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ON A NONLINEAR DIFFERENTIAL PROBLEM WITH SECOND-ORDER DIFFERENCES*

BY

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Abstract. We investigate the existence, uniqueness and asymptotic behaviour of the strong and weak solutions for a nonlinear differential system with second-order differences, subject to an extreme condition and initial data. A generalization of Minty's Theorem for the characterization of maximal monotone operators is also presented.

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Key words: differential system, extreme condition, maximal monotone operator, Cauchy problem, strong solution, weak solution.

1. Introduction. Let H be a real Hilbert space with the scalar product $\langle \cdot, \cdot \rangle$ and the associated norm $\| \cdot \|$. We consider the nonlinear differential system with second-order differences

$$\text{(S)} \qquad \begin{cases} u_j'(t) + \dfrac{v_{j+1}(t) - 2v_j(t) + v_{j-1}(t)}{h_j^2} + c_j A(u_j(t)) \ni f_j(t) \\ v_j'(t) - \dfrac{u_{j+1}(t) - 2u_j(t) + u_{j-1}(t)}{\overline{h}_j^2} + d_j B(v_j(t)) \ni g_j(t), \end{cases}$$

$$0 < t < T, \ \ j = \overline{1, N}, \ \text{ in } \ H,$$

with the extreme condition

$$\text{(EC)} \qquad \begin{pmatrix} u_1(t) - u_0(t) \\ -u_{N+1}(t) + u_N(t) \\ -v_1(t) + v_0(t) \\ v_{N+1}(t) - v_N(t) \end{pmatrix} \in \Lambda \begin{pmatrix} v_1(t) \\ v_N(t) \\ u_1(t) \\ u_N(t) \end{pmatrix}, \ \ 0 < t < T,$$

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and the initial data

$$\text{(ID)} \qquad u_j(0) = u_{j0}, \;\; v_j(0) = v_{j0}, \;\; j = \overline{1,N},$$

where $N \in I\!N$, $N \geq 2$, $T > 0$, c_j, d_j, h_j, $\overline{h}_j > 0$, for all $j = \overline{1,N}$, Λ and A, B are operators in H^4 and H, respectively, which satisfy some assumptions.

The above problem is a discrete version with respect to x (with $H = \mathbb{R}^n$) of the nonlinear system

$$\text{(S)}_0 \qquad \left\{ \begin{array}{l} \dfrac{\partial u}{\partial t}(t,x) + \dfrac{\partial^2 v}{\partial x^2}(t,x) + c(x)A(u(t,x)) \ni f(t,x) \\ \dfrac{\partial v}{\partial t}(t,x) - \dfrac{\partial^2 u}{\partial x^2}(t,x) + d(x)B(v(t,x)) \ni g(t,x), \end{array} \right.$$

$$0 < x < 1, \;\; 0 < t < T, \;\; \text{in } \mathbb{R},$$

subject to the boundary condition

$$\text{(BC)}_0 \qquad \left(\begin{array}{c} \dfrac{\partial u}{\partial x}(t,0) \\ -\dfrac{\partial u}{\partial x}(t,1) \\ -\dfrac{\partial v}{\partial x}(t,0) \\ \dfrac{\partial v}{\partial x}(t,1) \end{array} \right) \in \Lambda \left(\begin{array}{c} v(t,0) \\ v(t,1) \\ u(t,0) \\ u(t,1) \end{array} \right), \quad 0 < t < T$$

and the initial data

$$\text{(IC)}_0 \qquad u(0,x) = u_0(x), \;\; v(0,x) = v_0(x), \;\; 0 < x < 1.$$

This problem and some generalizations of it (with higher-order partial derivatives, time dependent coefficients in the system $(\text{S})_0$ or extra functions in $(\text{BC})_0$) have been studied in [12], [16]. The condition $(\text{BC})_0$ is a general one. By making suitable choices of Λ we deduce many classical boundary conditions. For $\Lambda = \partial l$, where $l : \mathbb{R}^4 \to (-\infty, +\infty]$, we obtain the following exemples:

1) If $l((u,v,x,y)^T) = \left\{ \begin{array}{ll} 0, & \text{if } u = a, \;\; v = b, \;\; x = c \text{ and } y = d, \\ +\infty, & \text{in rest,} \end{array} \right.$

then $(\text{BC})_0$ becomes $v(t,0) = a$, $v(t,1) = b$, $u(t,0) = c$, $u(t,1) = d$.

2) If $l((u,v,x,y)^T) = \begin{cases} au - bv, & \text{if } x = c \text{ and } y = d, \\ +\infty, & \text{in rest,} \end{cases}$

then $(BC)_0$ gives us $u(t,0) = c$, $u(t,1) = d$, $\frac{\partial u}{\partial x}(t,0) = a$, $\frac{\partial u}{\partial x}(t,1) = b$.

3) If $l((u,v,x,y)^T) = \begin{cases} dy, & \text{if } u = a,\ v = b,\ x = c, \\ +\infty, & \text{in rest,} \end{cases}$

then we obtain $v(t,0) = a$, $v(t,1) = b$, $u(t,0) = c$, $\frac{\partial v}{\partial x}(t,1) = d$.

4) If $l((u,v,x,y)^T) = \begin{cases} au - bv - cx, & \text{if } y = d, \\ +\infty, & \text{in rest,} \end{cases}$

then $(BC)_0$ becomes $u(t,1) = d$, $\frac{\partial u}{\partial x}(t,0) = a$, $\frac{\partial u}{\partial x}(t,1) = b$, $\frac{\partial v}{\partial x}(t,0) = c$.

5) If $l((u,v,x,y)^T) = au - bv - cx + dy$, then $(BC)_0$ gives us $\frac{\partial u}{\partial x}(t,0) = a$, $\frac{\partial u}{\partial x}(t,1) = b$, $\frac{\partial v}{\partial x}(t,0) = c$, $\frac{\partial v}{\partial x}(t,1) = d$.

Some nonlinear differential problems with first-order differences have been investigated in the papers [13]-[15]. For other differential and difference equations in abstract spaces we refer the reader to the papers [6], [11], [17] and the monographs [1]-[3], [8]-[10].

The paper is organized as follows: in Section 2 we shall introduce and study the operators associated to our problem. In Section 3 we shall investigate the existence, uniqueness and asymptotic behaviour of the solutions to the problem (S)+(EC)+(ID), (see the main results of the paper: Theorem 4 and Theorem 6). In Section 4 we shall finally present a generalization of Minty's Theorem, which gives a characterization of maximal monotone operators on Hilbert spaces. In the proofs of our theorems we shall use some results related to monotone operators and nonlinear evolution equations of monotone type in Hilbert spaces (see the monographs [5], [7], [9], [10]).

