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NAVIER-STOKES EQUATIONS WITH POTENTIALS. FEEDBACK STABILIZATION*

BY

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Abstract. From existence and uniqueness theorems for the Navier-Stokes equations perturbed with a maximal monotone operator, we infer feedback stability results in invariance conditions for the Navier-Stokes system.

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1. Introduction. In this paper we consider Navier-Stokes equations with potentials, that is perturbed with a maximal monotone operator. After establishing the functional framework in Section 1, in Section 2 we give existence and uniqueness results for strong and weak solutions of the Navier-Stokes perturbed system. The proofs use the semigroupal approach, which has been developed in the Navier-Stokes case in [2], [4]. Section 3 studies several applications of the previous theorems in the feedback stabilization of the Navier-Stokes equations. We are concerned with the exponential stabilization together with the preservation of the solution in a given closed convex set.

Let $T > 0$ and let $\Omega \subset \mathbf{R}^d,\ d = 2, 3$ be an open and bounded domain, with a smooth boundary $\partial\Omega$ (of class C^2, e.g.). Consider the Navier-Stokes

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equations with "no-slip" boundary conditions,

$$(1)\qquad \begin{cases} \dfrac{\partial y}{\partial t} - \nu\Delta y + (y\cdot\nabla)y + \nabla\pi = f_0 & , \text{ in } \Omega\times(0,T), \\ \text{div } y = 0 & , \text{ in } \Omega\times(0,T), \\ y = 0 & , \text{ on } \partial\Omega\times(0,T), \\ y(\cdot,0) = y_0 & , \text{ in } \Omega, \end{cases}$$

where $y = (y_1, y_2, \dots, y_d)$ is the velocity field, p is the scalar pressure, the density of external forces is $f_0 = (f_{01}, f_{02}, \dots, f_{0d})$ and the constant $\nu > 0$ is the kinematic viscosity coefficient. This system describes the motion of an incompressible homogenous Newtonian viscous fluid.

We will use the standard spaces (see, e.g., [7], [11], [12])

$$H = \{y \in (L^2(\Omega))^d; \text{div } y = 0 \text{ in } \Omega, y\cdot n_{\partial\Omega} = 0 \text{ on } \partial\Omega\},$$

$$V = \{y \in (H_0^1(\Omega))^d; \text{div } y = 0 \text{ in } \Omega\}.$$

H is a real Hilbert space endowed with L^2-norm $|\cdot|$ and V is a real Hilbert space endowed with H_0^1-norm $\|\cdot\| = |\nabla\cdot|$. Moreover, denoting by V' the dual space of V and considering H identified with its own dual, we have $V \subset H \subset V'$ algebraically and topologically with compact injections.

Here, $(\cdot,\cdot)$ denotes the scalar product of H and the pairing between V and its dual V'. The norm of V' is denoted by $\|\cdot\|_{V'}$.

Let $A \in L(V, V')$ (the space of linear continuous operators from V in V'), $(Ay, z) = \sum_{i=1}^d \int_\Omega \nabla y_i \cdot \nabla z_i dx,\ \forall y, z \in V$.

We have $(Ay, y) = \|y\|^2,\ \forall y \in V$. We set $D(A) = \{y \in V; Ay \in H\}$ and denote again by A the restriction of A to H.

Let $b : V\times V\times V \to \mathbf{R}$ be the trilinear continuous functional defined by

$$b(y,z,w) = \sum_{i,j=1}^d \int_\Omega y_i \frac{\partial z_j}{\partial x_i} w_j dx,\ \forall y,z,w \in V.$$

The functional b satisfies (see, e.g., [7], [11], [12])

$$(2)\qquad b(y,w,w) = 0,\ b(y,z,w) = -b(y,w,z),\ \forall y,z,w \in V,$$

$$(3)\qquad |b(y,z,w)| \le C\|y\|\|z\|_a|w|,\ \forall y,w \in V, z \in D(A),$$

where $a \in (1,2)$ for $d = 2$ and $a = 3/2$ for $d = 3$,

(4) $$|b(y,z,w)| \leq C|y|^{\frac{1}{2}}\|y\|^{\frac{1}{2}}|z|^{\frac{1}{2}}\|z\|^{\frac{1}{2}}\|w\|,\ \forall y,z,w \in V \text{ (for } d=2),$$

(5) $$|b(y,z,w)| \leq C|y|^{\frac{1}{2}}\|y\|^{\frac{1}{2}}\|z\|\|w\|,\ \forall y,z,w \in V \text{ (for } d=3).$$

Let $B : V \to V'$ be defined by

$$(By, w) = b(y,y,w),\ \forall y,w \in V.$$

In this setting, equations (1) may be rewritten

(6) $$\begin{cases} \dfrac{dy}{dt}(t) + \nu Ay(t) + By(t) = f(t), & t \in (0,T) \\ y(0) = y_0, \end{cases}$$

where $f = Pf_0,\ P : (L^2(\Omega))^d \to H$ is the Leray projection.

Consider the linear continuous operator $A_0 \in L(V,V')$,

(7) $$(A_0 y, w) = b(y, y_e, w) + b(y_e, y, w),\ \forall y,w \in V,$$

where $y_e \in D(A)$ is fixed. Then $A_0 : D(A_0) = D(A) \subset H \to H$ and for all $\varepsilon > 0$ there exists a constant $\mathcal{C}(\varepsilon) > 0$ such that $|A_0 y| \leq \varepsilon|Ay| + \mathcal{C}(\varepsilon)|y|,\ \forall y \in D(A)$.

