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ON THE UNSTABLE INITIAL COMPLEMENT IN EXPONENTIAL DICHOTOMY ON THE HALF-LINE*

BY

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Abstract. The paper emphasizes the recent progress in the study of exponential dichotomy on the half-line in terms of input-output techniques. We discuss the uniqueness problem of the projections family in exponential dichotomy on the half-line. We point out that if a an evolution family is uniformly exponentially dichotomic with respect to a family of projections, then it remains uniformly exponentially dichotomic with a family of projections determined by the arbitrary choice of the initial unstable subspace.

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1. Introduction. One of the most spectacular technique in the study of the existence of exponential dichotomy on the half-line is the input-output method (see [1]-[10]). The main idea is to associate with an evolution family on the half-line an equation and to obtain the uniform exponential dichotomy in terms of the solvability of the associated equation between two spaces - the input space and the output space. Compared to the study of exponential dichotomy on the real line, when we investigate the existence of dichotomy on the half-line we should suppose that the initial stable subspace is closed and complemented. Then, one chooses a complement at the initial moment for the stable subspace and all the investigation is done with respect to that subspace (see [3]-[9]).

Another interesting aspect is related to the exponential growth of the evolution family whose dichotomy is studied. If the evolution family is with

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uniform exponential growth then the appropriate concept of dichotomy is that of uniform exponential dichotomy (see [4], [6]-[9]). If the evolution family is with non-uniform exponential growth then it is natural to determine sufficient conditions for the existence of its non-uniform exponential dichotomy (see [3]). Moreover, we have shown in our papers that admissibility is a very powerful concept, because it implies the existence of the exponential dichotomy corresponding to each case of exponential growth.

The aim of this paper is to bring together our contributions published in the last few years, concerning the study of the exponential dichotomy on the half-line in terms of admissibility. We will give a parallel description of the discrete and continuous case providing technical comments on the qualitative properties of both cases. We will point out that generally the family of projections in exponential dichotomy on the half-line is not uniquely determined. Moreover, we show that if a an evolution family is uniformly exponentially dichotomic with respect to a family of projections, then it remains uniformly exponentially dichotomic with a family of projections determined by the arbitrary choice of the initial unstable subspace.

2. Discrete dichotomy and admissibility on the half-line. Let X be a Banach space. The norm on X and $\mathcal{B}(X)$ - the Banach algebra of all bounded linear operators will be denoted by $||\cdot||$. We denote by I the identity operator on X.

Let $\Delta = \{(m,n) \in \mathbb{N} \times \mathbb{N} : m \geq n\}$.

Definition 2.1. A family $\Phi = \{\Phi(m,n)\}_{(m,n)\in\Delta}$ of bounded linear operators on X is called *discrete evolution family* if the following properties are satisfied:
(i) $\Phi(n,n) = I$, for all $n \in \mathbb{N}$;
(ii) $\Phi(m,n)\Phi(n,k) = \Phi(m,k)$, for all $(m,n),(n,k) \in \Delta$;
(iii) there exist $M \geq 1$ and $\omega > 0$ such that

$$||\Phi(m,n)|| \leq Me^{\omega(m-n)}, \quad \forall (m,n) \in \Delta.$$

Definition 2.2. A discrete evolution family $\Phi = \{\Phi(m,n)\}_{(m,n)\in\Delta}$ is said to be *uniformly exponentially dichotomic* if there exist a family of projections $\{P(n)\}_{n\in\mathbb{N}}$ and two constants $K, \nu > 0$ such that:
(i) $\Phi(m,n)P(n) = P(m)\Phi(m,n)$, for every $(m,n) \in \Delta$;
(ii) the restriction $\Phi(m,n)_{|} : Ker\ P(n) \to Ker\ P(m)$ is an isomorphism, for every $(m,n) \in \Delta$;

(iii) $||\Phi(m,n)x|| \leq Ke^{-\nu(m-n)}||x||$, for all $x \in Im\ P(n)$ and all $(m,n) \in \Delta$;
(iv) $||\Phi(m,n)y|| \geq \frac{1}{K}\ e^{\nu(m-n)}||y||$, for all $y \in Ker\ P(n)$ and all $(m,n) \in \Delta$.

In what follows, we will present the connections between exponential dichotomy and discrete admissibility. Therefore, we will use the following standard notations:

(i) If $p \in [1,\infty)$ we denote by $\ell^p(\mathbb{N},X)=\{s : \mathbb{N}\to X: \sum_{k=0}^{\infty} ||s(k)||^p < \infty\}$ which is Banach space with respect to the norm $||s||_p = (\sum_{k=0}^{\infty} ||s(k)||^p)^{1/p}$.

