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PROPERTIES OF REGULARITY WITH RESPECT TO THE VIETORIS TOPOLOGY*

BY

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Abstract. In [1], chapter III, DINCULEANU has studied different types of regularity for finitely additive set functions taking values in Banach spaces. Using some of his ideas, we have defined and studied in [3] the properties of regularity with respect to the Hausdorff topology for $\mathcal{P}_f(X)$-valued multisubmeasures, where X is a real normed space and $\mathcal{P}_f(X)$ the family of all non-void, closed subsets of X. In the present paper, our intention is to define and study different concepts of regularity of $\mathcal{P}_f(X)$-valued set multifunctions, where X is a linear topological space and $\mathcal{P}_f(X)$ is endowed with the Vietoris topology. If, particularly, X is a real normed space, we also establish for $\mathcal{P}_f(X)$-valued multisubmeasures some relationships between certain types of regularity defined with respect to the Hausdorff topology, and the types that we have defined, with respect to the Vietoris topology.

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1. Introduction. Let T be a locally compact Hausdorff space, $\mathcal{C}$ a ring of subsets of T and (X,τ) a linear topological space. We consider the family $\mathcal{P}_0(X)$ of nonvoid subsets of X, the family $\mathcal{P}_f(X)$ of nonvoid, closed subsets of X, the family $\mathcal{P}_{bf}(X)$ of all nonvoid, bounded, closed subsets of X and the family $\mathcal{P}_k(X)$ of nonvoid, compact subsets of X.

On $\mathcal{P}_f(X)$ we consider the Minkowski addition " $\overset{\bullet}{+}$ " defined by:

$$M \overset{\bullet}{+} N = \overline{M+N}, \text{ for every } M,N \in \mathcal{P}_f(X).$$

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Definition 1.1. A multivalued set function $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is said to be a *multisubmeasure* if:

a) $\mu(\emptyset) = \{0\}$,

b) $\mu(A \cup B) \subseteq \mu(A) \overset{\bullet}{+} \mu(B)$, for every $A, B \in \mathcal{C}$ with $A \cap B = \emptyset$ (or, equivalently, for every $A, B \in \mathcal{C}$) and

c) $\mu(A) \subseteq \mu(B)$, for every $A, B \in \mathcal{C}$ with $A \subseteq B$.

that is, μ is *monotone.*

Examples 1.2. 1) If $\nu : \mathcal{C} \to \mathbb{R}_+$ is a submeasure in the sense of Drewnowski [2], then the multivalued set function $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R})$, defined by

$$\mu(A) = [0, \nu(A)], \text{ for every } A \in \mathcal{C},$$

is a multisubmeasure, called *the multisubmeasure induced by* ν.

2) If m_1, $m_2 : \mathcal{C} \to \mathbb{R}_+$, m_1 is finitely additive and m_2 is a submeasure on $\mathcal{C}$, then the multivalued set function $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R})$, defined by:

$$\mu(A) = [-m_1(A), m_2(A)], \text{ for every } A \in \mathcal{C},$$

is a multisubmeasure on $\mathcal{C}$ (see [4] for other examples of multisubmeasures).

Now, let $\mathcal{B}_0$ (respectively, $\mathcal{B}'_0$) be the Baire δ-ring (respectively, σ-ring) generated by the G_δ-compact sets (that is, compact sets which are countable intersections of open sets) and $\mathcal{B}$ (respectively, $\mathcal{B}'$) be the borelian δ-ring (respectively, σ-ring) generated by the compact sets of T. Obviously, $\mathcal{B}_0 \subset \mathcal{B}, \mathcal{B}_0 \subset \mathcal{B}'_0$, $\mathcal{B} \subset \mathcal{B}'$ and $\mathcal{B}'_0 \subset \mathcal{B}'$.

We shall also use the following notations which appear in [6], chapter 1:

$M^- = \{C \in \mathcal{P}_0(X); M \cap C \neq \emptyset\}$,
$M^+ = \{C \in \mathcal{P}_0(X); C \subseteq M\}$,
$S_{UV} = \{D^+; D \in \tau\}$ and $S_{LV} = \{D^-; D \in \tau\}$.

We remind from [6] that the Vietoris topology, denoted by $\widehat{\tau}_V$, is the topology on $\mathcal{P}_0(X)$ which has as a subbase the class $S_{UV} \cup S_{LV}$. Also, $\widehat{\tau}_V^+$ (respectively, $\widehat{\tau}_V^-$) denotes the topology which has as a subbase the class S_{UV} (respectively, S_{LV}).

It is easy to observe that if $U, V_1, \ldots, V_n \in \tau$, then the family of subsets

$$B_{U,V_1,\ldots,V_n} = \{M \in \mathcal{P}_0(X); M \subseteq U, M \cap V_k \neq \emptyset, k = \overline{1,n}\}$$

is a base for the topology $\widehat{\tau}_V$.

