n \ge k\}$ is relatively compact in Y, for $k = 1, 2, \ldots$. We consider first the case when Y is a separable space. From Lemma 2.2 we get

$$\begin{aligned} &\beta_Y(\{v_n(t); n \ge k\}\ ) \le \beta_Y\left(\left\{\int_0^t S_B(t-s)G(z_n(\sigma_n(s)))ds; n \ge k\right\}\right) \\ &+\beta_Y\left(\left\{\int_0^t S_B(\theta_n(t,s))\mathcal{G}_n^Y(s)ds, n \ge k\right\}\right) \\ &\le \int_0^t \beta_Y(\{S_B(t-s)G(u_n(\sigma_n(s)), v_n(\sigma_n(s))); n \ge k\})ds \\ &+ \int_0^t \beta_Y(\{S_B(\theta_n(t,s))\mathcal{G}_n^Y(s); n \ge k\})ds. \end{aligned}$$

Since $\Pi_Y\{\ (u_n(\sigma_n(t)), v_n(\sigma_n(t))); n \ge k\ \} = \{v_n(\sigma_n(t)); n \ge k\ \}$, from (4.6) we get

$$\begin{aligned} &\beta_Y(\{S_B(t-s)G(u_n(\sigma_n(s)), v_n(\sigma_n(s))); n \ge k\}) \\ &\le l(t-s)\omega(\beta_Y(\{v_n(\sigma_n(s)); n \ge k\})). \end{aligned} \tag{4.8}$$

As $\|\mathcal{G}_n^Y(s)\|_Y \le \varepsilon_n$, for $n = 1, 2, \ldots$ and $\|S_B(t)\|_Y \le Me^{at}$ for each $t \ge 0$, from (4.8) we deduce

$$\begin{aligned} &\beta_Y(\{v_n(t); n \ge k\}\ ) \le \int_0^t l(t-s)\omega(\beta_Y(\{v_n(\sigma_n(s)); n \ge k\}))ds \\ &+MTe^{aT}\varepsilon_k \\ &\le \int_0^t m\omega\left(\beta_Y(\{v_n(s); n \ge k\}) + \beta_Y(\{v_n(\sigma_n(s)) - v_n(s); n \ge k\})\right)ds \\ &+MTe^{aT}\varepsilon_k, \end{aligned}$$

where $m = \sup_{t\in[\,0,T\,]} l(t)$. Thus,

$$\beta_Y(\{v_n(t); n \geq k\}\ ) \leq \int_0^t m\omega\left(\beta_Y(\{v_n(s); n \geq k\}) + \varepsilon_k\right) ds + MTe^{aT}\varepsilon_k,$$

for each $k = 1, 2, \ldots$ and $t \in [\,0, T\,]$.

Set $y_k(t) = \beta_Y(\{v_n(t); n \geq k\}) + \varepsilon_k$ and $\gamma_k = (MTe^{aT} + 1)\varepsilon_k$, for $k = 1, 2, \ldots$, and $\omega_0 = m\omega$. Hence, we deduce that $y_k(t) \leq \gamma_k + \int_0^t \omega_0(y_k(s))ds$, for $k = 1, 2, \ldots$ and $t \in [\,0, T\,]$. By Remark 2.4, and Lemma 2.3, diminishing $T > 0$, if necessary, we have $\lim_k y_k(t) = 0$ uniformly for $t \in [\,0, T\,]$, which means that $\lim_k \beta_Y(\{v_n(t); n \geq k\}) = 0$ uniformly for $t \in [\,0, T\,]$. Now, using Remark 2.3, we get that for each $t \in [\,0, T\,]$, $\{v_n(t);\ n = 1, 2, ...\}$ is relatively compact in Y.

If Y is not separable, in view of Remark 2.2, there exists a separable and closed subspace W of Y such that

$$S_B(t)\eta,\ S_B(r)G(u_n(\sigma_n(s))), v_n(\sigma_n(s)),\ S_B(\theta_n(r,s))\mathcal{G}_n^Y(s) \in W$$

for $n = 1, 2, \ldots$ and a.e. for $t, r, s \in [\,0, T\,]$. In view of (4.6), Lemma 2.1 and of the monotonicity of ω, we get

$$\begin{aligned}\beta_W(S_B(t)G(C)) &\leq 2\beta_Y(S_B(t)G(C)) \leq 2l(t)\omega(\beta_Y(\Pi_Y C))\\ &\leq 2l(t)\omega(\beta_W(\Pi_Y C))\end{aligned}$$

for each $t > 0$ and each $C \subseteq D_{\mathcal{X}}(\zeta, \rho) \cap \mathcal{K} \cap (X \times W)$. We repeat now the arguments in the separable case with β_Y replaced by β_W and ω replaced by 2ω and we obtain that for each $t \in [\,0, T\,]$, $\{v_n(t);\ n = 1, 2, \ldots\}$ is relatively compact in Y. From now on the proof follows the very same lines as those of the proof of Theorem 3.2, with the help of (4.5) (in case when X is separable and then without this restriction). □

We pass now to the proof of the sufficiency of the Theorem 3.4.

