ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" DIN IAŞI (S.N.)
MATEMATICĂ, Tomul LIV, 2008, f.2

FOURIER TRANSFORM OF THE HEAT KERNEL ON THE HEISENBERG GROUP*†

BY

MARIAN-DUMITRU PANŢIRUC

Abstract. In this paper an explicit formula is given for the Fourier transform of the "heat" kernel on the Heisenberg group at every point of the dual space, as given in Folland. By a result of Siebert we obtain for each of the above representation a strongly continuous contraction semigroup. For these semigroups the formula of their infinitesimal generator is given.

Mathematics Subject Classification 2000: 43A30, 60B15.

Key words: Heisenberg group, Fourier transform, convolution semigroups, locally compact groups.

1. Preliminaries. The Heiseberg group H_n is the simplest noncommutative nilpotent Lie group. Its Laplacian Δ_H is not elliptic but a result of Hörmander implies that it is hypoelliptic. H_n is of interest in many domains of analysis and geometry: nilpotent Lie group theory, hypoelliptic second order PDE, probability theory of degenerate diffusion process etc.

The Heisenberg Laplacian generates a convolution semigroup whose measures are absolutely continuous with respect to the Haar measure. An explicit formula was derived by Hulanicki [5] using representation theory, and by Gaveau [6] using probability theory.

It is known that for a vaguely continuous convolution semigroup on a locally compact Abelian group the Fourier transform of its measures can be

---

*Supported in part by CNCSIS-UEFISCSU, contract TD II, code 202

†This paper was presented at The 6th Congress of Romanian mathematicians, July 2007 and Conference on Potential Theory and Stochastic, Albac, Romania, September 2007

written as the exponential of some continuous negative definite function on the dual group.

Performing a Fourier transform in the non-Abelian case is more difficult and requires notions of representation theory. For a more extensive introduction in representation theory see FOLLAND [1], ch. 3. By a result of SIEBERT [2] the Fourier transform of the measures of a vaguely continuous convolution semigroup on a locally compact group at each representation is a strongly continuous contraction semigroup.

An explicit formula for the Fourier transform of the convolution semigroup that Δ_H generates on H_n will be given, together with the infinitesimal generator of the contraction semigroup obtained.

We recall first some definitions and results from [1].

Definition 1.1. *A (**unitary**) **representation** of a locally compact group G is a homomorphism π from G to the multiplicative group of unitary operators on a Hilbert space H_π continuous in the strong operator topology, that is a map $\pi : G \to \mathcal{U}(H_\pi)$ such that*

- $\pi(xy) = \pi(x)\pi(y)$;
- $\pi(x^{-1}) = \pi^*(x) = [\pi(x)]^{-1}$;
- $x \mapsto \pi(x)u$ *is continuous from G to H_π for every $u \in H_\pi$.*

A representation π is called irreducible if the only subspaces of H_π that are invariant for each $\pi(x),\ x \in G$ are the trivial ones. Two representations π_1 and π_2 of G on H_{π_1} and H_{π_2} respectively are (unitary) equivalent if there exists an unitary operator $T : H_{\pi_1} \to H_{\pi_2}$ such that $T\pi_1(x) = \pi_2(x)T$ for all $x \in G$

The **dual space** of a locally compact group G is the space of all classes of equivalence of irreducible representations of G and will be denoted by $\hat{G}$. Several structures were proposed for the dual space. The one that yield better results is that of measurable space with the so called Mackey-Borel structure. If G is a **type I** group or equivalently, the Mackey-Borel structure on G is standard then there is a measurable field of representations over $\hat{G}$, $(\rho_\pi)_{[\pi]\in\hat{G}}$ such that $\rho_\pi \in [\pi]$. This way we can identify the points in $\hat{G}$ with representations in this measurable field.

Suppose we fix a measurable field of representations as above. Then, the Fourier transform of a bounded measure $\mu \in M_b(G)$ is the measurable field of operators over $\hat{G}$ given by:

$$\hat{\mu}(\pi) = \int_G \pi(x^{-1}) d\mu(x).$$

Consequently the Fourier transform of $f \in L^1(G)$ is the measurable filed of operators over $\hat{G}$:

$$\hat{f}(\pi) = \int_G f(x)\pi(x^{-1})dx.$$

It is easy to verify the following properties of the Fourier transform:

$$\widehat{a\mu + b\nu}(\pi) = a\hat{\mu}(\pi) + b\hat{\nu}(\pi),$$

$$\widehat{\mu * \nu}(\pi) = \hat{\nu}(\pi)\hat{\mu}(\pi),$$

$$\widehat{f^*}(\pi) = \hat{f}(\pi)^*$$

$$\widehat{L_x f}(\pi) = \hat{f}(\pi)\pi(x^{-1}), \quad \widehat{R_x f}(\pi) = \pi(x)\hat{f}(\pi).$$

Definition 1.2. *A family of bounded, positive measures $(\mu_t)_{t>0}$ on a locally compact group G is called a vaguely continuous convolution semigroup on G if:*

1. $\mu_t(G) \leq 1$ *for all* $t > 0$,
2. $\mu_t * \mu_s = \mu_{t+s}$ *for all* $s, t > 0$,
3. $\mu_t \to \varepsilon_0$ *vaguely as* $t \to 0$, *where* ε_0 *is the point-mass at the neutral element of* G.