Also, the family of all subsets

$$B_U = \{M \in \mathcal{P}_0(X); M \subseteq U\},$$

respectively,

$$B_{V_1,\ldots,V_n} = \{M \in \mathcal{P}_0(X); M \cap V_k \neq \emptyset, k = \overline{1,n}\},$$

is a base for $\widehat{\tau}_V^+$, respectively, for $\widehat{\tau}_V^-$.

2. Regularity with respect to the Vietoris topology. Let (X, τ) be a linear topological space, $\mathcal{C}$ a ring of subsets of a locally compact Hausdorff space T, $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ a monotone set multifunction and $A \in \mathcal{C}$ be an arbitrary set.

In [3] we have introduced and studied different types of regularity for $\mathcal{P}_f(X)$-valued multisubmeasures, with respect to the Hausdorff topology $\widehat{\tau}_H$ induced by the Hausdorff pseudometric h, where X is a real normed space.

In the following, we introduce for $\mathcal{P}_f(X)$ -valued monotone set multifunctions, where X is a linear topological space, different types of regularity with respect to the Vietoris topology (see also [5]).

Definition 2.1. A is said to be:

i) R_l^+*-regular* with respect to μ if for every $U \in \tau$, with $\mu(A) \subset U$, there exists a compact set $K \in \mathcal{C}, K \subset A$ so that $\mu(B) \subset U$, for every $B \in \mathcal{C}$, with $K \subset B \subset A$.

ii) R_r^+*-regular* with respect to μ if for every $U \in \tau$, with $\mu(A) \subset U$, there exists an open set $D \in \mathcal{C}$ so that $A \subset D$ and $\mu(B) \subset U$, for every $B \in \mathcal{C}$, with $A \subset B \subset D$.

iii) R^+*-regular* with respect to μ if for every $U \in \tau$, with $\mu(A) \subset U$, there exist a compact set $K \in \mathcal{C}$ and an open set $D \in \mathcal{C}$ such that $K \subset A \subset D$ and $\mu(B) \subset U$, for every $B \in \mathcal{C}$, with $K \subset B \subset D$.

iv) R_r^-*-regular* with respect to μ if for every $V_1, V_2, \ldots, V_n \in \tau$, with $\mu(A) \cap V_j \neq \emptyset$, for every $j = \overline{1,n}$, there exists an open set $D \in \mathcal{C}$ so that $A \subset D$ and $\mu(B) \cap V_j \neq \emptyset$, for every $j = \overline{1,n}$ and every $B \in \mathcal{C}$, with $A \subset B \subset D$.

v) R_l^-*-regular* with respect to μ if for every $V_1, V_2, \ldots, V_n \in \tau$ with $\mu(A) \cap V_j \neq \emptyset$, for every $j = \overline{1,n}$, there exists a compact set $K \in \mathcal{C}, K \subset A$ such that $\mu(B) \cap V_j \neq \emptyset$, for every $B \in \mathcal{C}, K \subset B \subset A$ and every $j = \overline{1,n}$.

vi) R^-*-regular* with respect to μ if for every $V_1, V_2, \ldots, V_n \in \tau$, with $\mu(A) \cap V_j \neq \emptyset$, for every $j = \overline{1,n}$, there exist a compact set $K \in \mathcal{C}$ and an open set $D \in \mathcal{C}$ so that $K \subset A \subset D$ and for every $B \in \mathcal{C}$, with $K \subset B \subset D$, we have $\mu(B) \cap V_j \neq \emptyset$, for every $j = \overline{1,n}$.

vii) R_l*-regular* with respect to μ if for every $U, V_1, \ldots, V_n \in \tau$ such that $\mu(A) \subset U$ and $\mu(A) \cap V_j \neq \emptyset$, for every $j = \overline{1,n}$, there exists a compact set $K \in \mathcal{C}, K \subset A$ so that for every $B \in \mathcal{C}$, with $K \subset B \subset A$, we have $\mu(B) \subset U$ and $\mu(B) \cap V_j \neq \emptyset$, for every $j = \overline{1,n}$.

viii) R_r*-regular* with respect to μ if for every $U, V_1, V_2, \ldots, V_n \in \tau$, with $\mu(A) \subset U$ and $\mu(A) \cap V_j \neq \emptyset$, for every $j = \overline{1,n}$, there exists an open set $D \in \mathcal{C}$, with $A \subset D$, so that for every $B \in \mathcal{C}$, with $A \subset B \subset D$, we have $\mu(B) \subset U$ and $\mu(B) \cap V_j \neq \emptyset$, for every $j = \overline{1,n}$.

ix) *R-regular* with respect to μ if for every $U, V_1, \ldots, V_n \in \tau$ such that $\mu(A) \subset U$ and $\mu(A) \cap V_j \neq \emptyset$, for every $j = \overline{1,n}$, there exist a compact set $K \in \mathcal{C}$ and an open set $D \in \mathcal{C}, K \subset A \subset D$ so that for every $B \in \mathcal{C}$, with $K \subset B \subset D$, we have $\mu(B) \subset U$ and $\mu(B) \cap V_j \neq \emptyset$, for every $j = \overline{1,n}$.