Proof. As in the previous theorems, we will use Lemma 2.4 to produce a sequence of ε_n-approximate solutions to the Cauchy problem (3.1). Let $\zeta = (\xi, \eta) \in \mathcal{K}$, let $\rho \in (\,0, r\,]$, $N > 0$, $M \geq 1$ and $a > 0$ as in proof of Theorem 4.1. Let $(\varepsilon_n)_n \downarrow 0$ be a sequence in $(0, 1)$ and let $((\sigma_n, \theta_n, \mathcal{G}_n, z_n))_n$ of $(\varepsilon_n)_n-$ approximate solutions defined on $[\,0, T\,]$ whose existence is ensured by the Lemma 2.4. Then

$$z_n(t) = \mathcal{S}(t)\zeta + \int_0^t \mathcal{S}(t-s)\mathcal{F}(z_n(\sigma_n(s)))ds + \int_0^t \mathcal{S}(\theta_n(t,s))\mathcal{G}_n(s)ds$$

for each $n \in \mathbb{N}$ and $t \in [\,0,T\,]$, which means that (u_n, v_n) satisfies

(4.9)
$$\begin{cases} u_n(t) = S_A(t)\xi + \displaystyle\int_0^t S_A(t-s)F(z_n(\sigma_n(s)))ds + \int_0^t S_A(\theta_n(t,s))\mathcal{G}_n^X(s)ds \\ v_n(t) = S_B(t)\eta + \displaystyle\int_0^t S_B(t-s)G(z_n(\sigma_n(s)))ds + \int_0^t S_B(\theta_n(t,s))\mathcal{G}_n^Y(s)ds, \end{cases}$$

where $\mathcal{G}_n(t) = (\mathcal{G}_n^X(t), \mathcal{G}_n^Y(t))$ for each $n = 1, 2, ...$ and $t \in [\,0,T\,]$. Since the C_0-semigroup $\{S_B(t) : Y \to Y;\, t \geq 0\}$ is compact, the family $\{G(z_n(\sigma_n(\cdot)));\, n \in \mathbb{N}\}$ is uniformly bounded on $[\,0,T\,]$ so uniformly integrable subset in $L^1(0,T;Y)$ and $\|\mathcal{G}_n^Y(s)\|_Y \leq \varepsilon_n$, from Theorem 8.4.2 in Vrabie [11] it follows that $\{v_n;\, n = 1, 2, ...\}$ is relatively compact in $C([\,0,T\,];Y)$. Since $\lim_n \sigma_n(t) = t$ uniformly for $t \in [\,0,T\,]$ it follows that there exists $v \in C([\,0,T\,];Y)$ such that, on a subsequence at least, $\lim_n v_n(t) = v(t)$ and $\lim_n v_n(\sigma_n(t)) = v(t)$ uniformly for $t \in [\,0,T\,]$.

At this point let us consider the problem:

$$\begin{cases} u'(t) = Au(t) + F(u(t), v(t)), \quad t \geq 0 \\ u(0) = \xi, \end{cases} \tag{4.10}$$

where $v \in C([\,0,T\,];Y)$ is as above. Since A is the infinitesimal generator of a C_0-semigroup and F is continuous on $X \times Y$ and globally Lipschitz with respect to its first argument, it follows that there exists an unique mild solution $u : [\,0,T\,] \to X$, i.e.

$$u(t) = S_A(t)\xi + \int_0^t S_A(t-s)F(u(s), v(s))ds \tag{4.11}$$

for each $t \in [\,0,T\,]$. We will prove next that $\lim_n u_n(t) = u(t)$ uniformly for $t \in [\,0,T\,]$. Indeed, we have

$$\|u_n(\sigma_n(t)) - u(t)\|_X \leq \|u_n(\sigma_n(t)) - u_n(t)\|_X + \|u_n(t) - u(t)\|_X. \tag{4.12}$$