Let us introduce the perturbed Navier-Stokes system

(8) $$\begin{cases} \dfrac{dy}{dt}(t) + (\nu A + A_0 + \mathcal{G})y(t) + By(t) + \Phi(y(t)) \ni f(t),\ t \in (0,T) \\ y(0) = y_0, \end{cases}$$

where $\mathcal{G} \in L(H,H)$ and $\Phi \subset H \times H$ is a maximal monotone operator. Such a nonlinear term $\mathcal{G} + \Phi$ arises usually as a feedback nonlinear controller.

Suppose Φ satisfies the following hypotheses:

- (h_1) $\Phi = \partial\varphi$, where $\varphi : H \to \overline{\mathbf{R}}$ is a lower semicontinuous proper convex function (hence Φ is a maximal monotone operator in $H \times H$);
- (h_2) $0 \in D(\Phi)$;
- (h_3) there exist two constants $\gamma \geq 0,\ \alpha \in \left(0, \frac{1}{\nu}\right)$ such that

(9) $$(Au, \Phi_\lambda(u)) \geq -\gamma(1 + |u|^2) - \alpha|\Phi_\lambda(u)|^2,\ \forall \lambda > 0, \forall u \in D(A),$$

where $\Phi_\lambda = \frac{1}{\lambda}(I - (I + \lambda\Phi)^{-1}) : H \to H$ is the Yosida approximation of Φ.

We consider the classical definition of the maximal monotone operator. We will denote $|\Phi(u)| = \inf\{|z|; z \in \Phi(u)\}$, where $u \in D(\Phi)$.

In the sequel, the symbol $\rightharpoonup$ will be used to denote the convergence in the weak topology, while the strong convergence will be denoted by $\to$. Also, we will denote by the same symbol C various positive constants.

2. Existence and uniqueness results for perturbed Navier-Stokes equations. This section contains definitions and existence theorems for strong and weak solutions.

Definition 2.1. *Let $y_0 \in V$ and $f \in L^2(0,T;H)$. By a strong solution for the perturbed Navier-Stokes equations* (8) *we mean a function*

$$y \in L^2(0,T;D(A)) \cap L^\infty(0,T;V) \cap C([0,T];H),\ \textit{satisfying}\ \frac{dy}{dt} \in L^2(0,T;H),$$

for which there exists a selection $\eta \in L^2(0,T;H)$, $\eta(t) \in \Phi(y(t))$ a.e. $t \in (0,T)$, such that

$$\begin{cases} \dfrac{dy}{dt}(t) + (\nu A + A_0 + \mathcal{G})y(t) + By(t) + \eta(t) = f(t),\ \text{a.e. } t \in (0,T) \\ y(0) = y_0. \end{cases} \tag{10}$$

Definition 2.2. *Let $y_0 \in H$ and $f \in L^2(0,T;V')$. By a weak solution for the perturbed Navier-Stokes equations* (8) *we mean a function*

$$y \in L^2(0,T;V) \cap C_w([0,T];H),\ \textit{satisfying}\ \frac{dy}{dt} \in L^1_{loc}(0,T;V'),$$

for which there exists a selection $\eta \in L^2(0,T;V')$, $\eta(t) \in \Phi(y(t))$ a.e. $t \in (0,T)$, such that

$$\begin{cases} \left(\dfrac{dy}{dt}(t), v\right) + \nu((y(t),v)) + b(y(t),y(t),v) + b(y(t),y_e,v) + b(y_e,y(t),v) \\ \qquad +(\mathcal{G}y(t),v) + (\eta(t),v) = (f(t),v),\ \text{a.e. } t \in (0,T),\ \forall v \in V \\ y(0) = y_0. \end{cases} \tag{11}$$

The space $C_w([0,T];H)$ is a subspace of $L^\infty(0,T;H)$ consisting of weakly continuous functions; in particular the initial condition in (11) is taken in this sense.

2.1. Main results. The main existence result for strong solutions is

Theorem 2.1. *Let $T > 0$ and let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be an open, bounded set, with a smooth boundary. Let $\Phi = \partial\varphi \subset H \times H$ satisfy the hypotheses $(h_1) - (h_3)$. Assume that φ is bounded on sets of $D(A) \cap D(\Phi)$ which are bounded in V. Let $\mathcal{G} \in L(H, H)$. Let $y_0 \in \overline{D(A) \cap D(\Phi)}^V$ and $f \in L^2(0, T; H)$. If $d = 2$, then there exists a unique*

$$y \in C([0, T]; H) \cap L^2(0, T; D(A)) \cap L^\infty(0, T; V), \text{ with } \frac{dy}{dt},\ By \in L^2(0, T; H),$$

which verifies

$$\begin{cases} \dfrac{dy}{dt}(t) + (\nu A + A_0 + \mathcal{G})y(t) + By(t) + \Phi(y(t)) \ni f(t), \ \ a.e.\ t \in (0, T) \\ y(0) = y_0. \end{cases} \tag{12}$$

If $d = 3$, then the solution y exists on some interval $[0, T_0)$, where

$$T_0 = T_0 \left(\|f\|^2_{L^2(0,T;H)}, \|y_0\|^2 \right) \le T.$$

Hence, strong solutions are unique and global in the dimension $d = 2$, respectively local in the dimension $d = 3$.