Remarks 2.2.

i) Every set $A \in \mathcal{C}$ is R_l^+-regular with respect to μ, because if $B \in \mathcal{C}$, $K \subset B \subset A$, then $\mu(B) \subset \mu(A) \subset U$.

ii) Every open set $D \in \mathcal{C}$ is R_r^+-regular and R_r-regular with respect to μ.

iii) Every set $A \in \mathcal{C}$ is R_r^--regular with respect to μ because, if $\mu(A) \subset \mu(B)$ and $\mu(A) \cap V_j \neq \emptyset$, then $\mu(B) \cap V_j \neq \emptyset$, for every $j = \overline{1,n}$.

iv) Every compact set $K \in \mathcal{C}$ is R_l^--regular and R_l-regular with respect to μ.

We present in the sequel some relationships among all these types of regularity.

Theorem 2.3. *i) A is* R_l^-*-regular with respect to* μ *if and only if A is* R_l*-regular with respect to* μ*;*

ii) A is R_r^+*-regular with respect to* μ *if and only if A is* R_r*-regular with respect to* μ;

iii) If A is R^-*-regular with respect to* μ*, then A is* R_l^-*-regular with respect to* μ*;*

iv) If A is R^+-regular with respect to μ, then A is R_r^+-regular with respect to μ;

v) If A is R_r-regular and R_l-regular with respect to μ, then A is R-regular with respect to μ;

vi) If A is R-regular with respect to μ, then A is R^+-regular and R^--regular with respect to μ.

Proof. i) *The if part.*

Let $U \in \tau$ and $V_1, V_2, \ldots, V_n \in \tau$ be so that $\mu(A) \subset U$ and $\mu(A) \cap V_j \neq \emptyset$, for every $j = \overline{1,n}$. Because A is R_l^--regular, there exists a compact set $K \in \mathcal{C}, K \subset A$ so that, for every $B \in \mathcal{C}$, with $K \subset B \subset A$, we have $\mu(B) \cap V_j \neq \emptyset$, for every $j = \overline{1,n}$. Since $B \subset A$ and $\mu(A) \subset U$, then $\mu(B) \subset U$, so A is R_l-regular.

The only if part is straightforward, since R_l-regularity always implies R_l^--regularity.

ii) *The if part.*

Let $U \in \tau$ and $V_1, V_2, \ldots, V_n \in \tau$ be such that $\mu(A) \subset U$ and $\mu(A) \cap V_j \neq \emptyset$, for every $j = \overline{1,n}$. Because A is R_r^+-regular, there exists an open set $D \in \mathcal{C}, A \subset D$ so that for every $B \in \mathcal{C}$, with $A \subset B \subset D$, we have $\mu(B) \subset U$.

But $A \subset B$ and $\mu(A) \cap V_j \neq \emptyset$, hence $\mu(B) \cap V_j \neq \emptyset$, for every $j = \overline{1,n}$. Consequently, A is R_r-regular.

The only if part. R_r-regularity always implies R_r^+-regularity.

iii) Let $V_1, V_2, \ldots, V_n \in \tau$ be such that $\mu(A) \cap V_j \neq \emptyset$, for every $j = \overline{1,n}$. Since A is R^--regular, there exist a compact set $K \in \mathcal{C}$ and an open set $D \in \mathcal{C}$ so that $K \subset A \subset D$ and for every $B \in \mathcal{C}$, with $K \subset B \subset D$, we have $\mu(B) \cap V_j \neq \emptyset$, for every $j = \overline{1,n}$.

If $B \in \mathcal{C}, K \subset B \subset A$, then $K \subset B \subset D$, so $\mu(B) \cap V_j \neq \emptyset$, for every $j = \overline{1,n}$, which means that A is indeed R_l^--regular.

iv) Let $U \in \tau$ be so that $\mu(A) \subset U$.

Because A is R^+-regular, there exist a compact set $K \in \mathcal{C}$ and an open set $D \in \mathcal{C}$ so that $K \subset A \subset D$ and $\mu(B) \subset U$, for every $B \in \mathcal{C}$, with $K \subset B \subset D$.

If $B \in \mathcal{C}, A \subset B \subset D$, then $K \subset B \subset D$, hence $\mu(B) \subset U$, that is, A is R_r^+-regular.

v) Let $U \in \tau$ and $V_1, V_2, \ldots, V_n \in \tau$ be so that $\mu(A) \subset U$ and $\mu(A) \cap V_j \neq \emptyset$, for every $j = \overline{1,n}$. Since A is R_r-regular, there exists an open

set $D \in \mathcal{C}, A \subset D$ such that for every $B \in \mathcal{C}$, with $A \subset B \subset D$, we have $\mu(B) \subset U$ and $\mu(B) \cap V_j \neq \emptyset$, for every $j = \overline{1,n}$.