From (4.9) and (4.11), it follows that

$$\begin{aligned} \|u_n(t) - u(t)\|_X &\leq \int_0^t \|S_A(\theta_n(t,s))\|_{L(X)}\|\mathcal{G}_n^X(s)\|_X ds \\ &+ \int_0^t \|S_A(t-s))\|_X \|F(u_n(\sigma_n(s)), v_n(\sigma_n(s))) - F(u(s), v(s))\|_X ds \end{aligned}$$

for each $t \in [\,0,T\,]$. Using (iii), (vi) and the Lipschitz's condition for F, from (4.12) we obtain

$$\begin{aligned}
&\|u_n(\sigma_n(t)) - u(t)\|_X \le Me^{aT}\varepsilon_n \\
&+Me^{aT}\int_0^t \|F(u(s),v_n(\sigma_n(s))) - F(u(s),v(s))\|_X ds \\
&+LMe^{aT}\int_0^t \|u_n(\sigma_n(s)) - u(s)\|_X ds.
\end{aligned} \tag{4.13}$$

for each $t \in [\,0,T\,]$. On the other hand, $\varepsilon_n \downarrow 0$, $v_n(\sigma_n(s)) \to v(s)$ uniformly for $s \in [\,0,T\,]$ and F is continuous. So, for each $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that, for each $n \in \mathbb{N}$, $n \ge n_0$ and for each $t \in [\,0,T\,]$ we have:

$$Me^{aT}\varepsilon_n + Me^{aT}\int_0^t \|F(u(s),v_n(\sigma_n(s))) - F(u(s),v(s))\|_X ds \le \varepsilon.$$

In view of (4.13) and the last inequality, we deduce

$$\|u_n(\sigma_n(t)) - u(t)\|_X \le \varepsilon + LMe^{aT}\int_0^t \|u_n(\sigma_n(s)) - u(s)\|_X ds$$

for all $n \in \mathbb{N}$, $n \ge n_0$ and $t \in [\,0,T\,]$. Gronwall's Lemma implies $\|u_n(\sigma_n(t)) - u(t)\|_X \le \varepsilon e^{LMTe^{aT}}$ for each $n \in \mathbb{N}$, $n \ge n_0$ and each $t \in [\,0,T\,]$. Therefore $\lim_n u_n(\sigma_n(t)) = u(t)$ uniformly for $t \in [\,0,T\,]$. Taking into account (i) we deduce that $\lim_n u_n(t) = u(t)$ uniformly for $t \in [\,0,T\,]$. Thanks to (ii), we deduce that $(u(t),v(t)) \in \mathcal{K}$ for each $t \in [\,0,T\,]$ and thus, passing to the limit for $n \to \infty$ in (4.9) we deduce that (u,v) is a mild solution of (1.1). The proof is complete. □

5. An example for a predator-pray system. Let $\Omega \subseteq \mathbb{R}^n$, $n = 1,2,\dots,$ be a bounded domain with C^2 boundary Γ, let $a \ge 0$, $b > 0$, $p > 0$, $q > 0$, let $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}_+$ and $g : \mathbb{R}\times\mathbb{R} \to \mathbb{R}_-$ be two continuous functions and let us consider the following general *predator-pray system*

$$\begin{cases}
u_t(t,x) = a\Delta u - pu(t,x) + f(u(t,x),v(t,x)) & (t,x) \in Q_{\tau,\infty} \\
v_t(t,x) = b\Delta v + qv(t,x) + g(u(t,x),v(t,x)) & (t,x) \in Q_{\tau,\infty} \\
u(t,x) = v(t,x) = 0 & (t,x) \in \Sigma_{\tau,\infty}, \\
u(\tau,x) = \xi(x), \quad v(\tau,x) = \eta(x) & x \in \Omega,
\end{cases} \tag{5.1}$$

where $0 \leq \tau < T \leq \infty$, $Q_{\tau,T} = (\tau, T) \times \Omega$, $\Sigma_{\tau,T} = (\tau, T) \times \Gamma$, Δ is the usual Laplace operator, i.e. $\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$ and $\xi, \eta \in L^2(\Omega)$.

Let $\widetilde{f} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ and $\widetilde{g} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_-$ be two continuous functions such that

$$\begin{cases} f(u,v) \leq \widetilde{f}(u,v) \\ g(u,v) \geq \widetilde{g}(u,v) \end{cases} \tag{5.2}$$

for each $(u, v) \in \mathbb{R} \times \mathbb{R}$. Let us consider also the comparison predator-pray system

$$\begin{cases} u_t(t,x) = a\Delta u - pu(t,x) + \widetilde{f}(u(t,x), v(t,x)) & (t,x) \in Q_{0,\infty} \\ v_t(t,x) = b\Delta v + qv(t,x) + \widetilde{g}(u(t,x), v(t,x)) & (t,x) \in Q_{0,\infty} \\ u(t,x) = v(t,x) = 0 & (t,x) \in \Sigma_{0,\infty}, \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x) & x \in \Omega, \end{cases} \tag{5.3}$$

where $u_0, v_0 \in L^2(\Omega)$, $u_0(x) \geq 0, v_0(x) \geq 0$ a.e. for $x \in \Omega$. Let $(\widetilde{u}, \widetilde{v}) : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+ \times \mathbb{R}_+$ be a mild solution of (5.3).