The assumptions that we shall use in the sequel are

(H1) a) The operators $A : D(A) \subseteq H \to H$, $B : D(B) \subseteq H \to H$ are maximal monotone, possibly multivalued.

b) $D(A) = D(B) = H$.

c) There exist $a_0 > 0$, $b_0 > 0$ such that

i) for all u_1, $u_2 \in D(A)$, $\gamma_1 \in A(u_1)$, $\gamma_2 \in A(u_2)$ we have
$\langle \gamma_1 - \gamma_2, u_1 - u_2 \rangle \geq a_0 \|u_1 - u_2\|^2$;

ii) for all v_1, $v_2 \in D(B)$, $\delta_1 \in B(v_1)$, $\delta_2 \in B(v_2)$ we have
$\langle \delta_1 - \delta_2, v_1 - v_2 \rangle \geq b_0 \|v_1 - v_2\|^2$.

(H2) a) The operator $\Lambda : D(\Lambda) \subseteq H^4 \to H^4$ is maximal monotone, possibly multivalued.

b) $D(\Lambda) = H^4$.

(H3) a) $(\text{int} D(\Lambda)) \cap (D(B)^2 \times D(A)^2) \neq \emptyset$.

b) $D(\Lambda)\cap[(\text{int}D(B))^2\times(\text{int}D(A))^2]\neq\emptyset$.
c) $0\in\text{int}(D(\Lambda)-(D(B)^2\times D(A)^2))$.
(H4) The constants $h_j>0$, $\overline{h}_j>0$, for all $j=\overline{1,N}$.
(H5) The constants $c_j>0$, $d_j>0$, for all $j=\overline{1,N}$.

2. Maximal monotone operators associated to (S)+(EC)+(ID). In this section we introduce and study some operators associated to our problem. We consider the Hilbert space $X=H^{2N}=\{(u_1,u_2,\dots,u_N,v_1,v_2,\dots,v_N)^T;\ u_j,\ v_j\in H,\ j=\overline{1,N}\}$ with the scalar product

$$\langle(u_1,\dots,u_N,v_1,\dots,v_N)^T,(\overline{u}_1,\dots,\overline{u}_N,\overline{v}_1,\dots,\overline{v}_N)^T\rangle_X$$

$$=\sum_{j=1}^{N}h_j^2\langle u_j,\overline{u}_j\rangle+\sum_{j=1}^{N}\overline{h}_j^2\langle v_j,\overline{v}_j\rangle$$

and the corresponding norm $\|\cdot\|_X$.

We introduce the operator $\mathcal{A}_1:D(\mathcal{A}_1)=X\to X$,

$$\mathcal{A}_1((u_1,u_2,\dots,u_N,v_1,v_2,\dots,v_N)^T)$$
$$=\left(\frac{v_2-2v_1}{h_1^2},\frac{v_3-2v_2+v_1}{h_2^2},\dots,\frac{v_N-2v_{N-1}+v_{N-2}}{h_{N-1}^2},\right.$$
$$\frac{-2v_N+v_{N-1}}{h_N^2},-\frac{u_2-2u_1}{\overline{h}_1^2},-\frac{u_3-2u_2+u_1}{\overline{h}_2^2},\dots,$$
$$\left.-\frac{u_N-2u_{N-1}+u_{N-2}}{\overline{h}_{N-1}^2},-\frac{-2u_N+u_{N-1}}{\overline{h}_N^2}\right)^T,$$

and the operator $\mathcal{A}_2:D(\mathcal{A}_2)\subseteq X\to X$, $D(\mathcal{A}_2)=\{(u_1,\dots,u_N,v_1,\dots,v_N)^T,(v_1,v_N,u_1,u_N)^T\in D(\Lambda)\}$,

$$\mathcal{A}_2((u_1,\dots,u_N,v_1,\dots,v_N)^T)$$
$$=\left\{\left(\frac{v_0}{h_1^2},0,\dots,0,\frac{v_{N+1}}{h_N^2},-\frac{u_0}{\overline{h}_1^2},0,\dots,0,-\frac{u_{N+1}}{\overline{h}_N^2}\right)^T,\right.$$
$$\left.(u_1-u_0,-u_{N+1}+u_N,-v_1+v_0,v_{N+1}-v_N)^T\in\Lambda((v_1,v_N,u_1,u_N)^T)\right\}.$$

Lemma 1. *If the assumption* (H4) *hold, then the operator* $\mathcal{A}_1$ *is maximal monotone in* X.

Proof. The operator $\mathcal{A}_1$ is everywhere defined, single-valued, linear and continuous. Besides $\mathcal{A}_1$ is monotone, because

$$\begin{aligned}\langle \mathcal{A}_1(U), U\rangle_X &= \sum_{j=1}^{N} h_j^2 \langle \frac{v_{j+1}-2v_j+v_{j-1}}{h_j^2}, u_j\rangle \\ &+ \sum_{j=1}^{N} \overline{h}_j^2 \langle \frac{-u_{j+1}+2u_j-u_{j-1}}{\overline{h}_j^2}, v_j\rangle \\ &= \sum_{j=1}^{N-1}\langle v_{j+1},u_j\rangle - 2\sum_{j=1}^{N}\langle v_j,u_j\rangle + \sum_{j=2}^{N}\langle v_{j-1},u_j\rangle \\ &- \sum_{j=1}^{N-1}\langle u_{j+1},v_j\rangle + 2\sum_{j=1}^{N}\langle u_j,v_j\rangle - \sum_{j=2}^{N}\langle u_{j-1},v_j\rangle = 0,\end{aligned}$$

where $u_0 = v_0 = 0$, $u_{N+1} = v_{N+1} = 0$, for all $U = (u_1, \dots, u_N, v_1, \dots, v_N)^T \in X$.

By using [[7], Proposition 2.4], we deduce that $\mathcal{A}_1$ is maximal monotone.□

Now we recall the following theorem, a generalization of Minty's Theorem to product spaces with weights, that we shall use in Lemma 2. For the proof see [Theorem 7, from [15]].