If we perform the Fourier transform for the measures of a convolution semigroup on G we will clearly have a semigroup property for the family of operators $(\widehat{\mu_t}(\pi))_{t>0}$ on H_π for each π:

$$\widehat{\mu_t * \mu_s}(\pi) = \hat{\mu}_{t+s}(\pi), \quad for\ all\ s, t > 0.$$

By [2], Prop. 3.1 we have even more: if we denote by $\widehat{\mu_0} = I$, the identity operator on H_π then $(\widehat{\mu_t}(\pi))_{t\geq 0}$ is a strongly continuous semigroup of operators over H_π for every unitary representation π of G.

Remark 1.1. If G is a locally compact group then all its irreducible representations are 1-dimensional and so the representation space can be chosen $\mathbb{C}$ for each such representation. Moreover, each irreducible representation can be identified with a continuous homomorphism $\gamma : G \to \mathbb{T}$

which allows the dual space to be organized as a group with pointwise multiplication. Endowing $\hat{G}$ with the topology of compact convergence on G, the dual space becomes a locally compact Abelian group called the **dual group** of G.

In particular if $G = \mathbb{R}^n$, the dual group $\widehat{\mathbb{R}^n} \simeq \mathbb{R}^n$ and the Fourier transform has the classic, well-known formula. For more details on the Abelian case see Berg and Forst [7].

However, if $(\mu_t)_{t>0}$ is a convolution semigroup on a LCA-group G then another result is available for the Fourier transform of the measures μ_t, that is more precise (see [7]):

Theorem 1.1. *If $(\mu_t)_{t>0}$ is a (vaguely continuous)convolution semigroup on a LCA-group G, then there exists a uniquely determined continuous, negative definite function ψ on the dual group, $\hat{G}$ such that*

$$\hat{\mu}_t(\gamma) = exp(-t\psi(\gamma)) \qquad for\ t > 0\ and\ \gamma \in \hat{G}.$$

Conversely, given a continuous, negative definite function ψ on $\hat{G}$, then the formula above determines a convolution semigroup $(\mu_t)_{t>0}$ on G.

As an example, for the brownian semigroup on $\mathbb{R}^n$:

$$\mu_t = (4\pi t)^{-\frac{n}{2}} exp(-\frac{\|x\|^2}{4t})dx \qquad for\ x \in \mathbb{R}^n$$

the Fourier transform is $\hat{\mu}_t(y) = exp(-t\|y\|^2)$ and the associated negative definite function is $y \mapsto \|y\|^2$.

2. The Heisenberg group and the "heat" kernel. The Heisenberg group H_n is $\mathbb{R}^{2n+1} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ with the composition law:

$$(x,\xi,t)(x',\xi',t') = \big(x + x', \xi + \xi', t + t' + \frac{1}{2}(x'\xi - x\xi')\big) \tag{1}$$

where $x\xi$ is the usual scalar product in $\mathbb{R}^n$. One verifies easily that H_n is a *non-Abelian* group, has the unit 1=(0,0,0) and the inverse of an element $X = (x,\xi,t) \in H_n$ is $(-x,-\xi,-t)$. It is also obvious that H_n endowed with the usual topology on $\mathbb{R}^{2n+1}$ is a locally compact group and the Haar measure for this group is the Lebesgue measure on $\mathbb{R}^{2n+1}$.

H_n has a second order differential operator, Δ_H which unlike the usual Laplacian is not elliptic; it is a starting point for many research in analysis and PDE's. The Heisenberg Laplacian is

$$\Delta_H = \frac{1}{2}\sum_{i=1}^{n}(X_i^2 + \Xi_i^2)$$

where

$$X_i = \frac{\partial}{\partial x_i} + \frac{1}{2}\xi_i\frac{\partial}{\partial t} \; and \quad \Xi_i = \frac{\partial}{\partial \xi_i} - \frac{1}{2}x_i\frac{\partial}{\partial t}$$

The heat kernel for Δ_H was first calculated independently by Gaveau in [6] and Hulanicki in [5] as an answer to the requirements:

$$\begin{cases} \dfrac{\partial K_s}{\partial s} = \Delta_H K_s \\ \\ \lim\limits_{s\to 0+} K_s(x,\xi,t) = \delta(x,\xi,t) \end{cases}.$$

The Gaveau-Hulanicki result is:

$$K_s(x,\xi,t) = \frac{1}{(4\pi s)^{n+1}}\int_{-\infty}^{+\infty} exp(-f(x,\xi,t,\tau)/2s)V(\tau)d\tau$$

where

$$f(x,\xi,t,\tau) = -it\tau + (\|x\|^2 + \|\xi\|^2)\frac{\tau}{4}ctgh\frac{\tau}{2}$$

and

$$V(\tau) = \left(\frac{\tau/2}{sh\frac{\tau}{2}}\right)^n.$$

In order to compute the Fourier transform of the measures $K_s(x,\xi,t)dxd\xi dt$ we need the irreducible representations of H_n. They can be deduced using the "Mackey machine" - a way of producing irreducible representations for a group G inducing them from those of a normal subgroup (for details see [1]). All irreducible representations of the Heisenberg group H_n are equivalent to one of the following:

- 1-dimensional :

$$\pi_{b,\beta} : H_n \to \mathbb{T},\; \pi_{b,\beta}(x,\xi,t) = e^{2\pi i(bx+\beta\xi)}$$

with $b,\beta \in \mathbb{R}^n$

- ∞-dimensional:

$$\rho_h : H_n \to \mathcal{U}(L^2(\mathbb{R}^n)),\ [\rho_h(x,\xi,t)f](y) = e^{2\pi i h(t+\frac{1}{2}x\xi - y\xi)} f(y-x)$$

where h is a non-zero real number.

Because H_n is obviously a type I group, the dual space can be identified with the measurable field of representations $\{\pi_{b,\beta};\rho_h\}_{b,\beta,h}$. Thus the Fourier transform of a bounded measure $\mu \in M_b(H_n)$ is a measurable field over $\widehat{H_n}$ given by:

- the complex values

$$\hat{\mu}(\pi_{b,\beta}) = \int_{H_n} \pi_{b,\beta}(-x,-\xi,-t)d\mu(x,\xi,t) = \int_{H_n} e^{-2\pi i(bx+\beta\xi)} d\mu(x,\xi,t)$$

- and the operators on $L^2(\mathbb{R}^n)$:

$$\begin{aligned}[\hat{\mu}(\rho_h)\phi](y) &= \int_{H_n} [\rho_h(-x,-\xi,-t)\phi](y)d\mu(x,\xi,t) \\ &= \int_{H_n} e^{2\pi i h(-t+\frac{1}{2}x\xi+y\xi)}\phi(y+x)d\mu(x,\xi,t).\end{aligned}$$

Remark 2.1. If $f \in L^1(H_n)$ the complex values $\hat{f}(\pi_{b,\beta})$:

$$\begin{aligned}\hat{f}(\pi_{b,\beta}) &= \int_{H_n} \pi_{b,\beta}(-x,-\xi,-t)f(x,\xi,t)dxd\xi dt \\ &= \int_{H_n} e^{-2\pi i(bx+\beta\xi)} f(x,\xi,t)dxd\xi dt\end{aligned}$$

are actually the euclidian Fourier transform of f at the point $(b,\beta,0)$.

Proposition 2.1. *The Fourier transform of K_s is given by:*

$$\widehat{K_s}(\pi_{b,\beta}) = e^{-4\pi^2(\|b\|^2+\|\beta\|^2)}$$

and

$$[\widehat{K_s}(\rho_h)\phi](y) = \left(\frac{h}{sh(4\pi hs)}\right)^{\frac{n}{2}} \cdot \int_{\mathbb{R}^n} \phi(x)e^{-\frac{\pi h}{2}(a\|x-y\|^2+\frac{1}{a}\|x+y\|^2)}dx.$$

Proof. For the representations $\pi_{b,\beta}$ we have:

$$\widehat{K}_s(\pi_{b,\beta}) = \int_{H_n} e^{-2\pi i(bx+\beta\xi)} K_s(x,\xi,t)dxd\xi dt$$

where $dxd\xi dt$ is the Haar measure on the Heisenberg group, and we recall again that it is actually the usual Lebesgue measure on $\mathbb{R}^{2n+1}$. So,

$$\begin{aligned}\widehat{K}_s(\pi_{b,\beta}) &= \int_{\mathbb{R}^{2n+1}} (4\pi s)^{-n-1} \int_{\mathbb{R}} e^{i\frac{t\tau}{2s}} \left(\frac{\tau/2}{sh(\tau/2)}\right)^n \cdot \\ &\cdot \prod_{j=1}^{n} e^{-2\pi i(b_j x_j+\beta_j\xi_j)} e^{-(x_j^2+\xi_j^2)\frac{\tau}{8s}ctgh(\frac{\tau}{2})} d\tau dxd\xi dt.\end{aligned}$$

One can easily see that we cannot change the order of integration with respect to the variable t but there's no problem to perform it with the variables τ, x, ξ. We can write then:

$$\begin{aligned}\widehat{K}_s(\pi_{b,\beta}) &= \frac{1}{(4\pi s)^{n+1}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\frac{t\tau}{2s}} \cdot \left(\frac{\tau/2}{sh(\tau/2)}\right)^n \cdot \\ &\cdot \prod_{j=1}^{n} \int_{\mathbb{R}^2} e^{-2\pi i(b_j x_j+\beta_j\xi_j)} e^{-(x_j^2+\xi_j^2)\frac{\tau}{8s}ctgh(\frac{\tau}{2})} dx_j d\xi_j d\tau dt.\end{aligned}$$