Using the viability result of Theorem 3.5, we obtain a sufficient conditions in order that, for each $(\xi, \eta) \in L^2(\Omega) \times L^2(\Omega)$, with

$$\begin{cases} 0 \leq \xi(x) \leq \widetilde{u}(\tau, x) \\ \eta(x) \geq \widetilde{v}(\tau, x) \end{cases} \tag{5.4}$$

a.e. for $x \in \Omega$, the predator-pray system (5.1) has at least one solution $(u, v) : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+ \times \mathbb{R}_+$, such that, for each $t \in [\tau, \infty)$, we have

$$\begin{cases} 0 \leq u(t,x) \leq \widetilde{u}(t,x) \\ v(t,x) \geq \widetilde{v}(t,x) \end{cases} \tag{5.5}$$

a.e. for $x \in \Omega$.

Let $\mathcal{K} \subseteq \mathbb{R} \times L^2(\Omega) \times L^2(\Omega)$ be defined by

$$\mathcal{K} = \left\{(t,u,v) \in \mathbb{R}_+ \times L^2(\Omega) \times L^2(\Omega);\ (u,v) \text{satisfy (5.7) below}\right\}. \tag{5.6}$$

$$\begin{cases} 0 \leq u(x) \leq \widetilde{u}(t,x) \\ v(x) \geq \widetilde{v}(t,x) \end{cases} \tag{5.7}$$

a.e. for $x \in \Omega$.

Theorem 5.1. *Let $\Omega \subseteq \mathbb{R}^n$, $n = 1, 2, \ldots$, be a bounded domain with C^2 boundary Γ, $a \geq 0$, $b > 0$ and let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$, $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_-$, $\widetilde{f} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$, $\widetilde{g} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_-$ be continuous such that* (5.2) *are satisfy. Assume that f is globally Lipschitz with respect to its first argument and for each $(u_0, v_0) \in \mathbb{R} \times \mathbb{R}$, $u \mapsto \widetilde{f}(u, v_0)$ and $v \mapsto \widetilde{g}(u_0, v)$ are nondecreasing, $u \mapsto \widetilde{g}(u, v_0)$ and $v \mapsto \widetilde{f}(u_0, v)$ are nonincreasing and there exist the constants $c_i \geq 0$, $i = 1, \ldots, 5$ such that $|\widetilde{f}(u, v)| \leq c_1|u| + c_2$, and $|\widetilde{g}(u, v)| \leq c_3|u| + c_4|v| + c_5$ for each $(u, v) \in \mathbb{R} \times \mathbb{R}$. Let $(u_0, v_0) \in L^2(\Omega) \times L^2(\Omega)$ with $u_0(x) \geq 0$ and $v_0(x) \geq 0$ a.e. for $x \in \Omega$ and let $(\widetilde{u}, \widetilde{v}) : \mathbb{R}_+ \to L^2(\Omega) \times L^2(\Omega)$ be a global mild solution of* (5.3) *with $\widetilde{u}(t, x) \geq 0$ for each $t \geq 0$ and a.e. for $x \in \Omega$. Let $\mathcal{K}$ be defined by* (5.6). *Then, for each $(\tau, \xi, \eta) \in \mathcal{K}$, the problem* (5.1) *has at least one global mild solution $(u, v) : [\,\tau, \infty) \to L^2(\Omega) \times L^2(\Omega)$ satisfying for each $\tau < \delta < T$:*

(i) $u, v \in C([\,\tau, T\,]; L^2(\Omega)) \cap L^2(\delta, T; H^2(\Omega)) \cap W^{1,1}(\delta, T; H_0^1(\Omega))$;

(ii) for each $t \in [\,\tau, \infty)$, $(t, u(t), v(t)) \in \mathcal{K}$.

The proof of this result follows by similar arguments from the Theorem 3.1 in Burlică-Roşu [3], obviously using Theorem 3.5. We remark that for $a = 0$ the first equation in (5.1) is degenerate.

For a systematic approach of various comparison results see Ladde *et al.* [7].

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