Theorem 1. *Let H be a Hilbert space with the scalar product $\langle\cdot,\cdot\rangle$, $p \in \mathbb{N}$, $a_i > 0$, for all $i = \overline{1,p}$, and $H^p = \underbrace{H \times \cdots \times H}_{p \text{ times}}$ the Hilbert space with the scalar product $\langle (u_1,\dots,u_p)^T, (v_1,\dots,v_p)^T\rangle_{H^p} = \sum_{i=1}^{p} a_i\langle u_i, v_i\rangle$. Let $A : D(A) \subseteq H^p \to H^p$ be a monotone operator, possibly multivalued. The operator A is maximal monotone if and only if for all $C = (\lambda_1,\dots,\lambda_p)^T \in \mathbb{R}^p$, $\lambda_i > 0$, for all $i = \overline{1,p}$ (equivalently there exists $C \in \mathbb{R}^p$, $C = (\lambda_1,\dots,\lambda_p)^T$, $\lambda_i > 0$, for all $i = \overline{1,p}$ such that) $R(CI + A) = H^p$. (We denote here by CI the operator $CI : H^p \to H^p$, $CI((u_1,\dots,u_p)^T) = (\lambda_1 u_1,\dots,\lambda_p u_p)^T$).*

Lemma 2. *If the assumptions* (H2)a *and* (H4) *hold, then the operator $\mathcal{A}_2$ is maximal monotone in X.*

Proof. By (H2)a, $D(\Lambda) \neq \emptyset$, and so $D(\mathcal{A}_2) \neq \emptyset$. We firstly observe that the operator $\mathcal{A}_2$ is monotone. Indeed we have

$$\langle Z - \overline{Z}, U - \overline{U}\rangle_X = \langle v_0 - \overline{v}_0, u_1 - \overline{u}_1\rangle + \langle v_{N+1} - \overline{v}_{N+1}, u_N - \overline{u}_N\rangle$$

$$+\langle -u_0+\overline{u}_0, v_1-\overline{v}_1\rangle+\langle -u_{N+1}+\overline{u}_{N+1}, v_N-\overline{v}_N\rangle$$

$$=\langle u_1-\overline{u}_1, v_0-v_1-\overline{v}_0+\overline{v}_1\rangle+\langle u_N-\overline{u}_N, v_{N+1}-v_N-\overline{v}_{N+1}+\overline{v}_N\rangle$$

$$+\langle v_1-\overline{v}_1, -u_0+u_1+\overline{u}_0-\overline{u}_1\rangle+\langle v_N-\overline{v}_N, -u_{N+1}+u_N+\overline{u}_{N+1}-\overline{u}_N\rangle$$

$$=\Big\langle \begin{pmatrix} u_1-u_0 \\ -u_{N+1}+u_N \\ -v_1+v_0 \\ v_{N+1}-v_N \end{pmatrix}-\begin{pmatrix} \overline{u}_1-\overline{u}_0 \\ -\overline{u}_{N+1}+\overline{u}_N \\ -\overline{v}_1+\overline{v}_0 \\ \overline{v}_{N+1}-\overline{v}_N \end{pmatrix}, \begin{pmatrix} v_1 \\ v_N \\ u_1 \\ u_N \end{pmatrix}-\begin{pmatrix} \overline{v}_1 \\ \overline{v}_N \\ \overline{u}_1 \\ \overline{u}_N \end{pmatrix}\Big\rangle_{H^4}\geq 0,$$

with $(v_1, v_N, u_1, u_N)^T \in D(\Lambda)$, $(\overline{v}_1, \overline{v}_N, \overline{u}_1, \overline{u}_N)^T \in D(\Lambda)$, $(u_1-u_0, -u_{N+1}+u_N, -v_1+v_0, v_{N+1}-v_N)^T \in \Lambda((v_1, v_N, u_1, u_N)^T)$, $(\overline{u}_1-\overline{u}_0, -\overline{u}_{N+1}+\overline{u}_N, -\overline{v}_1+\overline{v}_0, \overline{v}_{N+1}-\overline{v}_N)^T \in \Lambda((\overline{v}_1, \overline{v}_N, \overline{u}_1, \overline{u}_N)^T)$, for all $U=(u_1, \ldots, u_N, v_1, \ldots, v_N)^T$, $\overline{U}=(\overline{u}_1, \ldots, \overline{u}_N, \overline{v}_1, \ldots, \overline{v}_N)^T \in D(\mathcal{A}_2)$, $Z \in \mathcal{A}_2(U)$, $\overline{Z} \in \mathcal{A}_2(\overline{U})$.

In what follows we shall use Theorem 1 to prove that $\mathcal{A}_2$ is a maximal monotone operator. We shall show that there exists an element $C=(c_i)_{i=\overline{1,2N}} \in \mathbb{R}^{2N}$, $c_i>0$, for all $i=\overline{1,2N}$, such that $R(CI+\mathcal{A}_2)=X$. We consider $C=(\frac{1}{h_1^2}, \cdots, \frac{1}{h_N^2}, \frac{1}{\overline{h}_1^2}, \cdots, \frac{1}{\overline{h}_N^2})^T \in \mathbb{R}^{2N}$ and we shall prove that for any $Y=(x_1, \ldots, x_N, y_1, \ldots, y_N)^T \in X$, the equation

$$(CI+\mathcal{A}_2)(U) \ni Y \tag{1}$$

has a solution $U=(u_1, \ldots, u_N, v_1, \ldots, v_N)^T \in D(\mathcal{A}_2)$.

Equation (1) is equivalent to

$$\begin{cases} u_1+v_0=h_1^2x_1, \;\; u_j=h_j^2x_j, \;\; j=\overline{2,N-1}, \;\; u_N+v_{N+1}=h_N^2x_N, \\ v_1-u_0=\overline{h}_1^2y_1, \;\; v_j=\overline{h}_j^2y_j, \;\; j=\overline{2,N-1}, \;\; v_N-u_{N+1}=\overline{h}_N^2y_N, \end{cases} \tag{2}$$

if $N \geq 3$ or only the above first and last relations if $N=2$, with $(u_1-u_0, -u_{N+1}+u_N, -v_1+v_0, v_{N+1}-v_N)^T \in \Lambda((v_1, v_N, u_1, u_N)^T)$.

From the above relations we obtain

$$\begin{pmatrix} v_1 \\ v_N \\ u_1 \\ u_N \end{pmatrix}+\begin{pmatrix} -u_0 \\ -u_{N+1} \\ v_0 \\ v_{N+1} \end{pmatrix}=\begin{pmatrix} \overline{h}_1^2y_1 \\ \overline{h}_N^2y_N \\ h_1^2x_1 \\ h_N^2x_N \end{pmatrix}. \tag{3}$$

The equation (3) is equivalent to

$$(4)\qquad \begin{pmatrix} v_1 \\ v_N \\ u_1 \\ u_N \end{pmatrix} + \Lambda \begin{pmatrix} v_1 \\ v_N \\ u_1 \\ u_N \end{pmatrix} + \mathcal{T} \begin{pmatrix} v_1 \\ v_N \\ u_1 \\ u_N \end{pmatrix} \ni \begin{pmatrix} \overline{h}_1^2 y_1 \\ \overline{h}_N^2 y_N \\ h_1^2 x_1 \\ h_N^2 x_N \end{pmatrix},$$

where $\mathcal{T} : H^4 \to H^4$ is the operator defined by $\mathcal{T}((v_1, v_N, u_1, u_N)^T) = (-u_1, -u_N, v_1, v_N)^T$. The operator $\mathcal{T}$ is single-valued, everywhere defined, continuous and monotone; indeed we have

$$\langle \mathcal{T}((v_1, v_N, u_1, u_N)^T), (v_1, v_N, u_1, u_N)^T \rangle_{H^4}$$

$$= -\langle u_1, v_1 \rangle - \langle u_N, v_N \rangle + \langle u_1, v_1 \rangle + \langle u_n, v_N \rangle = 0.$$

Because Λ is maximal monotone, we deduce, by [[5], Corollary 1.3, Chapter II] that $\Lambda + \mathcal{T}$ is a maximal monotone operator. So $R(I + \Lambda + \mathcal{T}) = H^4$, that is the equation (4) has a solution $(v_1, v_N, u_1, u_N)^T \in D(\Lambda)$.