Because A is R_l-regular, there exists a compact set $K \in \mathcal{C}, K \subset A$ so that, for every $B \in \mathcal{C}$, with $K \subset B \subset A$, we have $\mu(B) \subset U$ and $\mu(B) \cap V_j \neq \emptyset$, for every $j = \overline{1,n}$.

Consequently, $K \subset A \subset D$, and if $B \in \mathcal{C}$, with $K \subset B \subset D$, then $K \subset A \cap B \subset A$ and $A \subset A \cup B \subset D$. It follows that $\mu(A \cap B) \cap V_j \neq \emptyset$, hence $\mu(B) \cap V_j \neq \emptyset$, for every $j = \overline{1,n}$. Also, $\mu(A \cup B) \subset U$, which implies that $\mu(B) \subset U$. Therefore, A is R-regular.

vi) Obviously, R-regularity always implies R^+-regularity and R^--regularity. □

Corollary 2.4. *The following statements are equivalent:*

(*i*) *A is R-regular;*

(*ii*) *A is R^+-regular and R^--regular;*

(*iii*) *A is R_r^+-regular and R_l^--regular;*

(*iv*) *A is R_r-regular and R_l-regular.*

We also obtain the following:

Theorem 2.5. *i) If $\mathcal{C}$ is $\mathcal{B}_0$ or $\mathcal{B}$ (or, more general, $\mathcal{C}$ is a ring of subsets of T so that for every $A \in \mathcal{C}$, there exists an open set $D \in \mathcal{C}$, with $A \subset D$) and if A is R_l^--regular, then it is also R^--regular;*

ii) If $\mathcal{C} = \mathcal{B}_0$ or $\mathcal{B}$ (or, more general, $\mathcal{C}$ is a ring of subsets of T such that for every $A \in \mathcal{C}$, there exists a compact set $K \in \mathcal{C}$, with $K \subset A$) and if A is R_r^+-regular, then it is also R^+-regular.

Proof. i) Let $V_1, V_2, \ldots, V_n \in \tau$ be so that $\mu(A) \cap V_j \neq \emptyset$, for every $j = \overline{1,n}$. By the hypothesis, there exists a compact set $K \in \mathcal{C}, K \subset A$ so that $\mu(B) \cap V_j \neq \emptyset$, for every $j = \overline{1,n}$ and for every $B \in \mathcal{C}$, with $K \subset B \subset A$.

Let be an arbitrary open set $D \in \mathcal{C}$, with $A \subset D$. Then $K \subset A \subset D$; if we consider $B \in \mathcal{C}$, with $K \subset B \subset D$, then $K \subset A \cap B \subset A$, so $\mu(A \cap B) \cap V_j \neq \emptyset$, for every $j = \overline{1,n}$. Because $\mu(A \cap B) \subseteq \mu(B)$, it immediately follows the conclusion.

ii) Let $U \in \tau$ so that $\mu(A) \subset U$. By the hypothesis, there exists an open set $D \in \mathcal{C}, A \subset D$ such that $\mu(B) \subset U$, for every $B \in \mathcal{C}$, with $A \subset B \subset D$.

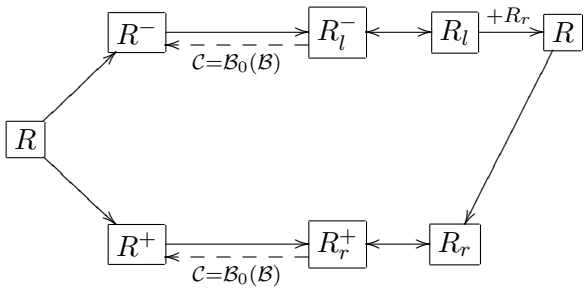
Let $K \subset A$ be an arbitrary, compact set of $\mathcal{C}$. Then $K \subset A \subset D$. If $B \in \mathcal{C}, K \subset B \subset D$, we get that $A \subset A \cup B \subset D$, so $\mu(A \cup B) \subset U$. Because $\mu(B) \subseteq \mu(A \cup B)$, then $\mu(B) \subset U$, hence A is R^+-regular. □

Corollary 2.6. *If $\mathcal{C} = \mathcal{B}_0$ or $\mathcal{B}$ (or, more general, $\mathcal{C}$ is a ring of subsets of T so that for every $A \in \mathcal{C}$, there exist a compact set $K \in \mathcal{C}$ and an open set $D \in \mathcal{C}$, with $K \subset A \subset D$), then the following statements are equivalent:*

a) i) A is R^+-regular;
ii) A is R_r-regular;
iii) A is R_r^+-regular;
b) i) A is R^--regular;
ii) A is R_l^--regular;
iii) A is R_l-regular.