(ii) Let $\ell^\infty(\mathbb{N},X) = \{s : \mathbb{N} \to X : \sup_{n\in\mathbb{N}} ||s(n)|| < \infty\}$ and respectively $c_0(\mathbb{N},X) = \{s : \mathbb{N} \to X : \lim_{n\to\infty} s(n) = 0\}$. $\ell^\infty(\mathbb{N},X)$ is a Banach space with respect to the norm

$$||s||_\infty = \sup_{n\in\mathbf{N}} ||s(n)||$$

and $c_0(\mathbb{N},X)$ is a linear subspace of $\ell^\infty(\mathbb{N},X)$. We set $c_{00}(\mathbb{N},X) := \{s \in c_0(\mathbb{N},X) : s(0) = 0\}$. Let $\ell^p_0(\mathbb{N},X) = \{s \in \ell^p(\mathbb{N},X) : s(0) = 0\}$.

In all what follows we will consider that $I(\mathbb{N},X)$ is one of the spaces:

$$c_{00}(\mathbb{N},X) \quad \text{or} \quad \ell^q_0(\mathbb{N},X), \quad \text{with } q \in [1,\infty)$$

and that $O(\mathbb{N},X)$ is one of the spaces:

$$c_0(\mathbb{N},X) \quad \text{or} \quad \ell^p(\mathbb{N},X), \quad \text{with } p \in [1,\infty].$$

The concept associated in the study of exponential dichotomy of discrete evolution families is the discrete admissibility. In order to introduce this concept, we associate with an evolution family $\Phi = \{\Phi(m,n)\}_{(m,n)\in\Delta}$, the difference equation:

$$(E_\Phi) \qquad \gamma(n+1) = \Phi(n+1,n)\gamma(n) + s(n+1), \quad n \in \mathbb{N}$$

with $\gamma \in O(\mathbb{N},X)$ and $s \in I(\mathbb{N},X)$, where $I(\mathbb{N},X)$ and $O(\mathbb{N},X)$ are some of the spaces introduced above.

Definition 2.3. The pair $(O(\mathbb{N},X), I(\mathbb{N},X))$ is said to be *admissible* for $\Phi = \{\Phi(m,n)\}_{(m,n)\in\Delta}$ if for every $s \in I(\mathbb{N},X)$ there exists $\gamma \in O(\mathbb{N},X)$ such the pair (γ,s) satisfies the equation (E_Φ).

Definition 2.4. A closed linear subspace $Y_1 \subset X$ is said to be *complemented* if there is a closed linear subspace $Y_2 \subset X$ such that

$$X = Y_1 \oplus Y_2.$$

The initial stable subspace is defined by

$$X_{1,O(\mathbf{N},X)} = \{x \in X : \Phi(\cdot,0)x \in O(\mathbb{N},X)\}.$$

The results published in the last few years concerning discrete exponential dichotomy on the half-line may be stated as:

Theorem 2.5. *Let $\Phi = \{\Phi(m,n)\}_{(m,n)\in\Delta}$ be a discrete evolution family on the Banach space X. Let $p, q \in [1,\infty)$, $p \geq q$ and $r \in (1,\infty]$. The following assertions are equivalent:*
(i) *Φ is uniformly exponentially dichotomic;*
(ii) *the pair $(c_0(\mathbb{N},X),\ c_{00}(\mathbb{N},X))$ is admissible for Φ and the subspace $X_{1,c_0(\mathbf{N},X)}$ is closed and complemented in X;*
(iii) *the pair $(l^\infty(\mathbb{N},X), c_{00}(\mathbb{N},X))$ is admissible for Φ and the subspace $X_{1,\ell^\infty(\mathbf{N},X)}$ is closed and complemented in X;*
(iv) *the pair $(\ell^p(\mathbb{N},X),\ \ell^q_0(\mathbb{N},X))$ is admissible for Φ and the subspace $X_{1,\ell^p(\mathbf{N},X)}$ is closed and complemented in X;*
(v) *the pair $(\ell^\infty(\mathbb{N},X),\ \ell^r_0(\mathbb{N},X))$ is admissible for Φ and the subspace $X_{1,\ell^\infty(\mathbf{N},X)}$ is closed and complemented in X.*

The equivalence $(i) \Longleftrightarrow (ii)$ was proved in [4] (see also [10]). The equivalence $(i) \Longleftrightarrow (iii)$ was obtained in [10] and the equivalence $(i) \Longleftrightarrow (iv)$ was proved in [8], using distinct techniques compared with those involved in the proofs in [4]. The equivalence $(i) \Longleftrightarrow (v)$ was obtained in [9].

