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CATEGORY VITALI SETS

BY

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Abstract. We introduce the concept of category Vitali Sets in a metric space which is motivated by a paper of ORLANDO and obtain various properties of such sets. In addition, we obtain some topological consequences

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Key words: Metric space, Baire category, category Vitali set, category accumulation point, countable intersection property in category.

1. Introduction. ORLANDO [5] proved an analogue of the classical Vitali covering theorem [2], p 262 with respect to Baire category of sets. In fact, a strengthened version of the Vitali covering theorem in R^n (n-dimensional Euclidean space) is now available: every Vitali cover of any set $E \subset R^n$ has a pairwise disjoint subfamily W such that $E \setminus \bigcup_{V \in W} V$ is a nowhere dense set in R^n having Lebesgue measure zero.

The motivation of the paper arises from Proposition 1[5] where the uncovered portion of a given set in a metric space with certain assumption is small in the sense of Baire category for each maximal pairwise disjoint subfamily selected from a given Vitali cover of the set. It is observed that the conclusion of Proposition 1[5] helps to generate a type of a set which provide several interesting and unforeseen conclusions. We introduce in this paper the term "category Vitali set" in a metric space and observe that there are sets which are category Vitali sets as well as sets which are not. We derive several properties of such sets with the help of open and closed sets, defined suitably in the metric space. Some properties of a topological space, that arises in consequence of study of category Vitali sets, are presented.

The results of the paper have no connection with those of [1], [3] and [4] where in [3] and [4] measure-theoretic approach are dominant.

2. Definitions and examples. Let (X, d) be a metric space which will be written simply as X when there is no confusion. Let $\mathcal{L}$ denote the family of all closed balls in X with positive radius. If $E \subset X$, then a non empty family v from $\mathcal{L}$ is called a *Vitali cover* of E if for each $x \in E$ and each $\varepsilon > 0$, there exists a ball $V \in v$ such that $x \in V$ and diam $(V) < \varepsilon$.

The following proposition and theorem are proved in [5].

Proposition A. *Let the metric space X satisfy the following assumption: $B = \overline{int(B)}$ for every closed ball B of X. Then given any Vitali cover v of any set $E \subset X$, we have that*

$$E \backslash \bigcup_{V \in W} V$$

is a nowhere dense set in X for each maximal pairwise disjoint family $W \subset v$.

Here and in future, the existence of a maximal pairwise disjoint subfamily is guaranteed by Zorn's lemma.

Theorem A. *In an arbitrary metric space (X, d), every Vitali cover v of a set $E \subset X$ contains a pairwise disjoint subfamily W such that $E \backslash \bigcup_{V \in W} V$ is a nowhere dense set in X.*

We introduce the definition of a category Vitali set as follows:

Definition 1. A set $P \subset X$ is said to be a *category Vitali Set* (in short, c-Vitali set), if given any Vitali cover v of P, $P \backslash \bigcup_{V \in W} V$ is a set of the first category for each maximal pairwise disjoint subfamily $W \subset v$.

We may note that from Theorem A and Definition 1, it does not follow that every subset of X is a c-Vitali set.

We shall, in general, not assume that X is complete or separable. We may note that the completeness of X implies that X is of second category and separability of X gives that the family W in Proposition A or Theorem A or Definition 1 is countable.

Further, for subsets A and B of X we shall write $A - B$ for $A \backslash B$. Sets A, B etc. are non-void subsets of X, unless otherwise stated, $int(A)$ means the interior of the set $A, \overline{A}$ denotes the closure of A and φ denotes the void set.

Example 1. A set of the first category is a c-Vitali set.

Example 2. If the condition $B = \overline{int(B)}$ of Proposition A is satisfied for every closed ball B in X, then every subset of X is a c-Vitali set.

The assumption $B = \overline{int(B)}$ of Proposition A holds automatically for every closed ball B of R^n, the n-dimensional Euclidean space. So

Example 3. Every subset of R^n is a c-Vitali set.