Using the relations (2) we have $u_j = h_j^2 x_j$, $v_j = \overline{h}_j^2 y_j$, $j = \overline{2, N-1}$, and with $(v_1, v_N, u_1, u_N)^T \in D(\Lambda)$ defined as above, we deduce that $U = (u_1, \ldots, u_N, v_1, \ldots, v_N)^T \in D(\mathcal{A}_2)$ is solution of equation (1). □

Now we define the operator $\mathcal{A} : D(\mathcal{A}) = D(\mathcal{A}_2) \subseteq X \to X$, $\mathcal{A}(U) = \mathcal{A}_1(U) + \mathcal{A}_2(U)$.

Lemma 3. *If the assumptions* (H2)a *and* (H4) *hold, then the operator $\mathcal{A}$ is maximal monotone in X.*

Proof. Because $\mathcal{A}_1$ is single-valued, everywhere defined, monotone and continuous (by Lemma 1), and $\mathcal{A}_2$ is maximal monotone (by Lemma 2), we deduce that $\mathcal{A}$ is maximal monotone in X. □

We now define the operator $\mathcal{B} : D(\mathcal{B}) \subseteq X \to X$, $D(\mathcal{B}) = D(A)^N \times D(B)^N$,

$$\mathcal{B}((u_1, u_2, \ldots, u_N, v_1, v_2, \ldots, v_N)^T) = \{(c_1\gamma_1, c_2\gamma_2, \ldots, c_N\gamma_N, d_1\delta_1, d_2\delta_2, \ldots, d_N\delta_N)^T,\ \gamma_i \in A(u_i),\ \delta_i \in B(v_i),\ i = \overline{1, N}\}.$$

Lemma 4. *If the assumptions* (H1)a, (H4) *and* (H5) *hold, then the operator $\mathcal{B}$ is maximal monotone in X.*

For the proof of Lemma 4 see [15].

Theorem 2. *If the assumptions* (H1)a, (H2)a, [(H2)b *or* (H3)a *or* (H3)c], (H4) *and* (H5) *hold, then the operator* $\mathcal{A}+\mathcal{B}$ *is maximal monotone.*

Proof. The operator $\mathcal{A}+\mathcal{B} : D(\mathcal{A})\cap D(\mathcal{B}) \subseteq X \to X$ has $D(\mathcal{A}+\mathcal{B}) = D(\mathcal{A})\cap D(\mathcal{B}) = \{(u_1,\dots,u_N,v_1,\dots,v_N)^T \in X,\ u_j \in D(A),\ v_j \in D(B),\ j=\overline{2,N-1},\ (v_1,v_N,u_1,u_N)^T \in D(\Lambda)\cap (D(B)^2\times D(A)^2)\}$. By the assumptions [(H2)b or (H3)a or (H3)c] we deduce that $D(\mathcal{A}+\mathcal{B})\neq\emptyset$.

First, we suppose that the assumption (H2)b holds. In this case $D(\Lambda) = H^4$, and so $D(\mathcal{A}) = X$ and $D(\mathcal{A}+\mathcal{B}) = D(\mathcal{B})$. Because $\mathcal{A}$ and $\mathcal{B}$ are monotone, then $\mathcal{A}+\mathcal{B}$ is also monotone. To prove that $\mathcal{A}+\mathcal{B}$ is maximal monotone, we shall show that for any $F_0 = (f_1^0,\dots,f_N^0,g_1^0,\dots,g_N^0)^T \in X$ the equation

$$U + \mathcal{A}(U) + \mathcal{B}(U) \ni F_0 \tag{5}$$

has a solution $U \in D(\mathcal{B})$.

For this aim, let $F_0 \in X$ be fixed. Equation (5) is equivalent to

$$\begin{cases} u_j + \dfrac{v_{j+1}-2v_j+v_{j-1}}{h_j^2} + c_jA(u_j) \ni f_j^0,\ \ j=\overline{1,N} \\ v_j - \dfrac{u_{j+1}-2u_j+u_{j-1}}{\overline{h}_j^2} + d_jB(v_j) \ni g_j^0,\ \ j=\overline{1,N}, \end{cases} \tag{6}$$

$$(u_1-u_0,-u_{N+1}+u_N,-v_1+v_0,v_{N+1}-v_N)^T \in \Lambda((v_1,v_N,u_1,u_N)^T). \tag{7}$$

We consider the following approximate problem

$$U^\lambda + \mathcal{A}(U^\lambda) + \mathcal{B}_\lambda(U^\lambda) \ni F_0,\ \ \lambda>0, \tag{8}$$

where $\mathcal{B}_\lambda((u_1,\dots,u_N,v_1,\dots,v_N)^T) = (c_1A_\lambda(u_1),\dots,c_NA_\lambda(u_N),d_1B_\lambda(v_1),\dots,d_NB_\lambda(v_N))^T$, with A_λ, B_λ the Yosida approximations of A, B respectively ($A_\lambda = \frac{1}{\lambda}(I-J_\lambda^A)$, $B_\lambda = \frac{1}{\lambda}(I-J_\lambda^B)$).