The property (iii) from Definition 2.1, meaning that the evolution family is with uniform exponential growth can be removed when we prove the sufficiency in the above equivalences. That means that the admissibility of any of the discrete pairs $(O(\mathbb{N},X), I(\mathbb{N},X))$ considered above implies the existence of the uniform exponential dichotomy of $\Phi = \{\Phi(m,n)\}_{(m,n)\in\Delta}$, in the hypothesis that the corresponding subspace $X_{1,O(\mathbf{N},X)}$ is closed and complemented in X. It follows that this sufficient condition for exponential dichotomy may be applied in the study of difference equations of the type

$$x(n+1) = A(n)x(n), \quad n \in \mathbb{N} \tag{A}$$

where $A(n)$ are arbitrary bounded linear operators on X. Then, the discrete evolution family associated with (A) is of the form

$$\Phi_A(m,n) = \begin{cases} I & , \quad m = n \\ A(m-1)\dots A(n) & , \quad m > n. \end{cases}$$

We note that Φ_A is with uniform exponential growth if and only if the equation (A) has uniformly bounded coefficients, i.e.

$$\sup_{n\in\mathbf{N}} ||A(n)|| < \infty.$$

The property (iii) from Definition 2.1 is used only to prove the necessity, specifically, that if $\Phi = \{\Phi(m,n)\}_{(m,n)\in\Delta}$ is uniformly exponentially dichotomic then the pair $(O(\mathbb{N},X), I(\mathbb{N},X))$ is admissible for Φ and the subspace $X_{1,O(\mathbf{N},X)}$ is closed and complemented in X.

Since in the case of difference equations, the discrete admissibility implies the existence of uniform exponential dichotomy, it follows that the appropriate and correct concept of exponential dichotomy for the case of difference equations is the *uniform exponential dichotomy.*

Another observation is that the admissibility of the pair $(\ell^p(\mathbb{N},X), \ell_0^q(\mathbb{N}, X))$ always implies the existence of the uniform exponential dichotomy, when the subspace $X_{1,\ell^p(\mathbf{N},X)}$ is closed and complemented in X. The admissibility of the pair $(\ell^p(\mathbb{N},X), \ell_0^q(\mathbb{N},X))$ is also a necessary condition for uniform exponential dichotomy if and only if $p \geq q$ and (A) has uniformly bounded coefficients (see [8]).

Theorem 2.6. *If a discrete evolution family $\Phi = \{\Phi(m,n)\}_{(m,n)\in\Delta}$ on a Banach space X is uniformly exponentially dichotomic with respect to a family of projections $\{P(n)\}_{n\in\mathbf{N}}$, then:*
(i) $\sup_{n\in\mathbf{N}} ||P(n)|| < \infty$;
(ii) $Im\ P(0) = X_{1,O(\mathbf{N},X)}$.

For the proof we refer to [4] and [8]. The property given by (ii) is significant and shows that for an exponentially dichotomic discrete evolution family, the initial stable subspaces coincide.

3. Exponential dichotomy of evolution families and discrete admissibility. Let X be a Banach space.

Definition 3.1. A family $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ of bounded linear operators on X is called *evolution family* if the following properties hold:
(i) $U(t,t) = I$, for all $t \in \mathbb{R}_+$;
(ii) $U(t,s)U(s,t_0) = U(t,t_0)$, for all $t \geq s \geq t_0 \geq 0$;
(iii) there exists $M \geq 1$ and $\omega > 0$ such that $||U(t,t_0)|| \leq Me^{\omega(t-t_0)}$, for all $t \geq t_0 \geq 0$;

(iv) for every $x \in X$ and every $t_0 \in \mathbb{R}_+$ the mapping $s \mapsto U(s,t_0)x$ is continuous on $[t_0,\infty)$ and the mapping $t \mapsto U(t_0,t)x$ is continuous on $[0,t_0]$.

Definition 3.2. An evolution family $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ is said to be *uniformly exponentially dichotomic* if there exists a family of projections $\{P(t)\}_{t\geq 0}$ and two constants $K \geq 1$ and $\nu > 0$ such that:
(i) $U(t,t_0)P(t_0) = P(t)U(t,t_0)$, for all $t \geq t_0 \geq 0$;
(ii) $||U(t,t_0)x|| \leq Ke^{-\nu(t-t_0)}||x||$, for all $x \in ImP(t_0)$ and all $t \geq t_0 \geq 0$;
(iii) $||U(t,t_0)y|| \geq \frac{1}{K}\, e^{\nu(t-t_0)}||y||$, for all $y \in KerP(t_0)$ and all $t \geq t_0 \geq 0$;
(iv) the restriction $U(t,t_0)_| : KerP(t_0) \to KerP(t)$ is an isomorphism, for all $t \geq t_0 \geq 0$.