Through the following example, which is patterned after Example of [5], we show the existence of a set which is not a c-Vitali set.

Example 4. Let $Y = Y_1 \cup Y_2$, where $Y_1 = \{(0, y) \in R^2 : 0 \leq y \leq 1\}$ and $Y_2 = \{(1, y) \in R^2 : 0 \leq y \leq 1$ and y is rational $\}$.

Let a distance function d be defined on Y as follows: $d(a, b) = 2$ if $a, b \in Y_2$, $a \neq b$ and otherwise $d(a, b)$ is the Euclidean distance. Then (Y, d) is a complete metric space (see [6], Theorem 9.3.3, p 179) and so is of second category.

Let $E = Y_1$ and v be the family of all closed balls of Y. Then v is a Vitali cover of E. The subcollection W of v with center at a point of Y_2 and having radius 1 is clearly a maximal pairwise disjoint (countable) subfamily of v. Now E is an open subset of Y and so is of second category and the set $E - \bigcup_{V \in W} V$, obtained from E by removing a countable set, is also of second category. Thus E is not a c-Vitali set.

3. Open and closed sets. In this section, we introduce the notion of a category accumulation point of a set and in terms of this idea, closed and open sets in X.

Definition 2. A point $a \in X$ is called a *category accumulation point* (in short, c-accumulation point) of a set A if for every sequence $\{F_n\}$ from $\mathcal{L}$ with diameters tending to zero and $a \in int(F_n)$ for each n, there exists a subsequence $\{F_{n_k}\}$ such that $A \cap F_{n_k}$ is a set of the second category for each k.

Definition 3. The collection of all c-accumulation points of A is called the *category derived set* of A and is denoted by $D(A)$.

The proof of the following lemma is omitted.

Lemma 1. *Let $\{A : \alpha \in \Lambda\}$ be a family of subsets of X where Λ is an index set. Then*

(i) $D(\bigcap_a A_\alpha) \subset \bigcap_a D(A_\alpha)$ *and*
(ii) $D(\bigcup_{n=1}^N A_n) = \bigcup_{n=1}^N D(A_n)$
where $A_1, A_2, \ldots, A_N$ *belong to the family.*

In the usual way, we now write the definition of closed and open sets in X with the help of category derived set.

Definition 4. A set F is closed if $D(F) \subset F$. A set G is open if $X - G$ is closed.

Lemma 2. *X and the void set φ are closed and open sets.*

Theorem 1. *The family of all open sets of X forms a topology.*

The proof follows from lemmas 1 and 2. This topology will be denoted by T_c. Thus (X, T_c) is a topological space. The following theorem gives a characterisation of open sets.

Theorem 2. *A subset G is open if and only if for every $x \in G$, there exists a sequence $\{F_k\}$ from $\mathcal{L}$ with diameters tending to zero and $x \in int(F_k)$, $k = 1, 2, \ldots$, such that $F_k - (G \cap F_k)$ is a set of the first category in X for each k.*

Proof. Suppose that G is open and let $x \in G$. There exists a sequence $\{F_k\}$ from $\mathcal{L}$ with diameters tending to zero such that $x \in int(F_k)$ and $F_k \cap (X - G)$ is a set of the first category for $k = 1, 2, \ldots$. But $F_k \cap (X - G) = F_k - (G \cap F_k)$ and so this part is finished.

Conversely, suppose that the condition of the theorem holds. Since $(X - G) \cap F_k = F_k - G \cap F_k, (X - G) \cap F_k$ is of first category for each k.