Because A_λ, B_λ are everywhere defined ($D(A_\lambda)=D(B_\lambda)=H$), single-valued, monotone and continuous, we deduce that $\mathcal{B}_\lambda$ is also everywhere defined in X, single-valued, monotone and continuous for each $\lambda>0$. By Lemma 3 the operator $\mathcal{A}$ is maximal monotone and then $\mathcal{A}+\mathcal{B}_\lambda$ is maximal monotone, for each $\lambda>0$. Therefore for each $\lambda>0$ the problem (8) has

a solution $U^\lambda = (u_1^\lambda, \dots, u_N^\lambda, v_1^\lambda, \dots, v_N^\lambda)^T \in D(\mathcal{A})$. The problem (8) is equivalent to

$$(9)\qquad \begin{cases} u_j^\lambda + \dfrac{v_{j+1}^\lambda - 2v_j^\lambda + v_{j-1}^\lambda}{h_j^2} + c_j A_\lambda(u_j^\lambda) = f_j^0, \ \ j = \overline{1,N}, \\ v_j^\lambda - \dfrac{u_{j+1}^\lambda - 2u_j^\lambda + u_{j-1}^\lambda}{\overline{h}_j^2} + d_j B_\lambda(v_j^\lambda) = g_j^0, \ \ j = \overline{1,N}, \end{cases}$$

$$(10)\quad (u_1^\lambda - u_0^\lambda, -u_{N+1}^\lambda + u_N^\lambda, -v_1^\lambda + v_0^\lambda, v_{N+1}^\lambda - v_N^\lambda)^T \in \Lambda((v_1^\lambda, v_N^\lambda, u_1^\lambda, u_N^\lambda)^T.$$

Let $U^0 = (u_1^0, \dots, u_N^0, v_1^0, \dots, v_N^0)^T \in X$. We consider

$$(11)\qquad F_\lambda = (f_1^\lambda, \dots, f_N^\lambda, g_1^\lambda, \dots, g_N^\lambda)^T \in U^0 + \mathcal{A}(U^0) + \mathcal{B}_\lambda(U^0), \ \ \lambda > 0.$$

The set $\{\mathcal{B}_\lambda(U^0);\ \lambda > 0\}$ is bounded in the space X; indeed we have

$$\|\mathcal{B}_\lambda(U^0)\|_X^2 = \sum_{j=1}^N [h_j^2 c_j^2 \|A_\lambda(u_j^0)\|^2 + \overline{h}_j^2 d_j^2 \|B_\lambda(v_j^0)\|^2]$$
$$\le \sum_{j=1}^N [h_j^2 c_j^2 \|A^0(u_j^0)\|^2 + \overline{h}_j^2 d_j^2 \|B^0(v_j^0)\|^2] = \|\mathcal{B}^0(U^0)\|_X^2, \ \ \forall\, \lambda > 0,$$

(where $A^0 : D(A) \subseteq H \to H$ is the minimal section of A).

Using the above inequality, from (11) we deduce that $\|F_\lambda\|_X \le$ const. for all $\lambda > 0$, (we denote by const. several positive constant independent of λ). Now from (8) and (11) we obtain that $\|U^\lambda - U^0\|_X \le \|F_0 - F_\lambda\|_X$, so $\|U^\lambda\|_X \le$ const., for all $\lambda > 0$. Hence we deduce that the sets $\{u_j^\lambda;\ \lambda > 0\}$ and $\{v_j^\lambda;\ \lambda > 0\}$, $j = \overline{1,N}$ are bounded in H. By the assumption (H2)b we conclude that the operator Λ is bounded on bounded sets of H^4. In this way we obtain from the above considerations that $\{u_0^\lambda;\ \lambda > 0\}$, $\{v_0^\lambda;\ \lambda > 0\}$, $\{u_{N+1}^\lambda;\ \lambda > 0\}$ and $\{v_{N+1}^\lambda;\ \lambda > 0\}$ are also bounded in H. Now using (9) we deduce that the sets $\{c_j A_\lambda(u_j^\lambda);\ \lambda > 0\}$ and $\{d_j B_\lambda(v_j^\lambda);\ \lambda > 0\}$, $j = \overline{1,N}$ are bounded in H, therefore

$$(12)\qquad \|A_\lambda(u_j^\lambda)\| \le \text{const. and } \|B_\lambda(v_j^\lambda)\| \le \text{const.}, \ \ \forall\, \lambda > 0.$$

We shall prove in what follows that the sets $\{u_j^\lambda;\ \lambda > 0\}$ and $\{v_j^\lambda;\ \lambda > 0\}$, $j = \overline{1,N}$ are Cauchy sequences. For this, let $U^\lambda = (u_1^\lambda, \dots, u_N^\lambda, v_1^\lambda, \dots,$

$v_N^\lambda)^T$, $U^\mu = (u_1^\mu, \dots, u_N^\mu, v_1^\mu, \dots, v_N^\mu)^T \in X$ be solutions for (8) ($\lambda,\ \mu > 0$). Using the relation (8) we have

$$U^\lambda + Z^\lambda + \mathcal{B}_\lambda(U^\lambda) = F_0, \quad U^\mu + Z^\mu + \mathcal{B}_\mu(U^\mu) = F_0,$$

where $Z^\lambda \in \mathcal{A}(U^\lambda)$, $Z^\mu \in \mathcal{A}(U^\mu)$, and then we obtain $U^\lambda - U^\mu + Z^\lambda - Z^\mu + \mathcal{B}_\lambda(U^\lambda) - \mathcal{B}_\mu(U^\mu) = 0$. We multiply the above relation by $U^\lambda - U^\mu$ in X and after some computations we get

$$\sum_{j=1}^N [h_j^2 \|u_j^\lambda - u_j^\mu\|^2 + \overline{h}_j^2 \|v_j^\lambda - v_j^\mu\|^2]$$
$$\leq (\lambda + \mu) \sum_{j=1}^N [c_j h_j^2 \langle A_\lambda(u_j^\lambda), A_\mu(u_j^\mu)\rangle + d_j \overline{h}_j^2 \langle B_\lambda(u_j^\lambda), B_\mu(u_j^\mu)\rangle].$$

Then using (12) we obtain

$$\sum_{j=1}^N [\|u_j^\lambda - u_j^\mu\|^2 + \|v_j^\lambda - v_j^\mu\|^2] \leq \text{const.}(\lambda + \mu), \ \ \forall\, \lambda,\ \mu > 0.$$

By the last inequality we deduce that $\{u_j^\lambda;\ \lambda > 0\}$ and $\{v_j^\lambda;\ \lambda > 0\}$ are Cauchy sequences. Therefore there exist $u_j,\ v_j \in H$ such that $\lim_{\lambda \to 0} u_j^\lambda = u_j$, $\lim_{\lambda \to 0} v_j^\lambda = v_j$, strongly in H, for all $j = \overline{1, N}$.