We synthetize the above results in the folowing

Scheme 1 - X is a linear topological space



3. **Relationships between the types of regularity with respect to the Vietoris topology, and the types of regularity with respect to the Hausdorff topology**. In the sequel, we shall compare the types of regularity introduced for $\mathcal{P}_f(X)$-valued multisubmeasures, where X is a linear topological space, with respect to the Vietoris topology $\widehat{\tau}_V$, with those that we have defined in [3] with respect to the Hausdorff topology $\widehat{\tau}_H$ for multisubmeasures $\mu : \mathcal{C} \to \mathcal{P}_f(X)$, where X is a real normed space. Let h be the Hausdorff pseudometric on $\mathcal{P}_f(X)$, which becomes a metric on $\mathcal{P}_{bf}(X)$. It is well-known that

$$h(M, N) = \max\{e(M, N), e(N, M)\}, \text{ for every } M, N \in \mathcal{P}_f(X),$$

where $e(M,N)=\sup_{x\in M} d(x,N)$ and $d(x,N)$ is the distance from x to N with respect to the metric induced by the norm of X. We denote $|M|=h(M,\{0\})$, for every $M\in\mathcal{P}_f(X)$, where 0 is the origin of X.

Let then in the following $\mu:\mathcal{C}\to\mathcal{P}_f(X)$ be a multisubmeasure, where X is a real normed space, with the origin 0, d, the distance induced by the norm $\|\cdot\|$ of X and τ_d the topology induced by d. Let also $A\in\mathcal{C}$ be arbitrarily.

We now recall from [3] several definitions of regularity with respect to the Hausdorff topology for multisubmeasures:

Definition 3.1. I) A is said to be R^H-*regular* with respect to μ if for every $\varepsilon>0$, there exist a compact set $K\in\mathcal{C}, K\subset A$ and an open set $D\in\mathcal{C}, D\supset A$ so that $h(\mu(A),\mu(B))<\varepsilon$, for every $B\in\mathcal{C}$, with $K\subset B\subset D$;

II) A is said to be R_l^H-*regular* with respect to μ if for every $\varepsilon>0$, there exists a compact set $K\in\mathcal{C}, K\subset A$ such that $h(\mu(A),\mu(B))<\varepsilon$, for every $B\in\mathcal{C}, K\subset B\subset A$;

III) A is said to be R_r^H-*regular* with respect to μ if for every $\varepsilon>0$, there is an open set $D\in\mathcal{C}, D\supset A$ so that $h(\mu(A),\mu(B))<\varepsilon$, for every $B\in\mathcal{C}, A\subset B\subset D$.

Definition 3.2. μ is said to be R^H-*regular* (respectively, R_l^H-*regular*, R_r^H-*regular*) if every set $A\in\mathcal{C}$ is R^H-regular (respectively, R_l^H-regular, R_r^H-regular) with respect to μ.

Theorem 3.3. *i) If A is R_r^+-regular, then A is R_r^H-regular;*

ii) If, moreover, μ has compact, non-void values, then A is R_r^+-regular if and only if it is R_r^H-regular.

Proof. i) Suppose that A is R_r^+-regular with respect to μ. Let also $\varepsilon>0$. We denote $U=\bigcup_{x\in\mu(A)} S(x,\frac{\varepsilon}{2})$. Obviously, $\mu(A)\subset U$ and $U\in\tau_d$. By the hypothesis, there exists an open set $D\in\mathcal{C}$ so that $A\subset D$ and for every $B\in\mathcal{C}$, with $A\subset B\subset D$, we have $\mu(B)\subset\bigcup_{x\in\mu(A)} S(x,\frac{\varepsilon}{2})$.

We shall prove that

$$e(\mu(B),\mu(A))=\sup_{y\in\mu(B)} d(y,\mu(A))<\varepsilon.$$

Indeed, if $y \in \mu(B)$ is arbitrarily, then $y \in \bigcup_{x\in\mu(A)} S(x, \frac{\varepsilon}{2})$, hence there exists $x_y \in \mu(A)$ so that $y \in S(x_y, \frac{\varepsilon}{2})$. Consequently,

$$d(y, \mu(A)) = \inf_{x\in\mu(A)} d(y, x) \leq d(y, x_y) < \frac{\varepsilon}{2},$$

which implies that

$$e(\mu(B), \mu(A)) = \sup_{y\in\mu(B)} d(y, \mu(A)) \leq \frac{\varepsilon}{2} < \varepsilon.$$

Since $e(\mu(A), \mu(B)) = 0$, we finally obtain that $h(\mu(B), \mu(A)) < \varepsilon$; therefore, A is R_r^H-regular, as claimed.

ii) Because $\mu : \mathcal{C} \to \mathcal{P}_k(X)$, then (see [6], theorem 1.30), $\widehat{\tau}_H^+ = \widehat{\tau}_V^+$ and the conclusion follows immediately. □

Theorem 3.4. *i) If A is R_l^H-regular, then A is R_l^--regular;*

ii) If, moreover, μ has compact non-void values, then A is R_l^--regular if and only if it is R_l^H-regular.