So, x cannot be a c-accumulation point of $X - G$. Because $x \in G$ is arbitrary, $D(X - G) \subset X - G$ and G becomes open. This proves the theorem. □

Theorem 3. *If for a set $A, D(A) - A$ is of first category, then $D(A)$ is closed.*

Proof. We show that $B = X - D(A)$ is open. For this, let $\alpha \in B$. There exists $\{F_k\}$ from $\mathcal{L}$ with diameters tending to zero such that $\alpha \in int(F_k)$ and $A \cap F_k$ is of first category for each k. Since $F_k \cap D(A) \subset [F_k \cap (D(A) - A)] \cup [F_k \cap A]$, it follows that $F_k \cap D(A)$ is a first category subset for each

k. But $F_k \cap D(A) = (X - B) \cap F_k = F_k - (B \cap F_k)$ and so it follows from Theorem 2, that B is open. This proves the theorem. □

4. c-Vitali sets. Throughout this section, we assume that X is separable.

Theorem 4. *$A - D(A)$ is a c-Vitali set if and only if $A - D(A)$ is a set of the first category.*

Proof. If $A - D(A)$ is of first category, then $A - D(A)$ is a c-Vitali set by Example 1. Conversely, suppose that $A - D(A)$ is a c-Vitali set and let $\alpha \in A - D(A)$. There exists a sequence $\{F_i(\alpha)\}$ from $\mathcal{L}$ with diameters tending to zero such that $\alpha \in int(F_i(\alpha))$ and $A \cap F_i(\alpha)$ is of first category for each i. We consider the collection

$$v = [\{F_i(\alpha)\}, i = 1, 2, \dots; \alpha \in A - D(A)]$$

which is clearly a Vitali cover of $A - D(A)$. Let $\{F_n\}$ be a maximal pairwise disjoint sequence from v such that $[A - D(A)] - \bigcup_n F_n$ is a set of the first category. Since

$$A - D(A) \subset [(A - D(A)) - \bigcup_n F_n] \cup [(\bigcup_n F_n) \cap A],$$

the proof of the theorem is complete. □

The following corollaries follows from Theorem 4.

Corollary 1. *Let A be a c-Vitali set. If $D(A) = \varphi$, then A is a set of the first category.*

Corollary 2. *If A is of second category and $A - D(A)$ is a c-Vitali set, then except a first category subset of A, all the points of A are c-accumulation points of A.*

Theorem 5. *If $A - D(A)$ is a c-Vitali set, $A \cap D(A) = B$ and $B \cap D(X - A) = \varphi$, then B is open.*

Proof. Let $\alpha \in B$. There exists $\{F_k\}$ from $\mathcal{L}$ with diameters tending to zero, $\alpha \in int(F_k)$ and $F_k \cap (X - A)$ is a first category set for $k = 1, 2, \dots$. Now

$$F_k - (F_k \cap B) = [F_k \cap (A - D(A))] \cup [F_k \cap (X - A)],$$

so $F_k - (F_k \cap B)$ is a set of the first category, because $A - D(A)$ is of first category by Theorem 4. The theorem now follows from Theorem 2.

Theorem 6. *Let A be a c-Vitali set with $A \cap D(B) = \varphi$ for some set B. Then $A \cap B$ is a set of the first category.*

Proof. Let $\alpha \in A$. There exists $\{F_k(\alpha)\}$ from $\mathcal{L}$ with diameters tending to zero such that $\alpha \in int(F_k(\alpha))$ and $F_k(\alpha) \cap B$ is a set of the first category for $k = 1, 2, \ldots$.

The collection $[\{F_k(\alpha)\}, k = 1, 2, \ldots ; \alpha \in A]$ is clearly a Vitali cover of A.