The product under the integral is equal to

$$\begin{aligned}&\prod_{j=1}^{n} \left(\int_{\mathbb{R}} e^{-2\pi i b_j x} e^{-x^2\frac{\tau}{8s}ctgh(\frac{\tau}{2})} dx \cdot \int_{\mathbb{R}} e^{-2\pi i \xi_j x} e^{-x^2\frac{\tau}{8s}ctgh(\frac{\tau}{2})} dx\right) \\ &= \prod_{j=1}^{n} \left(\sqrt{\frac{\pi}{\frac{\tau}{8s}ctgh(\frac{\tau}{2})}} e^{-\frac{8\pi^2 b_j^2 s}{\tau ctgh(\tau/2)}} \cdot \sqrt{\frac{\pi}{\frac{\tau}{8s}ctgh(\frac{\tau}{2})}} e^{-\frac{8\pi^2 \beta_j^2 s}{\tau ctgh(\tau/2)}}\right) \\ &= \frac{(8\pi s)^n}{\tau^n ctgh^n \frac{\tau}{2}} e^{-\frac{8\pi^2 s(\|b\|^2+\|\beta\|^2)}{\tau ctgh(\tau/2)}}.\end{aligned}$$

Thus,

$$\widehat{K}_s(\pi_{b,\beta}) = \frac{1}{4\pi s} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\frac{t\tau}{2s}} \frac{1}{ch^n \frac{\tau}{2}} e^{-\frac{8\pi^2 s(\|b\|^2+\|\beta\|^2)}{\tau ctgh(\tau/2)}} d\tau dt.$$

Changing the variable $\tau \mapsto 2s\tau$ we obtain:

$$\widehat{K}_s(\pi_{b,\beta}) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{it\tau} \frac{1}{ch^n(s\tau)} \cdot e^{-\frac{4\pi^2(b^2+\beta^2)}{\tau ctgh(s\tau)}} d\tau dt = \frac{1}{2\pi}\mathcal{F}\mathcal{F}g(0)$$

where $g(\tau)=\frac{1}{ch^n(s\tau)}\cdot e^{-\frac{4\pi^2(\|b\|^2+\|\beta\|^2)}{\tau ctgh(s\tau)}}$ and $\mathcal{F}g$ is the Fourier transform of g. It is obvious that $g(0)=exp(-4\pi^2 s(\|b\|^2+\|\beta\|^2))$ and then

$$\widehat{K_s}(\pi_{b,\beta})=e^{-4\pi^2 s(\|b\|^2+\|\beta\|^2)}.$$

To prove the formula for the representations ρ_h let $\phi\in L^2(\mathbb{R}^n)$. We have:

$$\begin{aligned}
[\widehat{K_s}(\rho_h)\phi](y) &= \int_{H_n} e^{2\pi ih(-t+\frac{1}{2}x\xi+y\xi)}\phi(y+x)K_s(x,\xi,t)dxd\xi dt\\
&= \frac{1}{(4\pi s)^{n+1}}\int_{\mathbb{R}} e^{-2\pi iht}\int_{\mathbb{R}} e^{i\frac{t\tau}{2s}}V(\tau)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\phi(y+x)\cdot\\
&\quad \cdot e^{-\pi ih\xi(x+2y)}e^{-(\|x\|^2+\|\xi\|^2)\frac{\tau}{8s}ctgh\frac{\tau}{2}}dxd\xi d\tau dt.
\end{aligned}$$

Denote by $g(\tau)=V(\tau)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\phi(y+x)e^{-\pi ih\xi(x+2y)}e^{-(\|x\|^2+\|\xi\|^2)\frac{\tau}{8s}ctgh\frac{\tau}{2}}dxd\xi$ and change the variable $\tau\mapsto 2s\tau$. We obtain:

$$[\widehat{K_s}(\rho_h)\phi](y)=\frac{2s}{(4\pi s)^{n+1}}\int_{\mathbb{R}} e^{-2\pi iht}\int_{\mathbb{R}} e^{it\tau}g(2s\tau)d\tau dt$$

and we recognize now a Fourier transform and its inverse for the function g at the point $4\pi hs$. Thus,

$$\begin{aligned}
[\widehat{K_s}(\rho_h)\phi](y) &= \frac{1}{(4\pi s)^n}g(4\pi hs)\\
&= \left(\frac{h}{2sh(2\pi hs)}\right)^n\int_{\mathbb{R}^n}\phi(x+y)e^{-\|x\|^2\frac{\pi h}{2}ctgh(2\pi hs)}\cdot\\
&\quad \cdot\int_{\mathbb{R}^n} e^{-\pi ih\xi(x+2y)}e^{-\|\xi\|^2\frac{\pi h}{2}ctgh(2\pi hs)}d\xi dx\\
&= \left(\frac{h}{2sh(2\pi hs)}\right)^n\int_{\mathbb{R}^n}\phi(x+y)e^{-\|x\|^2\frac{\pi h}{2}ctgh(2\pi hs)}\cdot\\
&\quad \cdot\prod_{j=1}^n\sqrt{\frac{2\pi}{\pi hctgh(2\pi hs)}}e^{-\frac{\pi h(x_j+2y_j)}{2ctgh(2\pi hs)}}dx\\
&= \left(\frac{h}{2sh(2\pi hs)}\right)^{n/2}\int_{\mathbb{R}^n}\phi(x+y)\cdot\\
&\quad \cdot e^{-\frac{\pi h}{2}\left(ctgh(2\pi hs)\|x\|^2+tgh(2\pi hs)\|x+2y\|^2\right)}dx.
\end{aligned}$$