Proof. i) Let $V_1, V_2, \ldots, V_n \in \tau_d$ be so that $\mu(A) \cap V_j \neq \emptyset$, for every $j = \overline{1,n}$. Because $V_1 \in \tau_d$ and $\mu(A) \cap V_1 \neq \emptyset$, there exists $x_1 \in \mu(A)$ and $\varepsilon_1 > 0$ such that $S(x_1, \varepsilon_1) \subset V_1$.

Applying the R_l^H-regularity of A, we get that for ε_1 there is a compact set $K_1 \in \mathcal{C}$, so that $K_1 \subset A$ and for every $B \in \mathcal{C}$ with $K_1 \subset B \subset A$, we have

$$e(\mu(A), \mu(B)) = \sup_{x\in\mu(A)} d(x, \mu(B)) < \varepsilon_1.$$

Since

$$d(x_1, \mu(B)) = \inf_{y\in\mu(B)} d(x_1, y) < \varepsilon_1,$$

there exists $y_1 \in \mu(B)$ so that $d(x_1, y_1) < \varepsilon_1$. Consequently, $\mu(B) \cap S(x_1, y_1) \neq \emptyset$, which implies that $\mu(B) \cap V_1 \neq \emptyset$.

Similarly, for $V_2, \ldots, V_n$, we get the existence of the compact sets $K_2, \ldots, K_n \in \mathcal{C}$, so that, for every $j = \overline{2,n}$, we have $K_j \subset A$ and $\mu(B) \cap V_j \neq \emptyset$, for every $B \in \mathcal{C}$, with $K_j \subset B \subset A$.

Let then $K = \bigcup_{j=1}^n K_j$. Obviously, K is a compact set, $K \in \mathcal{C}$, $K \subset A$ and for every $B \in \mathcal{C}$, with $K \subset B \subset A$, we also have $K_j \subset B \subset A$, so $\mu(B) \cap V_j \neq \emptyset$, for every $j = \overline{1,n}$. This means that, indeed, A is R_l^--regular.

ii) Since $\mu : \mathcal{C} \to \mathcal{P}_k(X)$ and from [6], theorem 1.30, we know that $\widehat{\tau}_H^- = \widehat{\tau}_V^-$, then we have the conclusion. □

Corollary 3.5. *If μ has non-void, compact values, then $A \in \mathcal{C}$ is R-regular (with respect to $\widehat{\tau}_V$) if and only if A is R^H-regular.*

Proof. According to corollary 2.4, A is R-regular if and only if it is R_r^+-regular and R_l^--regular (with respect to $\widehat{\tau}_V$). But, since $\mu : \mathcal{C} \to \mathcal{P}_k(X)$, then R_r^+-regularity (in $\widehat{\tau}_V$) is equivalent to R_r^H-regularity and R_l^--regularity (in $\widehat{\tau}_V$) is equivalent to R_l^H-regularity. On the other hand, we know from [3] that A is R^H-regular if and only if it is R_l^H-regular and R_r^H-regular, hence we have the conclusion. □

We observe that we can also indicate another proof based on the fact that, if $\mu : \mathcal{C} \to \mathcal{P}_k(X)$, then $\widehat{\tau}_H = \widehat{\tau}_V$ (see [6], theorem 1.30).

Corollary 3.6. *If $\mathcal{C}$ is $\mathcal{B}_0$ or $\mathcal{B}$ and μ has compact values, then the following statements are equivalent:*

i) A is R^--regular (with respect to $\widehat{\tau}_V$) if and only if it is R_l^H-regular;

ii) A is R^+-regular (with respect to $\widehat{\tau}_V$) if and only if it R_r^H-regular.

Theorem 3.7. *i) If A is R^+-regular, then*

$(*)$ *for every $\varepsilon > 0$, there exist a compact set $K \in \mathcal{C}$ and an open set $D \in \mathcal{C}$ such that $K \subset A \subset D$ and $e(\mu(B), \mu(A)) < \varepsilon$, for every $B \in \mathcal{C}$, with $K \subset B \subset D$;*

ii) If, moreover, μ has compact values, then A is R^+-regular if and only if it has the property $(*)$.

Proof. i) Let A be R^+-regular with respect to μ. Let also be $\varepsilon > 0$. We denote $U = \bigcup_{x \in \mu(A)} S(x, \frac{\varepsilon}{2})$. Obviously, $\mu(A) \subset U$ and $U \in \tau_d$. Using the hypothesis, we get that there are a compact set $K \in \mathcal{C}$ and an open set $D \in \mathcal{C}$ so that $K \subset A \subset D$ and $\mu(B) \subset \bigcup_{x \in \mu(A)} S(x, \frac{\varepsilon}{2})$, for every $B \in \mathcal{C}$ with $K \subset B \subset D$.