Then $(v_1^\lambda, v_N^\lambda, u_1^\lambda, u_N^\lambda)^T \to (v_1, v_N, u_1, u_N)^T$, as $\lambda \to 0$, $\{(u_1^\lambda - u_0^\lambda, -u_{N+1}^\lambda + u_N^\lambda, -v_1^\lambda + v_0^\lambda, v_{N+1}^\lambda - v_N^\lambda)^T;\ \lambda > 0\}$ is bounded in H^4, so on a subsequence (denoted in the same way) we have $(u_1^\lambda - u_0^\lambda, -u_{N+1}^\lambda + u_N^\lambda, -v_1^\lambda + v_0^\lambda, v_{N+1}^\lambda - v_N^\lambda)^T \rightharpoonup (x, y, z, w)^T$, as $\lambda \to 0$ (we denote by $\rightharpoonup$ the weak convergence), with $(x, y, z, w)^T \in H^4$. Because $(u_1^\lambda - u_0^\lambda, -u_{N+1}^\lambda + u_N^\lambda, -v_1^\lambda + v_0^\lambda, v_{N+1}^\lambda - v_N^\lambda)^T \in \Lambda((v_1^\lambda, v_N^\lambda, u_1^\lambda, u_N^\lambda)^T)$ and Λ is demicontinuous, we deduce that $(v_1, v_N, u_1, u_N)^T \in D(\Lambda)$, $(x, y, z, w)^T \in \Lambda((v_1, v_N, u_1, u_N)^T)$. We denote by $u_0 = u_1 - x$, $u_{N+1} = u_N - y$, $v_0 = v_1 + z$, $v_{N+1} = w + v_N$. Then $(x, y, z, w)^T = (u_1 - u_0, -u_{N+1} + u_N, -v_1 + v_0, v_{N+1} - v_N)^T$ and so we obtain (7).

By $u_j^\lambda \to u_j$ and $v_j^\lambda \to v_j$, as $\lambda \to 0$, for all $j = \overline{1, N}$, we have $J_\lambda^A(u_j^\lambda) \to u_j$, $J_\lambda^B(v_j^\lambda) \to v_j$, as $\lambda \to 0$, for all $j = \overline{1, N}$. Because $\{A_\lambda(u_j^\lambda);\ \lambda > 0\}$, $\{B_\lambda(v_j^\lambda);\ \lambda > 0\}$ are bounded (by (12)), and A, B are demiclosed, we deduce that $u_j \in D(A)$, $v_j \in D(B)$, $A_\lambda(u_j^\lambda) \rightharpoonup p_j$, as $\lambda \to 0$, $B_\lambda(v_j^\lambda) \rightharpoonup q_j$,

as $\lambda \to 0$ (eventually on some subsequences), and $p_j \in A(u_j)$, $q_j \in B(v_j)$, $j = \overline{1,N}$. By passing to $\lambda \to 0$ in (9) and (10) we deduce that $U = (u_1, \ldots, u_N, v_1, \ldots, v_N)^T \in D(\mathcal{B})$ is a solution for (6) and (7), that is the equation (5) has a solution. Hence $\mathcal{A} + \mathcal{B}$ is maximal monotone in X.

In this first case we can also deduce the above conclusion in a different and direct way. Namely, if (H2)b holds, then the operator $\mathcal{A}$ is everywhere defined. Because $\mathcal{A}$ and $\mathcal{B}$ are maximal monotone (Lemma 3 and Lemma 4) and $(\text{int}D(\mathcal{A})) \cap D(\mathcal{B}) = X \cap D(\mathcal{B}) = D(\mathcal{B}) \neq \emptyset$, we deduce by Rockafellar's Theorem (see [7], Corollaire 2.7) that the operator $\mathcal{A} + \mathcal{B}$ is maximal monotone.

If the assumption (H3)a holds, then we can easily deduce that $\text{int}D(\mathcal{A}) = \{(u_1, \ldots, u_N,\ v_1, \ldots, v_N)^T \in X,\ (v_1, v_N, u_1, u_N)^T \in \text{int}D(\Lambda)\}$ and so $\text{int}D(\mathcal{A}) \cap D(\mathcal{B}) \neq \emptyset$. Therefore using once again Rockafellar's Theorem we deduce that $\mathcal{A} + \mathcal{B}$ is maximal monotone.

If the assumption (H3)c holds, then it follows that $0 \in \text{int}(D(\mathcal{A}) - D(\mathcal{B}))$. Using now Attouch's Theorem (see [4], Theorem 1) we deduce that the operator $\mathcal{A} + \mathcal{B}$ is maximal monotone. □

Theorem 3. *If the assumptions* (H1)a, (H2)a, [(H1)b *or* (H3)b], (H4) *and* (H5) *hold, then the operator* $\mathcal{A} + \mathcal{B}$ *is maximal monotone.*

Proof. If (H1)b holds we can use a similar idee that we used in the proof of Theorem 2 (using Minty's Theorem and an associated aproximate problem) or we can deduce the conclusion of the theorem in a direct way, as we did in the last part of the proof of Theorem 2. For the second variant, using Lemma 3 and Lemma 4 we have that $\mathcal{A}$ and $\mathcal{B}$ are maximal monotone. Because $D(A) = D(B) = H$, we obtain $D(\mathcal{B}) = X$, and so $D(\mathcal{A}) \cap (\text{int}(D(\mathcal{B})) = D(\mathcal{A}) \cap X = D(\mathcal{A}) \neq \emptyset$. Using now Rockafellar' Theorem we deduce that the operator $\mathcal{A} + \mathcal{B}$ is maximal monotone.
If the assumption (H3)b holds, then it easily follows that $D(\mathcal{A}) \cap (\text{int}(D(\mathcal{B})) \neq \emptyset$. Then using the same theorem we deduce that $\mathcal{A} + \mathcal{B}$ is maximal monotone. □

3. Existence, uniqueness and asymptotic behaviour of the solutions to (S)+(EC)+(ID). Using the operators $\mathcal{A}$ and $\mathcal{B}$, our problem can be expressed as the following Cauchy problem in the space X

$$\text{(P)} \qquad \begin{cases} \dfrac{dU}{dt}(t) + \mathcal{A}(U(t)) + \mathcal{B}(U(t)) \ni F(t), \\ U(0) = U_0, \end{cases}$$

where $U = (u_1, \dots, u_N, v_1, \dots, v_N)^T$, $U_0 = (u_{10}, \dots, u_{N0}, v_{10}, \dots, v_{N0})^T$, $F = (f_1, \dots, f_N, g_1, \dots, g_N)^T$.