In the following we state an existence result for weak solutions in the case when Φ is the subdifferential of the indicator function of a convex set. Let $K \subset H$ be a closed convex set, satisfying $0 \in K$ and

$$(I + \lambda A)^{-1}(K) \subset K, \ \forall \lambda > 0. \tag{13}$$

Let I_K be the indicator function of the set K, which is

$$I_K : H \to H, \ I_K(w) = \begin{cases} 0 & , \text{ for } w \in K \\ +\infty & , \text{ for } w \notin K \end{cases}$$

and assume that Φ is the subdifferential of this function: $\Phi = \partial I_K$, where

$$\partial I_K(w) = \begin{cases} \emptyset & , w \notin K \\ \{0\} & , w \in \text{int } K \\ N_K(w) = \{y \in H; (y, w - z) \ge 0, \ \forall z \in K\} & , w \in \partial K. \end{cases}$$

Here $N_K(w)$ is the normal cone at K in w. It can be seen that $0 \in D(\Phi) = K$. The Yosida approximation of ∂I_K is

$$(\partial I_K)_\lambda(w) = \frac{1}{\lambda}(w - P_K w) = \frac{1}{\lambda} Q_K w, \ \forall \lambda > 0, \ \forall w \in H,$$

where the operator $P_K : H \to K$ is the orthogonal projection on K and $Q_K = I - P_K$. We also know that $(\partial I_K)_\lambda$ is the Fréchet differential of the regularisation of I_K,

$$(I_K)_\lambda(w) = \frac{1}{2\lambda}|w - P_K w|^2 = \frac{1}{2\lambda}|Q_K w|^2.$$

The following result on weak solutions takes place,

Theorem 2.2. *Let $T > 0$ and $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be an open, bounded set, with a smooth boundary. Let $K \subset H$ verify the above hypotheses. Assume, moreover, that*

(14) $\quad$ *P_K carries bounded sets of H in bounded sets*

$$in \begin{cases} (H^{1+\tau}(\Omega))^2 & , \textit{if } d = 2 \ \textit{(with } \tau > 0 \textit{ small enough)} \\ (H^{\frac{3}{2}}(\Omega))^3 & , \textit{if } d = 3. \end{cases}$$

Let $\mathcal{G} \in L(H, H)$, $y_0 \in K$ and $f \in L^2(0, T; H)$. Then the problem

(15)
$$\begin{cases} \dfrac{dy(t)}{dt} + (\nu A + A_0 + \mathcal{G})y(t) + By(t) + \partial I_K(y(t)) \ni f(t), \ t \in (0, T) \\ y(0) = y_0 \end{cases}$$

admits at least one solution $y \in L^2(0, T; V) \cap C_w([0, T]; H)$. Moreover, $\frac{dy}{dt} \in L^2(0, T; V')$ for $d = 2$, $\frac{dy}{dt} \in L^{\frac{4}{3}}(0, T; V')$ for $d = 3$, and $y(t) \in K$, $t \in [0, T)$. The solution is unique if $d = 2$.

Thus we have global weak solutions, with the uniqueness property in the bidimensional case.

2.2. Proofs. Let $\mathcal{G} \in L(H, H)$ and let $A_0 : D(A) \to H$ given by relation (7). Denote by $\mathcal{A}$ the perturbed operator

$$\mathcal{A} = \nu A + A_0 + \mathcal{G} : D(A) \subset H \to H.$$

The operator $-\nu A$ is the generator of an analytical C_0-semigroup on H. From the fact that A_0 and $\mathcal{G}$ are perturbations bounded with respect to A, it results ([10], Th. 2.1, Ch. 3, p. 80) that $-\mathcal{A}$ is also the generator of an analytical C_0-semigroup on H . We assume that $\mathcal{A}$ is positive ([13]) (otherwise $\mathcal{A}$ will be substituted by $\xi I + \mathcal{A}$, with $\xi > 0$ large enough). The following relations hold ([13], Th. 1.15.3, p. 103):

$$D(\mathcal{A}^\theta) = [D(\mathcal{A}), H]_{1-\theta} = [D(A), H]_{1-\theta} = D(A^\theta),\ \theta \in (0,1).$$

Moreover, $|\mathcal{A}\cdot|$ defines a norm equivalent to $|A\cdot|$ on $D(A)$.

Hence, it is sufficient to prove that the theorems obtained by substituting $\nu A + A_0 + \mathcal{G}$ with νA in Theorems 2.1, 2.2 hold. The correspondents of Theorem 2.1 are Theorems 2.2, 2.3 proven in [8].

Let us now prove the correspondent of Theorem 2.2 for the problem

$$(16)\qquad \begin{cases} \dfrac{dy(t)}{dt} + \nu Ay(t) + By(t) + \partial I_K(y(t)) \ni f(t) \quad , \ t \in (0,T) \\ y(0) = y_0. \end{cases}$$

Because the Yosida approximation of ∂I_K has a more convenient form, we will start from the approximate problem involving B_N, $(\partial I_K)_{\frac{1}{j}}$, with $N, j \in \mathbb{N}^*$ and having more regular initial data y_0^j, f_j.