By an obvious change of variable and denoting $a = ctgh(2\pi hs)$ we have:

$$[\widehat{K_s}(\rho_h)\phi](y) = \left(\frac{h}{2sh(2\pi hs)}\right)^{n/2} \int_{\mathbb{R}^n} \phi(x) e^{-\frac{\pi h}{2}\left(a\|x-y\|^2 + \frac{1}{a}\|x+y\|^2\right)} dx. \quad \square$$

Remark 2.2. *As one can easily see we have* $\widehat{K_s}(\rho_h) = \widehat{K_s}(\rho_{-h})$ *for each* $s, h > 0$

The values of the Fourier transform at the $\pi_{b,\beta}$ representations reminds of the classical brownian semigroup. For the values at the representations ρ_h we state some properties in the next two theorems. The theorems are proved only for the case $n = 1$ but they can be verified easily for the general case.

Theorem 2.1. *1.* $\widehat{K_s}(\rho_h)$ *is a bounded, self-adjoint operator on* $L^2(\mathbb{R}^n)$*;*
2. $\widehat{K_s}(\rho_h)$ *is positive,* $< \widehat{K_s}(\rho_h)\phi, \phi >= \|\widehat{K_{\frac{s}{2}}}(\rho_h)\phi\|^2$ *and* $\|\widehat{K_s}(\rho_h)\| < 1$.

Proof. 1. Continuity follows from the obvious relation

$$\|\hat{\mu}(\pi)\| \leq \|\mu\|, \quad \forall\ \mu \in M_b(G) \text{ and } [\pi] \in \hat{G}$$

and from the fact that $\|K_s\|_{L^1(H_n)} = 1$.

Let now ϕ and $\psi \in L^2(\mathbb{R}^n)$. We have:

$$\begin{aligned} < \widehat{K_s}(\rho_h)\phi, \psi >&= \int \widehat{K_s}(\rho_h)\phi(y)\overline{\psi(y)} dy \\ &= \int \int_{H_n} K_s(x, \xi, t) e^{2\pi i h(-t + \frac{1}{2}x\xi + y\xi)} \phi(y + x)\overline{\psi(y)} dx d\xi dt dy. \end{aligned}$$

Changing the variable $y + x \mapsto y$ the above integral is equal to:

$$\begin{aligned} &\int \int_{H_n} K_s(x, \xi, t) e^{2\pi i h(-t - \frac{1}{2}x\xi + y\xi)} \phi(y)\overline{\psi(y - x)} dx d\xi dt dy \\ &= \int \phi(y) \int_{H_n} K_s(x, \xi, t) \overline{e^{2\pi i h(t + \frac{1}{2}x\xi - y\xi)} \psi(y - x)} dx d\xi dt dy. \end{aligned}$$

$=< \phi, \widehat{K_s}(\rho_h)\psi >$ since K_s is a real-valued function. Hence, $\widehat{K_s}(\rho_h)$ is self-adjoint.

2. Indeed, using 1. and Theorem 2.2 we have:

$$< \widehat{K_s}(\rho_h)\phi, \phi >=< \hat{K}_{\frac{s}{2}}(\rho_h) \circ \hat{K}_{\frac{s}{2}}(\rho_h)\phi, \phi >$$

$$=< \widehat{K_{\frac{s}{2}}}(\rho_h)\phi, \widehat{K_{\frac{s}{2}}}(\rho_h)\phi >= \|\widehat{K_{\frac{s}{2}}}(\rho_h)\phi\|^2 \geq 0.$$

This implies on the other hand that

$$\|\hat{K}_s(\rho_h)\| = \|\hat{K}_{\frac{s}{2}}(\rho_h)\|^2, \quad \forall s > 0$$

and thus

$$\|\hat{K}_s(\rho_h)\| = \|\hat{K}_{\frac{s}{2^n}}(\rho_h)\|^{2^n}, \quad \forall s > 0, \; n \in \mathbb{N}.$$

Estimating the norm of $\hat{K}_s(\rho_h)$ we get:

$$\begin{aligned}
|\hat{K}_s(\rho_h)\phi(y)| &\leq \sqrt{\frac{h}{sh(4\pi hs)}} \int_{\mathbb{R}} |\phi(x)| \cdot e^{-\frac{\pi h}{2}a(x-y)^2} \cdot e^{-\frac{\pi h}{2}\frac{1}{a}(x+y)^2} dx \\
&\leq \sqrt{\frac{h}{sh(4\pi hs)}} \|\phi\|_{L^2(\mathbb{R})} \cdot \left(\int_{\mathbb{R}} e^{-\pi h a(x-y)^2} \cdot e^{-\pi h \frac{1}{a}(x+y)^2} dx \right)^{\frac{1}{2}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|\hat{K}_s(\rho_h)\phi\|^2_{L^2(\mathbb{R})} &\leq \frac{h}{sh(4\pi hs)} \|\phi\|^2_{L^2(\mathbb{R})} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\pi h a(x-y)^2} \cdot e^{-\pi h \frac{1}{a}(x+y)^2} dx dy \\
&\leq \frac{h}{2sh(2\pi hs)} \cdot \sqrt{\frac{1}{ha}} \cdot \sqrt{\frac{a}{h}},
\end{aligned}$$