Theorem 4. *Assume that the assumptions* (H1)a, (H2)a, [(H1)b *or* (H2)b *or* (H3)a *or* (H3)b *or* (H3)c], (H4) *and* (H5) *hold. If* $(v_{10}, v_{N0}, u_{10}, u_{N0})^T \in D(\Lambda) \cap (D(B)^2 \times D(A)^2)$, $u_{j0} \in D(A)$, $v_{j0} \in D(B)$, *for all* $j = \overline{1, N-1}$, $f_j,\ g_j \in W^{1,1}(0,T;H)$, *for all* $j = \overline{1,N}$, *then there exist unique functions* $u_j,\ v_j \in W^{1,\infty}(0,T;H)$, $j = \overline{1,N}$, *with* $(v_1(t), v_N(t), u_1(t), u_N(t))^T \in D(\Lambda) \times (D(B)^2 \times D(A)^2)$, $u_j(t) \in D(A)$, $v_j(t) \in D(B)$, *for all* $j = \overline{2, N-1}$, *for all* $t \in [0,T]$ *that verify the system* (S) *and the extreme condition* (EC) *for all* $t \in [0,T]$ *and the initial data* (ID). *Besides* $u_j,\ v_j$, $j = \overline{1,N}$ *are everywhere differentiable from right in the topology of* H *and*

$$\frac{d^+u_j}{dt}(t) = \left(f_j(t) - c_jA(u_j(t)) - \frac{v_{j+1}(t) - 2v_j(t) + v_{j-1}(t)}{h_j^2}\right)^0, \quad j = \overline{2, N-1},$$

$$\frac{d^+v_j}{dt}(t) = \left(g_j(t) - d_jB(v_j(t)) + \frac{u_{j+1}(t) - 2u_j(t) + u_{j-1}(t)}{\overline{h}_j^2}\right)^0, \quad j = \overline{2, N-1},$$

$$\begin{pmatrix} \frac{d^+u_1}{dt}(t) \\ \frac{d^+u_N}{dt}(t) \\ \frac{d^+v_1}{dt}(t) \\ \frac{d^+v_N}{dt}(t) \end{pmatrix} = \begin{pmatrix} f_1(t) - c_1A(u_1(t)) - \frac{v_2(t) - 2v_1(t) + v_0(t)}{h_1^2} \\ f_N(t) - c_NA(u_N(t)) - \frac{v_{N+1}(t) - 2v_N(t) + v_{N-1}(t)}{h_N^2} \\ g_1(t) - d_1B(v_1(t)) + \frac{u_2(t) - 2u_1(t) + u_0(t)}{\overline{h}_1^2} \\ g_N(t) - d_NB(v_N(t)) + \frac{u_{N+1}(t) - 2u_N(t) + u_{N-1}(t)}{\overline{h}_N^2} \end{pmatrix}^0, \tag{13}$$

for all $t \in [0,T)$, *with* $(u_1(t)-u_0(t), -u_{N+1}(t)+u_N(t), -v_1(t)+v_0(t), v_{N+1}(t)-v_N(t))^T \in \Lambda((v_1(t), v_N(t), u_1(t), u_N(t))^T)$, *for all* $t \in [0,T)$.

Proof. By Theorem 2 and Theorem 3 the operator $\mathcal{A}+\mathcal{B}$ is maximal monotone in the space X. Using [[5], Theorem 2.2, Corollary 2.1, Chapter III] we deduce that for $U_0 \in D(\mathcal{A}) \cap D(\mathcal{B})$ and $F \in W^{1,1}(0,T;X)$, the problem (P)≡(S)+(EC)+(ID) has a unique strong solution $U = (u_1, \ldots, u_N, v_1, \ldots, v_N)^T \in W^{1,\infty}(0,T;X)$, $U(t) \in D(\mathcal{A}) \cap D(\mathcal{B})$, for all $t \in [0,T)$. We can conclude that $U(T) \in D(\mathcal{A}) \cap D(\mathcal{B})$, by extending correspondingly the functions f_j, g_j, $j = \overline{1,N}$ and by considering the equation $(\mathrm{P})_1$ in the interval $[0, T+\varepsilon]$, with $\varepsilon > 0$. The solution U is everywhere differentiable from right and $\frac{d^+U}{dt}(t) = (F(t) - \mathcal{A}(U(t)) - \mathcal{B}(U(t)))^0$, for all $t \in [0,T)$, that is the relations (13) are verified. Moreover we have

$$\left\| \frac{d^+U}{dt}(t) \right\|_X \leq \|(F(0) - \mathcal{A}(U_0) - \mathcal{B}(U_0))^0\|_X + \int_0^t \left\| \frac{dF}{ds}(s) \right\|_X ds, \quad \forall\, t \in [0,T).$$

□

Remark 1. Under the assumptions (H1)a, (H2)a, [(H1)b or (H2)b or (H3)a or (H3)b or (H3)c], (H4) and (H5), if $U_0 \in \overline{D(\mathcal{A}) \cap D(\mathcal{B})}$ and $F \in L^1(0,T;X)$, then by [[5], Corollary 2.2, Chapter III], we deduce that the problem (P) has a unique weak solution $U \in C([0,T];X)$.

Theorem 5. *Assume that the assumptions* (H1)ac, (H2)a, [(H1)b *or* (H2)b *or* (H3)a *or* (H3)b *or* (H3)c], (H4) *and* (H5) *hold. Then for every* $F_0 \in X$ *the equation* $\mathcal{A}(U) + \mathcal{B}(U) \ni F_0$ *has a unique solution* $U \in D(\mathcal{A}) \cap D(\mathcal{B})$.

Proof. By the assumptions of the theorem, we deduce that the maximal monotone operator $\mathcal{A}+\mathcal{B}$ is strongly monotone. Indeed for all $U = (u_1, \ldots, u_N, v_1, \ldots, v_N)^T$, $\overline{U} = (\overline{u}_1, \ldots, \overline{u}_N, \overline{v}_1, \ldots, \overline{v}_N)^T \in D(\mathcal{A}) \cap D(\mathcal{B})$, $Z \in (\mathcal{A}+\mathcal{B})(U)$, $\overline{Z} \in (\mathcal{A}+\mathcal{B})(\overline{U})$ we have

$$\langle Z - \overline{Z}, U - \overline{U} \rangle_X \geq \sum_{j=1}^N h_j^2 c_j a_0 \|u_j - \overline{u}_j\|^2 + \sum_{j=1}^N \overline{h}_j^2 d_j b_0 \|v_j - \overline{v}_j\|^2$$

$$\geq M \left[\sum_{j=1}^N \left(h_j^2 \|u_j - \overline{u}_j\|^2 + \overline{h}_j^2 \|v_j - \overline{v}_j\|^2 \right) \right] = M \|U - \overline{U}\|_X^2,$$

where $M = \min\{c_j a_0, d_j b_0, \ \ j = \overline{1,N}\} > 0$.