Here $B_N : V \to V'$, $N \in \mathbb{N}^*$ is the truncation of B,

$$B_N y = \begin{cases} By & , \text{if } \|y\| \le N \\ \left(\frac{N}{\|y\|}\right)^2 By & , \text{if } \|y\| > N \end{cases}$$

and $(\partial I_K)_{\frac{1}{j}} = jQ_K$ is the Yosida approximation $(\partial I_K)_\lambda$ for $\lambda = \frac{1}{j}$; the sequence $\frac{1}{j} \searrow 0$. Passing to the limit, first with j, then with N, we will obtain a weak solution for problem (16).

So, let $y_0 \in K = \overline{D(A) \cap K}^H$ and $f \in L^2(0,T;H)$. Then there exist two sequences $(y_0^j)_{j\in\mathbb{N}^*} \subset D(A) \cap K$, $(f_j)_{j\in\mathbb{N}^*} \subset W^{1,1}(0,T;H)$, such that $y_0^j \to y_0$ in H and $f_j \to f$ in $L^2(0,T;H)$.

Using (13) and applying [6], Prop. 4.5, p. 131 for the single valued maximal monotone operator $A \subset H \times H$, with $\overline{D(A)}^H = H$, we get

$$(17)\qquad (Aw, (\partial I_K)_\lambda(w)) \ge 0,\ \forall w \in D(A),\ \forall \lambda > 0.$$

Making use of [1], Th. 1.4, Th. 1.6, p. 214-216, in [8] we have proven the following proposition (see [8], Proposition 3.3):

Proposition 2.1. *Let $N \in \mathbb{N}^*$ be fixed. Assume that $\Phi \subset H \times H$ is a maximal monotone operator verifying hypotheses (h_2), (h_3). Let $y_0 \in D(A) \cap D(\Phi)$ and $f \in W^{1,1}(0,T;H)$. Then for all $\lambda > 0$, there exists a unique strong solution*

$$y_N^\lambda \in W^{1,\infty}(0,T;H) \cap L^\infty(0,T;D(A)) \cap C([0,T];V)$$

for the problem

$$\begin{cases} \dfrac{dy_N^\lambda}{dt}(t) + \nu A y_N^\lambda(t) + B_N y_N^\lambda(t) + \Phi_\lambda(y_N^\lambda(t)) = f(t), \quad a.e.\ t \in (0,T) \\ y_N^\lambda(0) = y_0. \end{cases} \tag{18}$$

Moreover, y_N^λ is right differentiable, $\frac{d^+}{dt} y_N^\lambda(t)$ is right continuous and

$$\frac{d^+}{dt} y_N^\lambda(t) + \nu A y_N^\lambda(t) + B_N y_N^\lambda(t) + \Phi_\lambda(y_N^\lambda(t)) = f(t),\ \forall t \in [0,T). \tag{19}$$

The operator ∂I_K satisfies the hypotheses in Proposition 2.1 and, assuming $j, N \in \mathbb{N}^*$ fixed, we know now that the problem

$$\begin{cases} \dfrac{dy_N^j(t)}{dt} + \nu A y_N^j(t) + B_N y_N^j(t) + (\partial I_K)_{\frac{1}{j}}(y_N^j(t)) = f_j(t),\ \text{a.e. } t \in (0,T) \\ y_N^j(0) = y_0^j \end{cases} \tag{20}$$

has a unique solution $y_N^j \in W^{1,\infty}(0,T;H) \cap L^\infty(0,T;D(A)) \cap C([0,T];V)$.

Multiplying equation (20) by $y_N^j(t)$, integrating on $(0,t)$ and using (2) and the monotony of $(\partial I_K)_{\frac{1}{j}}$, we infer

$$\frac{1}{2}|y_N^j(t)|^2 + \nu \int_0^t \|y_N^j(s)\|^2 ds \le C,\ \text{a.e. } t \in (0,T), \tag{21}$$

where C does not depend on N, j. On the other part, multiplying (20) with $(\partial I_K)_{\frac{1}{j}}(y_N^j(t)) = \nabla(I_K)_{\frac{1}{j}}(y_N^j(t))$, integrating on $(0,t)$ and using (17), (2), (3), (14), (21), we obtain

$$\frac{1}{2}|Q_K y_N^j(t)|^2 + \frac{j}{2}\int_0^t |Q_K y_N^j(s)|^2 ds \le \frac{1}{j}\left(C + \int_0^t |f_j(s)|^2 ds\right).$$

Dropping the term $\frac{1}{2}|Q_K y_N^j(t)|^2 \geq 0$, we get $\frac{j}{2}\int_0^t |Q_K y_N^j(s)|^2 ds \leq \frac{C}{j}$, which is

$$\int_0^t |(\partial I_K)_{\frac{1}{j}}(y_N^j(s))|^2 ds \leq C, \ \text{a.e. } t \in (0,T), \tag{22}$$

where C does not depend on j, N.