which shows again the continuity of $\hat{K}_s(\rho_h)$ and

$$\|\hat{K}_s(\rho_h)\| \leq \frac{1}{\sqrt{2sh(4\pi hs)}}.$$

Letting s tend to ∞ we obtain

$$\lim_{s \to \infty} \|\hat{K}_s(\rho_h)\| = 0.$$

But, for a fixed $s > 0$ we have

$$\|\hat{K}_{2^n s}(\rho_h)\| = \|\hat{K}_s(\rho_h)\|^{2^n}, \quad \forall n \in \mathbb{N},$$

so as $n \to \infty$ the left side of the equality tend to 0 so necessarily

$$\|\hat{K}_s(\rho_h)\| < 1, \quad \forall s > 0.$$

□

Theorem 2.2. 1. $(\widehat{K_s}(\rho_h))_{s\geq 0}$ *is a strongly continuous contraction semigroup on* $L^2(\mathbb{R}^n)$ *for every* $h \neq 0$*;*
2. *The infinitesimal generator of the semigroup* $(\widehat{K_s}(\rho_h))_{s\geq 0}$ *is*

$$A_h f(y) = \Delta f(y) - 4\pi^2 h^2 \|y\|^2 f(y), \qquad for\ f \in C_c^\infty(\mathbb{R}^n).$$

Proof. 1. We can use the general formula $\widehat{\mu * \nu}(\pi) = \widehat{\nu}(\pi)\widehat{\mu}(\pi)$ to prove the semigroup relation but instead of that we will give here a direct proof. It is enough to consider $n = 1$.

Let $\phi \in L^2(\mathbb{R})$. Denoting by $a_1 = ctgh(2\pi h s_1)$ and $a_2 = ctgh(2\pi h s_2)$ we have:

$$\hat{K}_{s_1}[\hat{K}_{s_2}(\rho_h)\phi](y) = \sqrt{\frac{h}{sh(4\pi h s_1)}} \int_{\mathbb{R}} [\hat{K}_{s_2}(\rho_h)\phi](z) e^{-\frac{\pi h}{2} a_1 (z-y)^2} \cdot$$
$$\cdot e^{-\frac{\pi h}{2}\frac{1}{a_1}(z+y)^2} dz = \frac{h}{\sqrt{sh(4\pi h s_1) sh(4\pi h s_2)}} \int_{\mathbb{R}} \int_{\mathbb{R}} \phi(x) \cdot$$
$$\cdot e^{-\frac{\pi h}{2}(a_1(z-y)^2 + \frac{1}{a_1}(z+y)^2 + a_2(x-z)^2 + \frac{1}{a_2}(x+z)^2)} dx dz$$
$$= C \int_{\mathbb{R}} \phi(x) \cdot e^{\frac{-\pi h}{2}(x^2(a_2 + \frac{1}{a_2}) + y^2(a_1 + \frac{1}{a_1}))} \cdot \int_{\mathbb{R}} e^{-\frac{\pi h}{2}(z^2 A - 2zB)} dz dx$$

where

$$A = a_1 + \frac{1}{a_1} + a_2 + \frac{1}{a_2}\ ,$$
$$B = y(a_1 - \frac{1}{a_1}) + x(a_2 - \frac{1}{a_2})\ ,$$
$$C = \frac{h}{\sqrt{sh(4\pi h s_1) sh(4\pi h s_2)}}.$$

Thus,

$$\hat{K}_{s_1}[\hat{K}_{s_2}(\rho_h)\phi](y) = C \int_{\mathbb{R}} \phi(x) \cdot e^{\frac{-\pi h}{2}(x^2(a_2 + \frac{1}{a_2}) + y^2(a_1 + \frac{1}{a_1}))} \cdot f(x, y) dx$$

where

$$f(x, y) = \int_{\mathbb{R}} e^{-\frac{\pi h}{2}(z^2 A - 2zB)} dz = \int_{\mathbb{R}} e^{\frac{-\pi h}{2}(z\sqrt{A} - \frac{B}{\sqrt{A}})^2} \cdot e^{\frac{\pi h}{2}\cdot\frac{B^2}{A}} dz$$

$$f(x, y) = \sqrt{\frac{2}{hA}} \cdot e^{\frac{\pi h}{2}\cdot\frac{B^2}{A}}$$

But

$$C\cdot\sqrt{\frac{2}{hA}} = \sqrt{\frac{2h}{sh(4\pi hs_1)sh(4\pi hs_2)}}\cdot$$
$$\cdot\frac{1}{\sqrt{\frac{ch(2\pi hs_1)}{sh(2\pi hs_1)}+\frac{sh(2\pi hs_1)}{ch(2\pi hs_1)}+\frac{ch(2\pi hs_2)}{sh(2\pi hs_2)}+\frac{sh(2\pi hs_2)}{ch(2\pi hs_2}}}$$