Therefore the operator $\mathcal{A}+\mathcal{B}$ is coercive, and so $R(\mathcal{A}+\mathcal{B})=X$. We deduce that for $F_0=(f_1^0,\dots,f_N^0,g_1^0,\dots,g_N^0)^T\in X$, the equation $(\mathcal{A}+\mathcal{B})(U)\ni F_0$, which is equivalent to the following problem with second-order differences

$$
\begin{aligned}
&\frac{q_{j+1}-2q_j+q_{j-1}}{h_j^2}+c_jA(p_j)\ni f_j^0,\ \ j=\overline{1,N},\\
&-\frac{p_{j+1}-2p_j+p_{j-1}}{\overline{h}_j^2}+d_jB(q_j)\ni g_j^0,\ \ j=\overline{1,N},\\
&\begin{pmatrix} p_1-p_0\\ -p_{N+1}+p_N\\ -q_1+q_0\\ q_{N+1}-q_N\end{pmatrix}\in\Lambda\begin{pmatrix} q_1\\ q_N\\ p_1\\ p_N\end{pmatrix},
\end{aligned}
\tag{14}
$$

has a unique solution $U=(p_1,\dots,p_N,q_1,\dots,q_N)^T\in D(\mathcal{A})\cap D(\mathcal{B})$. □

Using now Theorem 5 and [[7], Theoreme 3.9] we obtain

Theorem 6. *Assume that the assumptions* (H1)ac, (H2)a, [(H1)b *or* (H2)b *or* (H3)a *or* (H3)b *or* (H3)c], (H4) *and* (H5) *hold,* $F\in L^1_{loc}(\mathbb{R}_+;X)$ *verifies the condition* $F(t)\to\widetilde{F}$, *as* $t\to\infty$, *strongly in* X, *and* $\widetilde{U}$ *is the unique solution of the system* (14) *with* $F_0=0$. *Then* $U(t)\to\widetilde{U}$, *as* $t\to\infty$, *strongly in* X, *where* $U(t)$, $t\ge 0$, *is an arbitrary weak solution of the equation* $(\mathrm{P})_1$. *More precisely*

$$\|U(t)-\widetilde{U}\|_X\le e^{-Mt}\|U(0)-\widetilde{U}\|_X+\int_0^t e^{M(s-t)}\|F(s)-\widetilde{F}\|_X ds,\ \ \forall\, t\ge 0.$$

If $\frac{dF}{dt}\in L^1(\mathbb{R}_+;X)$ *and* $U(0)\in D(\mathcal{A})\cap D(\mathcal{B})$ *then* $\lim_{t\to\infty}\|\frac{d^+U}{dt}(t)\|_X=0$ *and*

$$\int_0^\infty\left\|\frac{d^+U}{dt}(t)\right\|_X dt\le\frac{1}{M}\|((\mathcal{A}+\mathcal{B})(U(0))-F(0))^0\|_X+\frac{1}{M}\int_0^\infty\left\|\frac{dF}{dt}(t)\right\|_X dt.$$

4. A characterization of maximal monotone operators. In this section we shall present another generalization of Minty's Theorem (Theorem 8 below). Let H be a real Hilbert space with the scalar product $\langle\cdot,\cdot\rangle$ and the corresponding norm $\|\cdot\|$. First, we recall Minty's theorem.

Theorem 7. *Let $A : D(A) \subseteq H \to H$ be a monotone operator, possibly multivalued. Then A is maximal monotone if and only if for all $\lambda > 0$ (equivalently there exists a $\lambda > 0$ such that) $R(\lambda I + A) = H$.*

For the proof of the above theorem see [[5], Theorem 1.2, Chapter II] or [[7], Proposition 2.2].

Theorem 8. *Let $A : D(A) \subseteq H \to H$ be a monotone operator, possibly multivalued. Then A is maximal monotone if and only if for every single-valued, everywhere defined, hemicontinuous and strongly monotone operator $L : H \to H$, (equivalently there exists a single-valued, everywhere defined, hemicontinuous and strongly monotone operator $L : H \to H$ such that) $R(A + L) = H$.*

Proof. *"Only if" part.* Assume that A is maximal monotone, and let $L : H \to H$ be a single-valued, everywhere defined, hemicontinuous and strongly monotone (that is there exists $a > 0$ such that $\langle L(x) - L(y), x - y\rangle \geq a\|x - y\|^2, \ \forall\, x,\, y \in H$). Then the operator $L - aI$ is monotone, because $\langle (L - aI)(x), (L - aI)y, x - y\rangle = \langle L(x) - L(y), x - y\rangle - a\|x - y\|^2 \geq 0, \ \forall\, x,\, y \in H$.

Because A is maximal monotone and $L - aI$ is everywhere defined, single-valued, hemicontinuous and monotone, we deduce that $A + L - aI$ is maximal monotone. Then, using Theorem 7 we obtain $R(\lambda I + A + L - aI) = H$, for all $\lambda > 0$. Therefore for $\lambda = a$ we have $R(A + L) = H$.

"If" part. Assume now that for a single-valued, everywhere defined, hemicontinuous and strongly monotone operator $L : H \to H$, $(\langle L(x) - L(y), x - y\rangle \geq a\|x - y\|^2, \ \forall\, x,\, y \in H, \ a > 0)$, we have

$$R(A + L) = H. \tag{15}$$

We suppose by reductio ad absurdum that A is not maximal monotone, that is there exists $[x_0, y_0] \notin A$ such that

$$\langle x - x_0, y - y_0\rangle \geq 0, \ \forall\, [x, y] \in A, \tag{16}$$

(we identified the operator A with its graph).

We consider now the element $y_0 + L(x_0) \in H$. Then by (15) we deduce that there exists $[x_1, y_1] \in A$ such that $y_1 + L(x_1) = y_0 + L(x_0)$ or

$$y_1 - y_0 = L(x_0) - L(x_1). \tag{17}$$

In (16) we take $x = x_1$ and $y = y_1$. Then we obtain $\langle x_1 - x_0, y_1 - y_0 \rangle \geq 0$, and by (17) we deduce $\langle x_1 - x_0, L(x_1) - L(x_0) \rangle \leq 0$. Because L is strongly monotone from the above inequality we obtain

$$a\|x_1 - x_0\|^2 \leq \langle x_1 - x_0, L(x_1) - L(x_0) \rangle \leq 0.$$

So $x_1 = x_0$ and by (17) we have $y_1 = y_0$. Therefore $[x_0, y_0] \in A$, which is a contradiction to our hypothesis. Hence the operator A is maximal monotone. □

Remark 2. In the proof of the "If" part of Theorem 8 we didn't used the hemicontinuity of the operator L. We can also replace the strong monotonicity of L by a weakened assumption, by supposing that L is stricty monotone, that is for all x, $y \in H$, $x \neq y$, we have $\langle L(x) - L(y), x - y \rangle > 0$. Therefore we obtain the following theorem.

Theorem 9. *Let $A : D(A) \subseteq H \to H$ be a monotone operator, possibly multivalued. If there exists a single-valued, everywhere defined and strictly monotone operator $L : H \to H$ such that $R(A+L) = H$, then A is maximal monotone.*

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