Relations (4), (5) and (21) imply

$$\int_0^T \|B_N y_N^j(t)\|_{V'}^{\frac{4}{d}} dt \leq C, \ d = 2,3. \tag{23}$$

From (22), (23) and from the continuity of the operator $A : V \to V'$, we infer

$$\int_0^T \left\| \frac{dy_N^j}{dt}(t) \right\|_{V'}^{\frac{4}{d}} dt \leq C, \ d = 2,3. \tag{24}$$

On a subsequence, denoted again by $(y_N^j)_j$, we have for $j \to \infty$,

$$y_N^j \rightharpoonup y_N \text{ in } L^2(0,T;V), \ Ay_N^j \rightharpoonup Ay_N \text{ in } L^2(0,T;V'),$$

$$\frac{dy_N^j}{dt} \rightharpoonup \frac{dy_N}{dt} \text{ in } L^{\frac{4}{d}}(0,T;V'),$$

(actually, $y_N^j \to y_N$ in $L^2(0,T;V) \cap C([0,T];H)$),

$$B_N y_N^j \rightharpoonup B_N y_N \text{ in } L^{\frac{4}{d}}(0,T;V'),$$

$(\partial I_K)_{\frac{1}{j}}(y_N^j) \rightharpoonup \eta_N$ in $L^2(0,T;H)$, with $\eta_N(t) \in \partial I_K(y_N(t))$ a.e. $t \in (0,T)$.

Consequently, the problem

$$\begin{cases} \dfrac{dy_N(t)}{dt} + \nu A y_N(t) + B_N y_N(t) + \partial I_K(y_N(t)) \ni f(t), & \text{a.e. } t \in (0,T) \\ y_N(0) = y_0 \end{cases} \tag{25}$$

has a unique solution

$$y_N \in L^2(0,T;V) \cap C([0,T];H), \ \text{with } B_N y_N, \ \frac{dy_N}{dt} \in L^{\frac{4}{d}}(0,T;V'), \ d = 2,3,$$

$$\eta_N \in L^2(0,T;H), \text{ where } \eta_N = f - \frac{dy_N}{dt} - \nu A y_N - B_N y_N.$$

Moreover, y_N satisfies the estimates

$$\frac{1}{2}|y_N(t)|^2 + \nu \int_0^t \|y_N(s)\|^2 ds + \int_0^t |\eta_N(s)|^2 ds \le C, \text{ a.e. } t \in (0,T)$$

$$\int_0^T \left(\|B_N y_N(t)\|_{V'}^{\frac{4}{d}} + \left\| \frac{dy_N}{dt}(t) \right\|_{V'}^{\frac{4}{d}} \right) dt \le C, \ d = 2,3,$$

where C is positive and independent of N.

Hence, on a subsequence, denoted again by $(y_N)_N$, we have for $N \to \infty$ that

$$y_N \rightharpoonup y \text{ in } L^2(0,T;V), \ y_N \rightharpoonup y \text{ weakly * in } L^\infty(0,T;H),$$

$$Ay_N \rightharpoonup Ay \text{ in } L^2(0,T;V'), \ \frac{dy_N}{dt} \rightharpoonup \frac{dy}{dt} \text{ in } L^{\frac{4}{d}}(0,T;V'),$$

$$y_N \to y \text{ in } L^2(0,T;H) \text{ (from Aubin's compactness theorem)},$$

$$B_N y_N \rightharpoonup By \text{ in } L^{\frac{4}{d}}(0,T;V'),$$

$$\eta_N \rightharpoonup \eta \text{ in } L^2(0,T;H), \text{ with } \eta(t) \in \partial I_K(y(t)) \text{ a.e. } t \in (0,T).$$

This concludes the existence part of the theorem.

If $d = 2$ the solution of (16) is unique. Indeed, considering two solutions $y_1, \ y_2$, it results that

$$\frac{1}{2}\frac{d}{dt}|y_1(t) - y_2(t)|^2 + \nu\|y_1(t) - y_2(t)\|^2 + b(y_1(t) - y_2(t), y_1(t), y_1(t) - y_2(t))$$

$+(\eta_1(t) - \eta_2(t), y_1(t) - y_2(t)) = 0$, with $\eta_i(t) \in \partial I_K(y_i(t))$ a.e. $t \in (0,T), \ i = 1,2$.

From the monotony of ∂I_K and the properties of b, we deduce

$$\begin{cases} \frac{d}{dt}|y_1(t) - y_2(t)|^2 \le C\|y_1(t)\|^2 |y_1(t) - y_2(t)|^2 \\ (y_1 - y_2)(0) = 0 \end{cases}$$

and Gronwall's lemma implies $y_1 \equiv y_2$. $\square$

3. Feedback stabilization for the Navier-Stokes equations. We apply the results in Section 2 to the feedback stabilization of the Navier-Stokes system. The examples are taken from [9], so we will only sketch the proofs. The first example introduces a feedback controller distributed on the entire domain and the stability result is global. The second example uses a finite dimensional feedback controller supported on a subdomain; the stability result is local.

3.1. Example 1. Consider the controlled system

$$\begin{cases} \dfrac{dy}{dt}(t) + \nu Ay(t) + By(t) = f_e + u(t), \quad t > 0 \\ y(0) = y_0, \end{cases} \tag{26}$$

where $y_0 \in V$, $f_e \in H$.

Let $y_e \in D(A)$ be a stationary solution for (26), i.e. y_e verifies

$$\nu Ay_e + By_e = f_e. \tag{27}$$

Let $K \subset H$ be a closed convex set, satisfying $0 \in K$ and (13). We look for a feedback controller u such that $y(t) - y_e \in K, \forall t > 0$ and $\lim_{t\to\infty} |y(t) - y_e| = 0$ exponentially. The method we apply is that used in [3].