Let $B \in \mathcal{C}$, with $K \subset B \subset D$ be arbitrarily, but fixed.

Since $\mu(B) \subset \bigcup_{x\in\mu(A)} S(x, \frac{\varepsilon}{2})$, then for every $y \in \mu(B)$, there exists $x_y \in \mu(A)$ such that $d(y, x_y) < \frac{\varepsilon}{2}$; this implies

$$d(y, \mu(A)) = \inf_{x\in\mu(A)} d(y, x) \leq d(y, x_y) < \frac{\varepsilon}{2}.$$

Consequently,

$$e(\mu(B), \mu(A)) = \sup_{y\in\mu(B)} d(y, \mu(A)) \leq \frac{\varepsilon}{2} < \varepsilon.$$

ii) It only remains to prove that the property $(*)$ implies R^+-regularity. We first note that, since $0 \in \mu(A)$, everywhere appears the condition "$\mu(A) \subset U$, $U \in \tau_d$", it follows that, particularly, $U \in V(0)$, where by $V(0)$ we denote the system of all neighbourhoods of 0.

Let thus $r > 0$ be so that $\mu(A) \subset S(0, r)$, where $S(0, r) = \{x \in X; \|x\| < r\}$. Then $\sup_{x\in\mu(A)} \|x\| \leq r$ and, because $\mu(A)$ is a compact set, there exists $x_0 \in \mu(A)$ such that $\|x_0\| = \sup_{x\in\mu(A)} \|x\| < r$.

Then there is $r_1 < r$ so that $\|x\| < r_1 < r$, for every $x \in \mu(A)$.

By the hypothesis, for $\varepsilon = r - r_1 > 0$, there exist a compact set $K \in \mathcal{C}$ and an open set $D \in \mathcal{C}$ such that $K \subset A \subset D$ and $e(\mu(B), \mu(A)) < r - r_1$, for every $B \in \mathcal{C}$ with $K \subset B \subset D$.

We shall prove that $\mu(B) \subset S(0, r)$. Indeed, for every $y \in \mu(B)$, we have

$$d(y, \mu(A)) \leq \sup_{y\in\mu(B)} d(y, \mu(A)) = e(\mu(B), \mu(A)) < r - r_1,$$

so there exists $x_y \in \mu(A)$ such that $d(y, x_y) < r - r_1$. Therefore,

$$\|y\| \leq d(y, x_y) + \|x_y\| < r - r_1 + r_1 = r.$$

It follows that, indeed, $\mu(B) \subset S(0, r)$. □

Theorem 3.8. *If $A \in \mathcal{C}$, $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ and*

$(**)$ *for every $\varepsilon > 0$, there are a compact set $K \in \mathcal{C}$ and an open set $D \in \mathcal{C}$ such that $K \subset A \subset D$ and $e(\mu(A), \mu(B)) < \varepsilon$, for every $B \in \mathcal{C}$, with $K \subset B \subset D$,*

then A is R^--regular.

Proof. Let $V_1, V_2, \ldots, V_n \in \tau_d$ be so that $\mu(A) \cap V_j \neq \emptyset$, for every $j = \overline{1,n}$. For V_1, there exists $x_1 \in \mu(A)$ and $r_1 > 0$ so that $S(x_1, r_1) \subset V_1$. Applying the hypothesis for $\varepsilon = r_1$, we get that there are a compact set $K_1 \in \mathcal{C}$ and an open set $D_1 \in \mathcal{C}, K_1 \subset A \subset D_1$ such that $e(\mu(A), \mu(B)) < r_1$, for every $B \in \mathcal{C}, K_1 \subset B \subset D_1$.

Since

$$d(x_1, \mu(B)) \leq e(\mu(A), \mu(B)) < r_1,$$

there exists $y_1 \in \mu(B)$, with $d(x_1, y_1) < r_1$. Consequently, $\mu(B) \cap S(x_1, r_1) \neq \emptyset$, and, because $S(x_1, r_1) \subset V_1$, we finally get that $\mu(B) \cap V_1 \neq \emptyset$.

Similarly, we obtain the existence of the compact sets $K_2, ...K_n \in \mathcal{C}$ and of the open sets $D_2, ...D_n \in \mathcal{C}$ so that, for every $j = \overline{2,n}$, we have $K_j \subset A \subset D_j$ and $\mu(B) \cap V_j \neq \emptyset$, for every $B \in \mathcal{C}$, with $K_j \subset B \subset D_j$.