So, there exists a maximal countable pairwise disjoint subcollection $\{F_n\}$ such that $A - \bigcup_{n=1}^{\infty} F_n$ is of first category. Now

$$\begin{aligned} A \cap B &\subset [(A \cap B) - (\bigcup_{n=1}^{\infty} F_n) \cap B] \cup [(\bigcup_{n=1}^{\infty} F_n) \cap B] \\ &= [(A - \bigcup_{n=1}^{\infty} F_n) \cap B] \cup [(\bigcup_{n=1}^{\infty} F_n) \cap B] \\ &\subset [A - \bigcup_{n=1}^{\infty} F_n] \cup [\bigcup_{n=1}^{\infty} (F_n \cap B)]. \end{aligned}$$

Both the sets in the brackets in the right hand side are of first category and so the theorem is proved. □

Corollary 3. *Let A be a c-Vitali set. If $A \cap D(A) = \varphi$, then A is a set of the first category.*

In other words, if A is a c-Vitali set such that no point of A is a c-accumulation point of A, then A is of first category.

Theorem 7. *Let X be complete. Then $A - D(A)$ is a c-Vitali set if and only if A can be expressed as $A = G \cup H$ where G is of first category, $H \subset D(H)$ and $G \cap H = \varphi$.*

Proof. Suppose first that the condition of the theorem holds. Then by Lemma 1

$$A - D(A) = A - D(G \cup H) = A - [D(G) \cup D(H)] = A - D(H) \subset A - H = G.$$

Thus $A - D(A)$ is of first category and thus a c-Vitali set by Theorem 4.

Conversely, suppose that $A - D(A)$ is a c-Vitali set. Let $G = A - D(A)$ and $H = A \cap D(A)$. By Theorem 4, G is of first category. If $\alpha \in H$, then given an arbitrary sequence $\{F_n\}$ from X with diameters tending to zero, $\alpha \in int(F_n)$ for each n, there exists a subsequence $\{F_{n_k}\}$ such that $A \cap F_{n_k}$ is of second category for each k. For each k, we observe that $A \cap F_{n_k} = (G \cap F_{n_k}) \cup (H \cap F_{n_k})$ and so $H \cap F_{n_k}$ is of second category. This shows that $\alpha \in D(H)$ and the theorem is proved. □

Theorem 8. *A c-Vitali set E is closed and discrete (i.e. $E \cap D(E) = \varphi$) if and only if E is of first category.*

Proof. If E is of first category, then from definition, $D(E) = \varphi$ and so E is closed and discrete. Conversely, suppose that E is closed and discrete. Since E is closed, $D(E) \subset E$ and $E \cap D(E) = \varphi$ implies $D(E) = \varphi$. So, by Theorem 4, $E = E - D(E)$ is of first category. This proves the theorem. □

Theorem 9. *Suppose that E and $E - D(E)$ be c-Vitali sets. If $D(E)$ is of first category, then $D(E)$ is a void set.*

Proof. We have $E = [E - D(E)] \cup [E \cap D(E)]$ and thus by Theorem 4, E is of first category. By Theorem 8, E is closed and discrete. Thus $D(E)$ is void. □

5. Countable intersection property in category and c-Vitali sets. In this section, we show that the following idea of countable intersection property in category has some connection with c-Vitali sets. We introduce the following definitions.

Definition 5. Let $\mathcal{U}$ be a non-void family of subsets of X, the complement of each of which with respect to X is a closed ball. The family $\mathcal{U}$ is called a *complementary Vitali class* if for $U \in \mathcal{U}$ and for every $x \in X - U$, there exists a sequence $\{U_n\}$ from $\mathcal{U}$ such that $x \in X - U_n$ for each $n = 1, 2, \ldots$, with diameters of $X - U_n$ tending to zero.

Definition 6. Let $\mathcal{U}$ be a complementary Vitali class in X and $\bigcap_{n=1}^{\infty} U_n$ be a set of the second category for every maximal sequence $\{U_n\}$, $U_n \in \mathcal{U}, n = 1, 2, \ldots$ with pairwise disjoint complements. Then X is said to have the *countable intersection property in category* if $\bigcap_{U\in\mathcal{U}} U$ is of second category.

Theorem 10. *Let X be complete and separable. If $X - A$ is a c-Vitali set for every first category subset A, then X has the countable intersection property in category.*

Proof. Suppose that $X - A$ is a c-Vitali set for every set A of the first category. Let $\mathcal{U}$ be a complementary Vitali class satisfying the condition of Definition 6. We show that $\bigcap_{U\in\mathcal{U}} U = I$ (say) is a set of the second category.