Let $z = y - y_e$ and we take $u(t) \in -\theta z(t) - \partial I_K(z(t))$, with $\theta > 0$.

Then (26) becomes

$$\begin{cases} \dfrac{dz}{dt}(t) + \nu Az(t) + Bz(t) + A_0 z(t) + \theta z(t) + \partial I_K(z(t)) \ni 0, t > 0 \\ z(0) = z_0 = y_0 - y_e, \end{cases} \tag{28}$$

where the operator A_0 is introduced by (7), for y_e given by (27).

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded set, with a smooth boundary. Let $f_e \in H$ and $y_e \in D(A)$ solution to* (27). *Let $K \subset H$ verify the above conditions. Assume that $y_0 - y_e \in \overline{D(A) \cap K}^V$. Then, for $\theta > 0$ large enough, problem* (28) *has a unique solution*

$$z \in C([0,T]; H) \cap L^2(0,T; D(A)) \cap L^\infty(0,T; V),$$

$$\textit{with } \frac{dz}{dt} \in L^2(0,T; H), \textit{ for all } T > 0.$$

Moreover, $z(t) \in K$, $t > 0$ and there exists $\delta = \delta(\theta) > 0$ such that

$$\lim_{t\to\infty} |z(t)| e^{\delta t} = 0.$$

In order to prove stability we need a *global* solution for (28). The results we have for strong solutions state that, in dimension $d = 3$, we may get only *local* strong solutions. That is why the stability theorem in the three dimensional case is based on an existence result for weak solutions.

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^3$ be an open, bounded set, with a smooth boundary. Let $f_e \in H$ and $y_e \in D(A)$ subject to (27). Let $K \subset H$ verify the above conditions. Suppose that P_K carries bounded sets of H in bounded sets in $(H^{\frac{3}{2}}(\Omega))^3$ and that $y_0 - y_e \in K$. Then, for $\theta > 0$ large enough, problem (28) has at least one solution*

$$z \in L^2(0,T;V) \cap C_w([0,T];H), \text{ with } \frac{dz}{dt} \in L^{\frac{4}{3}}(0,T;V'), \text{ for all } T > 0.$$

Moreover, $z(t) \in K,\ t \geq 0$ and there exists $\delta = \delta(\theta) > 0$ such that

$$\lim_{t\to\infty} |z(t)|e^{\delta t} = 0.$$

Proof of Theorem 3.1. We take $f \equiv 0$, $\Phi = \partial I_K$ and $\mathcal{G} \in L(H,H)$, $\mathcal{G} = \theta I$ (where $I : H \to H$ is the identity application) in Theorem 2.1, with $\theta > 0$ a fixed arbitrary constant. We infer that for any $T > 0$, the system (28) has a unique solution z, from the required spaces, satisfying the invariance property $z(t) \in K,\ t > 0$.

In order to prove the stability, equation (28) is scalarly multiplied by z in H and this yields

$$\frac{1}{2}\frac{d}{dt}|z(t)|^2 + \nu\|z(t)\|^2 + \theta|z(t)|^2 \leq \frac{\nu}{2}\|z(t)\|^2 + \frac{C^2\|y_e\|^2}{2\nu}|z(t)|^2, \text{hence}$$

$$|z(t)|^2 \leq e^{-2\delta_1 t}|y_0 - y_e|^2, \text{ a.e. } t \geq 0,$$

where θ is chosen large enough, such that $\delta_1 = \theta - C^2\|y_e\|^2/(2\nu) > 0$. Then $\lim_{t\to\infty} |z(t)|e^{\delta t} = 0$ for $\delta \in (0, \delta_1)$. □

Proof of Theorem 3.2. Applying Theorem 2.2 for $f \equiv 0$ and for $\mathcal{G} \in L(H,H)$, $\mathcal{G} = \theta I$, with $\theta > 0$ a fixed arbitrary constant, it results that for any $T > 0$, (28) admits at least one weak solution z verifying the invariance property.

For the stability part, we start from the approximative problem

$$\text{(29)} \qquad \begin{cases} \dfrac{dz_N}{dt}(t) + \nu A z_N(t) + B_N z_N(t) + A_0 z_N(t) \\ \qquad\qquad + \theta z_N(t) + \partial I_K(z_N(t)) \ni 0,\ t > 0 \\ z_N(0) = z_0 = y_0 - y_e, \end{cases}$$

which for any $T > 0$ has a unique solution

$$z_N \in L^2(0,T;V) \cap C([0,T];H), \quad \frac{dz_N}{dt} \in L^{\frac{4}{3}}(0,T;V'), \text{ with}$$

$$z_N \rightharpoonup z \text{ in } L^2(0,T;V), \ z_N \rightharpoonup z \text{ weakly * in } L^\infty(0,T;H),$$

$$z_N \to z \text{ in } L^2(0,T;H).$$

We scalarly multiply (29) by $z_N(t)$ in H and we deduce

$$\frac{1}{2}\frac{d}{dt}|z_N(t)|^2 + \nu\|z_N(t)\|^2 + \theta|z_N(t)|^2 \le \frac{\nu}{2}\|z_N(t)\|^2 + \frac{C^2}{2\nu}\|y_e\|^2_{\frac{3}{2}}|z_N(t)|^2,$$

hence

$$|z_N(t)|^2 \le e^{-2\delta_1 t}|z_0|^2, \text{ a.e. } t \ge 0, \text{ for all } N \in \mathbb{N}^*,$$

where θ is chosen large enough, such that $\delta_1 = \theta - C^2\|y_e\|^2_{3/2}/(2\nu) > 0$. Passing to the limit with $N \to \infty$, we obtain $|z(t)|^2 \le e^{-2\delta_1 t}|z_0|^2$, a.e. $t \ge 0$. Taking $\delta \in (0,\delta_1)$, it results that $\lim_{t\to\infty} |z(t)|e^{\delta t} = 0$. □