and using basic properties of hyperbolic sine and cosine functions we get:

$$C\cdot\sqrt{\frac{2}{hA}}=\sqrt{\frac{h}{sh(4\pi h(s_1+s_2))}}.$$

It remains to prove that:

$$x^2(a_2+\frac{1}{a_2})+y^2(a_1+\frac{1}{a_1})-\frac{B^2}{A}=a(x-y)^2+\frac{1}{a}(x+y)^2,$$

where $a=ctgh(2\pi h(s_1+s_2))$. For this we will identify the coefficients so we need to show that:

$$for\ x^2:\quad a_2+\frac{1}{a_2}-\frac{(a_2-\frac{1}{a_2})^2}{a_1+\frac{1}{a_1}+a_2+\frac{1}{a_2}}=a+\frac{1}{a}$$

$$for\ y^2:\quad a_1+\frac{1}{a_1}-\frac{(a_1-\frac{1}{a_1})^2}{a_1+\frac{1}{a_1}+a_2+\frac{1}{a_2}}=a+\frac{1}{a}$$

$$and\ for\ xy:\quad \frac{(a_1-\frac{1}{a_1})(a_2-\frac{1}{a_2})}{a_1+\frac{1}{a_1}+a_2+\frac{1}{a_2}}=a-\frac{1}{a},$$

things that are easy to verify using again the properties of the hyperbolic functions sh and ch. Thus we have:

$$\hat{K}_{s_1}[\hat{K}_{s_2}(\rho_h)\phi](y)=\sqrt{\frac{h}{sh(4\pi h(s_1+s_2))}}\int_{\mathbb{R}}\phi(x)e^{-\frac{\pi h}{2}a(x-y)^2}\cdot e^{-\frac{\pi h}{2}\cdot\frac{1}{a}(x+y)^2}dx$$

$$=[\hat{K}_{s_1+s_2}\phi](y).$$

The fact that $(\widehat{K_s}(\rho_h))_{s\geq 0}$ is a strongly continuous semigroup follows easily from the vague convergence of measures $K_s(x,\xi,t)dxd\xi dt$ to the Dirac measure concentrated at $(0,0,0)$ as s tends to 0.

2. We need to determine

$$\lim_{s\to 0+} \frac{1}{s}\left([\widehat{K_s}(\rho_h)\phi](y) - \phi(y)\right)$$

pointwise for $y \in \mathbb{R}^n$ and for suitable functions $f \in L^2(\mathbb{R}^n)$. I will give here a complete calculus only for $n = 1$. For the general case it is similar. We therefore have to find out the following limit:

$$\lim_{s\to 0+} \frac{1}{s}\left(\sqrt{\frac{h}{sh(4\pi hs)}}\int_{\mathbb{R}} \phi(x)e^{-\frac{\pi h}{2}(a(x-y)^2+\frac{1}{a}(x+y))^2}dx - \phi(y)\right),$$

where $a = ctgh(2\pi hs)$. We can rewrite the above limit changing the variable $x \mapsto x + y$ in the following way:

$$\lim_{s\to 0+}\left\{\frac{1}{s}\left(\sqrt{\frac{h}{sh(4\pi hs)}}\int_{\mathbb{R}}(\phi(x+y)-\phi(x))e^{-\frac{\pi h}{2}(ax^2+\frac{1}{a}(x+2y)^2)}dx\right)\right.$$
$$\left.+\phi(y)\frac{1}{s}\left(\sqrt{\frac{h}{sh(4\pi hs)}}\int_{\mathbb{R}} e^{-\frac{\pi h}{2}(ax^2+\frac{1}{a}(x+2y)^2)}dx - 1\right)\right\}.$$

Take the second term of the sum of the limit. The exponent can be written as:

$$ax^2 + \frac{1}{a}(x+2y)^2 = (a+\frac{1}{a})\left(x + y\frac{2}{1+a^2}\right)^2 - \pi hy^2\frac{2a}{1+a^2}.$$

Changing the variable $x \mapsto x - y\frac{2}{1+a^2}$ the limit of the second term becomes:

$$\lim_{s\to 0+}\frac{1}{s}\left(\sqrt{\frac{h}{sh(4\pi hs)}}\cdot e^{-\pi hy^2\frac{2a}{1+a^2}}\int_{\mathbb{R}} e^{-\pi h\frac{1+a^2}{2a}x^2}dx - 1\right)$$

$$= \lim_{s\to 0+}\frac{1}{s}\left(\sqrt{\frac{h}{sh(4\pi hs)}}\cdot\sqrt{\frac{1}{\pi h}}\sqrt{\frac{2a}{1+a^2}}\cdot\sqrt{\pi}\cdot e^{-\pi hy^2\frac{2a}{1+a^2}} - 1\right).$$