If possible, let I be of the first category. For $U \in \mathcal{U}$, let $V = X - U$, then $\bigcap(X - V)$, i.e. $X - \cup V = V_0$ (say) is of first category. We consider the collection $v = [V : X - V \in \mathcal{U}]$ which is clearly a cover of $X - V_0$. We verify that v is a Vitali cover of $X - V_0$. Let $x \in V$ for some V. Then $x \in X - U$ for some $U \in \mathcal{U}$. Choose $\varepsilon > 0$ arbitrary. Then from the definition of the complementary Vitali class, there exists an $U_\varepsilon \in \mathcal{U}$ such that $x \in X - U_\varepsilon = V_\varepsilon$ (say) and diam $(V_\varepsilon) < \varepsilon$. So, v is a Vitali cover of $X - V_0$. Since $X - V_0$ is a c-Vitali set, there exists a maximal pairwise disjoint sequence $\{V_n\}$ from v such that $(X - V_0) - \bigcup_{n=1}^{\infty} V_n$ is a set of the first category. Since

$$X - \bigcup_{n=1}^{\infty} V_n \subset [(X - V_0) - \bigcup_{n=1}^{\infty} V_n] \cup V_0,$$

it follows that

$$X - \bigcup_{n=1}^{\infty} V_n = \bigcap_{n=1}^{\infty} (X - V_n) = \bigcap_{n=1}^{\infty} U_n, U_n = X - V_n \in \mathcal{U}$$

is a set of the first category. Clearly $\{U_n\}$ is a maximal sequence with pairwise disjoint complements. This implies a contradiction. Hence $\bigcap_{U\in\mathcal{U}} U$ is a set of the second category and X has the countable intersection property in category. This proves the theorem. □

Note 1. Example 3 and Theorem 10 show that R^n has the countable intersection property in category.

6. Topological consequences. In this section, we are interested to find several properties of the topological space (X, T_c) considered in Section 3.

Definition 7. If $A \subset X$, then $A \cup D(A)$ is the closure of A and is denoted by $cl(A)$.

Note 2. A is closed if and only if $A = cl(A)$.

Lemma 3. *$cl(A)$ is closed, provided $D(A) - A$ is of first category.*

Proof. We have by Lemma 1 and Theorem 3

$$D[cl(A)] = D(A) \cup D[D(A)] = D(A) \subset cl(A)$$

and thus $cl(A)$ is closed. □

Theorem 11. *If for a subset $A, D(A) - A$ is a set of the first category, then $cl(A) = \cap[F : F \supset A; F$ is closed$]$.*

Proof. Let $\overline{A}$ stand for the right hand expression. Since (X, T_c) is a topological space, it is clear that $\overline{A}$ is closed and so by Note 2

$$cl(A) \subset cl(\overline{A}) = \overline{A}.$$

Now $A \subset cl(A)$ and from lemma 3, $cl(A)$ is closed. So $\overline{A} \subset cl(A)$. Thus $\overline{A} = cl(A)$ and the theorem is proved. □

Note 3. From Theorem 11, it follows that $cl(A)$ coincides with the closure of A in the topological space (X, Tc), provided $D(A) - A$ is a set of the first category.

Theorem 12. *If X is complete, then (X, Tc) is not separable, provided X is not a finite set.*

Proof. Suppose that (X, Tc) is separable. So, there exists a countable subset, say A such that $cl_{Tc}(A) = X$ where $cl_{Tc}(A)$ is the closure of A in (X, Tc). Because A is a set of the first category, $D(A) = \varphi$ and so by Theorem 11, $cl(A) = X$, i.e. $X = cl(A) = A \cup D(A) = A$. But X is of second category. This contradiction proves the theorem. □

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