3.2. Example 2. Let $K = \text{linspan}\,\{e_i;\ i = \overline{1,r}\} \subset H$, where $e_1, e_2, \ldots, e_r \in H$, with $r \in \mathbb{N}^*$ are eigenfunctions for the Stokes operator A. The set K is convex, closed in H, $0 \in K$ and (13), (14) are satisfied.

Denote by $(\lambda_i)_{i=\overline{1,r}}$ the eigenvalues of the Stokes operator corresponding to the eigenfunctions $(e_i)_{i=\overline{1,r}}$. Then $Ae_i = \lambda_i e_i,\ i = \overline{1,r}$.

Let us consider the controlled system

$$\begin{cases} \dfrac{dy}{dt}(t) + \nu Ay(t) + By(t) = f_e + U(t), & t > 0 \\ y(0) = y_0, \end{cases} \tag{30}$$

where $y_0 \in V$, $f_e \in H$. Let $y_e \in D(A)$ be a stationary solution for (30), i.e. y_e verifies (27). We look for a feedback controller U such that $y(t) - y_e \in K$, $t > 0$ and $\lim_{t\to\infty} |y(t) - y_e| = 0$ exponentially.

Let $z = y - y_e$. We choose $U(t) \in PmR(P_K z(t)) - \partial I_K(z(t))$, where P is the Leray projection; if $\omega \subset\subset \Omega$, with a smooth boundary $\partial\omega$, then $R : K \to (L^2(\omega))^d$ is a linear continuous operator, and the application

$$m : (L^2(\omega))^d \to (L^2(\Omega))^d, \ (mu)(x) = \begin{cases} u(x) & , \ x \in \omega \\ 0 & , \ x \in \Omega \setminus \omega \end{cases}$$

extends the functions of the subdomain $(L^2(\omega))^d$ with 0 to the whole space.

Problem (30) becomes

(31)
$$\begin{cases} \dfrac{dz}{dt}(t) + (\nu A + A_0 + B)z(t) + \partial I_K(z(t)) - PmR(P_K z(t)) \ni 0,\ t > 0 \\ z(0) = y_0 - y_e, \end{cases}$$

where the operator A_0 was defined by (7) for y_e given by (27).

The following result takes place,

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be an open, bounded set, with a smooth boundary $\partial\Omega$ and let $\omega \subset\subset \Omega$, with a smooth boundary $\partial\omega$. Let $f_e \in H$, $y_e \in D(A)$ satisfying* (27) *and $K \subset H$ defined above. Then for all $\delta > 0$ there exist a linear continuous operator $R : K \to (L^2(\omega))^d$ and a constant $\rho > 0$ such that if $y_0 - y_e \in K$, with $|y_0 - y_e| < \rho$, then the problem* (31) *has at least one solution $z : [0, \infty) \to V$, having the property that for all $T > 0$,*

$$z \in L^2(0, T; V) \cap C_w([0, T]; H) \text{ and } \frac{dz}{dt} \in \begin{cases} L^2(0, T; V') & , \text{ if } d = 2 \\ L^{\frac{4}{3}}(0, T; V') & , \text{ if } d = 3 \end{cases};$$

moreover, $z(t) \in K$, $t \geq 0$ and $\lim_{t\to\infty} |z(t)|e^{\delta t} = 0$.

Proof of Theorem 3.3. From Theorem 2.2, it results that for any linear continuous operator $R : K \to (L^2(\omega))^d$ problem (31) has at least one weak solution for all $T > 0$; the solution stays in K. In order to be more precise how the stabilizing operator R appears, we introduce the problem

(32)
$$\begin{cases} \dfrac{dz}{dt}(t) + \nu Az(t) + A_0 z(t) + Bz(t) + \partial I_K(z(t)) - Pm\bar{u}(t) \ni 0,\ t > 0 \\ z(0) = y_0 - y_e. \end{cases}$$

We multiply scalarly equation (32) by e_i, $i = \overline{1, r}$ and obtain the finite dimensional differential system

$$\begin{cases} w'(t) + \mathcal{L}w(t) + \mathcal{B}w(t) = \mathcal{H}\bar{u}(t),\ t > 0 \\ w(0) = w_0, \end{cases} \tag{33}$$

where $w(t) = ((z(t), e_i))_{i=\overline{1,r}} \in \mathbb{R}^r$, $w_0 = ((y_0 - y_e, e_i))_{i=\overline{1,r}} \in \mathbb{R}^r$. The linear space K is identified with $\mathbb{R}^r$ by means of the basis $(e_i)_{i=\overline{1,r}}$. The linear operators $\mathcal{L} : \mathbb{R}^r \to \mathbb{R}^r$, $\mathcal{H} : (L^2(\omega))^d \to \mathbb{R}^r$ and the quadratic form $\mathcal{B} : \mathbb{R}^r \to \mathbb{R}^r$ are given by