Let then be the compact set $K = \bigcup_{j=1}^{n} K_j$ and the open set $D = \bigcap_{j=1}^{n} D_j$. Since $K \subset A \subset D$ and $K, D \in \mathcal{C}$, if $B \in \mathcal{C}$, with $K \subset B \subset D$, then for every $j = \overline{1,n}$, we have $K_j \subset B \subset D_j$, so $\mu(B) \cap V_j \neq \emptyset$. The proof is thus finished. □

Using theorems 3.7 and 3.8 we obtain the following result:

Corollary 3.9. *If $\mathcal{C}$ is $\mathcal{B}_0$ or $\mathcal{B}$, $\mu : \mathcal{C} \to \mathcal{P}_k(X)$ and $A \in \mathcal{C}$, then A is:*

i) R_r^H-regular if and only if it has the property $(*)$;

ii) R_l^H-regular if and only if it has the property $(**)$.

Proof. i) We apply corollary 2.6 and theorem 3.7.

ii) The only if part follows immediately, using corollary 2.6 and theorem 3.8.

Let us prove now the if part. If $\varepsilon > 0$, from the hypothesis there is a compact set $K \in \mathcal{C}$ so that $K \subset A$ and $h(\mu(B), \mu(A)) < \varepsilon$, for every $B \in \mathcal{C}$ with $K \subset B \subset A$.

On the other hand, if $\mathcal{C} = \mathcal{B}_0$ or $\mathcal{B}$, there exists an open set $D \in \mathcal{C}$, with $A \subset D$. Then $K \subset A \subset D$ and if $B \in \mathcal{C}, K \subset B \subset D$, we get that $K \subset A \cap B \subset A$, hence $h(\mu(A \cap B), \mu(A)) < \varepsilon$. Therefore, $e(\mu(A), \mu(A \cap B)) < \varepsilon$, which implies that $e(\mu(A), \mu(B)) < \varepsilon$. Consequently, A has indeed the property $(**)$. □

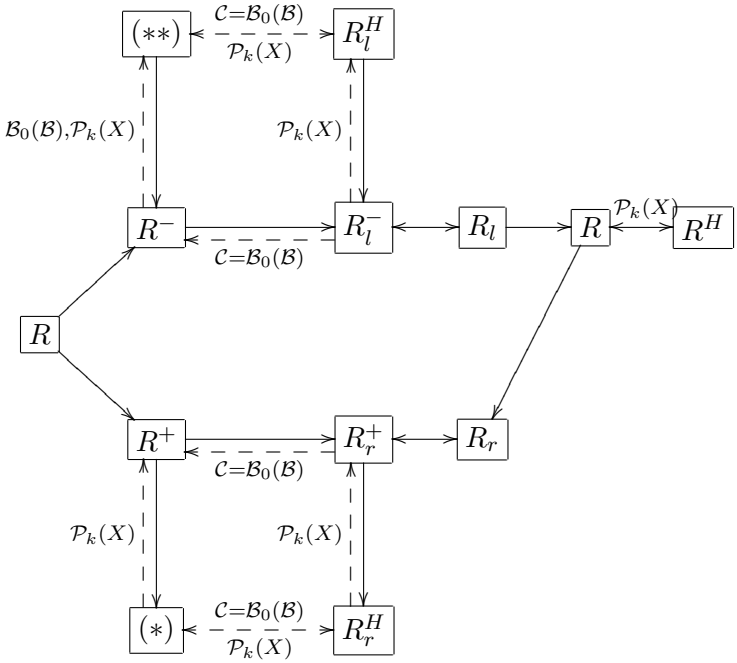
Remarks 3.10. Let us note that all the statements of corollary 3.9 can be directly obtained from the definitions for any multisubmeasure $\mu : \mathcal{C} \to \mathcal{P}_f(X)$, where $\mathcal{C}$ is $\mathcal{B}_0$ or $\mathcal{B}$.

For this particular case, using theorem 3.8 and corollaries 2.6 and 3.9, we also obtain:

Corollary 3.11. *If $\mu : \mathcal{C} \to \mathcal{P}_k(X)$, where $\mathcal{C}$ is $\mathcal{B}_0$ or $\mathcal{B}$ and $A \in \mathcal{C}$, then A is R^--regular if and only if it has the property $(**)$.*

We synthetize all the above results in the folowing:

Scheme 2 - X is a real normed space



(*) $\forall \varepsilon > 0, \exists K$ compact $\in \mathcal{C}, \exists\ D$ open $\in \mathcal{C}$, $K \subset A \subset D$ so that $e(\mu(B), \mu(A)) < \varepsilon, \forall B \in \mathcal{C}, K \subset B \subset D$.

(**) $\forall \varepsilon > 0, \exists K$ compact $\in \mathcal{C}, \exists\ D$ open $\in \mathcal{C}$, $K \subset A \subset D$ so that $e(\mu(A), \mu(B)) < \varepsilon, \forall B \in \mathcal{C}, K \subset B \subset D$.

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