But $a = ctgh(2\pi hs)$ so $\frac{2a}{1+a^2} = tgh(4\pi hs)$ so the last quantity is equal to

$$\lim_{s\to 0+}\frac{1}{s}\left(\frac{e^{-\pi hy^2 tgh(4\pi hs)}}{\sqrt{ch(4\pi hs)}} - 1\right) = -4\pi^2h^2y^2.$$

Returning to the first term in the initial limit one can approximate for appropriate functions, $\phi(x+y)-\phi(y)$ by $x\phi'(y)+\frac{1}{2}x^2\phi''(y)$. Thus we only need to calculate

$$\lim_{s\to 0+}\frac{1}{s}\sqrt{\frac{h}{sh(4\pi hs)}}e^{-\pi hy^2 tgh(4\pi hs)}\int_{\mathbb{R}}(x\phi'(y)$$
$$+\frac{1}{2}x^2\phi''(y))e^{-\pi h\frac{1+a^2}{2a}(x+y\cdot\frac{2}{1+a^2})^2}dx.$$

But, clearly, $\frac{hs}{sh(4\pi hs)}\to\frac{1}{4\pi}$ and $e^{-\pi hy^2 tgh(4\pi hs)}\to 1$ as $s\to 0+$. We can separate again the limit and we need to find:

$$A=\lim_{s\to 0}\frac{\phi'(y)}{2\pi s\sqrt{s}}\int_{\mathbb{R}}xe^{-\pi hctgh(4\pi hs)(x+y\frac{2}{1+a^2})^2}dx$$

and

$$B=\lim_{s\to 0}\frac{\phi''(y)}{2\pi s\sqrt{s}}\int_{\mathbb{R}}x^2e^{-\pi hctgh(4\pi hs)(x+y\frac{2}{1+a^2})^2}dx.$$

For A, making an obvious change of variable we get:

$$\begin{aligned} A &= -\lim_{s\to 0+}\frac{\phi'(y)}{2\pi s\sqrt{s}}\frac{2}{1+a^2}\cdot\frac{1}{\sqrt{\pi h}}\cdot\frac{1}{\sqrt{ctgh(4\pi hs)}}\cdot\sqrt{\pi} \\ &= -\frac{\phi'(y)}{\pi h}\lim_{s\to 0+}\frac{1}{s\sqrt{s}}\cdot\frac{sh^2(2\pi hs)}{ch(4\pi hs)}\cdot\sqrt{\frac{sh(4\pi hs)}{ch(4\pi hs)}}=0. \end{aligned}$$

For B we proceed in the same way and having in mind what we've already proved for A, we obtain:

$$\begin{aligned} B &= \lim_{s\to 0+}\frac{\phi''(y)}{4\sqrt{\pi}s\sqrt{s}}\int x^2e^{-\pi hctgh(4\pi hs)x^2}dx \\ &= \lim_{s\to 0+}\frac{\phi''(y)}{4\sqrt{\pi}s\sqrt{s}}\cdot\frac{\sqrt{sh(4\pi hs)}}{\sqrt{\pi hch(4\pi hs)}}\cdot\frac{sh(4\pi hs)}{\pi hch(4\pi hs)}\int x^2e^{-x^2}dx. \end{aligned}$$

We have then $B=\phi''(y)$ so for n=1 the infinitesimal generator which clearly depends on h is:

$$A_h\phi(y)=\phi''(y)-4\pi^2h^2y^2\phi(y).$$

In general, the infinitesimal generator is given by:

$$A_h\phi(y)=\Delta\phi(y)-4\pi^2h^2\|y\|^2\phi(y).$$

□

REFERENCES

1. Folland, G.B. – *A Course in Abstract Harmonic Analysis,* Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995.
2. Siebert, E. – *Fourier analysis and limit theorems for convolution semigroups on a locally compact group,* Adv. in Math. 39 (1981), no. 2, 111–154.
3. Barczy, M.; Pap, G. – *Fourier transform of a gaussian measure on the Heisenberg group*, arXiv:math/0511016v2, 2006.
4. Beals, R.; Gaveau, B.; Greiner, P. – *Hamilton-Jacobi theory and the heat kernel on Heisenberg groups,* J. Math. Pures Appl. (9) 79 (2000), no. 7, 633–689.
5. Hulanicki, A. – *The distribution of energy in the Brownian motion in the Gaussian field and analytic-hypoellipticity of certain subelliptic operators on the Heisenberg group,* Studia Math. 56 (1976), no. 2, 165–173.
6. Gaveau, B. – *Principe de moindre action, propagation de la chaleur et estimées sous elliptiques sur certains groupes nilpotents,* Acta Math. 139 (1977), no. 1-2, 95–153.
7. Berg, C.; Forst, G. – *Potential Theory on Locally Compact Abelian Groups*, Springer-Verlag, New York, 1975.
8. Siebert, E. – *A new proof of the generalized continuity theorem of Paul Lévy,* Math. Ann. 233 (1978), no. 3, 257–259.

Received: 17.I.2008

Faculty of Mathematics,
"Al. I. Cuza" University, Iaşi,
ROMÂNIA
pantirucmarian@yahoo.com