$$\mathcal{L}v = \left(\nu\lambda_i v_i + \sum_{j=1}^{r} g_{ij} v_j \right)_{i=\overline{1,r}}, \ \mathcal{B}v = \left(\sum_{j,k=1}^{r} c_{ijk} v_j v_k \right)_{i=\overline{1,r}},$$

where $v = (v_i)_{i=\overline{1,r}} \in \mathbb{R}^r$ and $g_{ij} = b(e_j, y_e, e_i) + b(y_e, e_j, e_i) \in \mathbb{R}$, $c_{ijk} = b(e_j, e_k, e_i) \in \mathbb{R},\ i, j, k = \overline{1, r}$;

$$\mathcal{H}u = ((mu, e_i))_{i=\overline{1,r}} \text{ for } u \in (L^2(\omega))^d.$$

Let us linearize the system (33) in 0,

$$\text{(34)} \qquad \begin{cases} w'(t) + \mathcal{L}w(t) = \mathcal{H}\bar{u}(t),\ t > 0 \\ w(0) = w_0, \end{cases}$$

and write the adjoint system

$$\text{(35)} \qquad p'(t) - \mathcal{L}^* p(t) = 0,\ t > 0,$$

where $\mathcal{L}^*$ is the adjoint of $\mathcal{L}$. From the following unique continuation property (see [5]),

(36) for p solution of the adjoint system (35),

$$\mathcal{H}^* p(t) = 0,\ t > 0 \Rightarrow p(t) = 0,\ t > 0$$

($\mathcal{H}^*$ is the adjoint of $\mathcal{H}$) we get ([14]) that the system (34) is exactly controllable, equivalently, completely stabilizable. Then for every $\xi \in \mathbb{R}$ (and let us fix a $\xi > 0$) there exists a linear continuous feedback $R : \mathbb{R}^r \to (L^2(\omega))^d$ such that the operator norm

$$\text{(37)} \qquad \|e^{-(\mathcal{L}-\mathcal{H}R)t}\|_{\mathcal{O}} \le Ce^{-\xi t},\ t \ge 0.$$

For the controller $\bar{u} = Rw$, the linearized system (34) becomes

$$\text{(38)} \qquad \begin{cases} w'(t) + (\mathcal{L} - \mathcal{H}R)w(t) = 0,\ t > 0 \\ w(0) = w_0, \end{cases}$$

and it is asimptotically stable.

It can be proven that the feedback $\bar{u} = Rw$ also stabilizes the nonlinear system (33). Indeed, by the variation of constants formula, the solution of the system

$$\left\{ \begin{array}{l} w'(t) + (\mathcal{L} - \mathcal{H}R)w(t) + \mathcal{B}w(t) = 0,\ t > 0 \\ w(0) = w_0, \end{array} \right. \tag{39}$$

is given by

$$w(t) = e^{-(\mathcal{L}-\mathcal{H}R)t}w_0 - \int_0^t e^{-(\mathcal{L}-\mathcal{H}R)(t-s)}\mathcal{B}w(s)ds,\ t \geq 0.$$

The application $\mathcal{B}$ is a quadratic form, hence there exists a constant $\gamma_0 > 0$ such that $|\mathcal{B}v| \leq \gamma_0|v|^2,\ \forall v \in \mathbb{R}^r$ ($|\cdot|$ denotes here the euclidian norm of $\mathbb{R}^r$). This inequality together with (37) implies that

$$e^{\xi t}|w(t)| \leq C|w_0| + C\gamma_0 \int_0^t e^{\xi s}|w(s)|^2 ds$$

and, by Gronwall's lemma,

$$|w(t)| \leq C|w_0|e^{-\xi t + \int_0^t C\gamma_0|w(s)|ds},\ t > 0. \tag{40}$$

Let now $\rho \in \left(0, \frac{\xi}{C\gamma_0}\right)$ and let w_0 such that $|w_0| < \frac{\rho}{C}$; assume, without loss of generality, that $C \geq 1$, such that $\frac{\rho}{C} \leq \rho$. By reducing *ad absurdum* we prove that

$$|w(t)| \leq \rho,\ \forall t \geq 0. \tag{41}$$

From (40) and (41) we get

$$|w(t)| \leq C|w_0|e^{(-\xi + C\gamma_0\rho)t} = C|w_0|e^{-\delta_1 t},\ \forall t \geq 0,\ \text{where } \delta_1 = \xi - C\gamma_0\rho > 0.$$

Then $\lim_{t\to\infty}|w(t)|e^{\delta t} = 0$ for any $\delta \in (0, \delta_1)$. We observe that δ is arbitrary in $(0, \xi)$, if ρ is small enough. Because ξ is also arbitrary, it results that δ is arbitrary in $(0, \infty)$.

Hence, for the controller $\bar{u} = Rw$ the solution w of problem (33) is defined on $[0, \infty)$ and $\lim_{t\to\infty}|w(t)|e^{\delta t} = 0$. By the invariance property of z, we have that for the controller $\bar{u} = RP_K z$ the solution z of problem (32) is, also, defined on $[0, \infty)$ and $\lim_{t\to\infty}|z(t)|e^{\delta t} = 0$. $\square$

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