Remark 3.3. If $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ is an evolution family on the Banach space X, then we associate with $\mathcal{U}$ the discrete evolution family:

$$\Phi_{\mathcal{U}}(m,n) = U(m,n), \quad \forall (m,n) \in \Delta.$$

A very important result establishes the connection between dichotomy and discrete dichotomy.

Theorem 3.4. *An evolution family $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ is uniformly exponentially dichotomic if and only if the discrete evolution family $\Phi_{\mathcal{U}}$ is uniformly exponentially dichotomic.*

For the proof we refer to [8]. It follows that on the half-line it is sufficient to study the discrete case. Thus, it makes sense to give:

Definition 3.5. We say that the discrete pair $(O(\mathbb{N},X), I(\mathbb{N},X))$ is *admissible* for $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ if it is admissible for $\Phi_{\mathcal{U}}$.

As a consequence of Theorem 2.5 and Theorem 3.4 we immediately deduce:

Theorem 3.6. *Let $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ be an evolution family on the Banach space X. Let $p,q \in [1,\infty)$, $p \geq q$ and $r \in (1,\infty]$. The following assertions are equivalent:*
(i) $\mathcal{U}$ is uniformly exponentially dichotomic;
(ii) the pair $(c_0(\mathbb{N},X),\ c_{00}(\mathbb{N},X))$ is admissible for $\mathcal{U}$ and the subspace $X_{1,c_0(\mathbf{N},X)}$ is closed and complemented in X;
(iii) the pair $(l^\infty(\mathbb{N},X), c_{00}(\mathbb{N},X))$ is admissible for $\mathcal{U}$ and the subspace $X_{1,\ell^\infty(\mathbf{N},X)}$ is closed and complemented in X;

(iv) the pair $(\ell^p(\mathbb{N},X),\ \ell^q_0(\mathbb{N},X))$ *is admissible for* $\mathcal{U}$ *and the subspace* $X_{1,\ell^p(\mathbf{N},X)}$ *is closed and complemented in* X*;*
(v) the pair $(\ell^\infty(\mathbb{N},X),\ \ell^r_0(\mathbb{N},X))$ *is admissible for* $\mathcal{U}$ *and the subspace* $X_{1,\ell^\infty(\mathbf{N},X)}$ *is closed and complemented in* X*.*

4. Exponential dichotomy and integral admissibility. It is obvious, that not in all cases the reduction to the discrete case is the real solution. Sometimes, it is easier to estimate an integral than the sum of a series. That is why the integral admissibility has an individual spectrum of applicability and why not, its own poetry. But, before speaking about this case, we need to remind some basic notations.

Let X be a Banach space. Let $L^p(\mathbb{R}_+,X)$ denote the space of all Bohner measurable functions $f:\mathbb{R}_+\to X$ with the property that $\int_0^\infty ||f(\tau)||^p\ d\tau<\infty$, which is a Banach space with respect to the norm:

$$||f||_p=\left(\int_0^\infty ||f(\tau)||^p\ d\tau\right)^{1/p}.$$

Let $C_b(\mathbb{R}_+,X)$ be the space of all continuous bounded functions $f:\mathbb{R}_+\to X$, and $C_0(\mathbb{R}_+,X)=\{f\in C_b(\mathbb{R}_+,X):\lim_{t\to\infty}f(t)=0\}$. These are Banach spaces with respect to the norm

$$||f||_\infty:=\sup_{t\geq 0}||f(t)||.$$

We set $C_{00}(\mathbb{R}_+,X)=\{f\in C_0(\mathbb{R}_+,X):f(0)=0\}$.

In all what follows we suppose that $V(\mathbb{R}_+,X)$ is one of the spaces

$$L^p(\mathbb{R}_+,X),\quad \text{with } p\in[1,\infty),\quad C_0(\mathbb{R}_+,X)\quad \text{or}\quad C_{00}(\mathbb{R}_+,X)$$

and $W(\mathbb{R}_+,X)$ is one of the spaces:

$$L^p(\mathbb{R}_+,X),\quad \text{with } p\in[1,\infty),\quad C_0(\mathbb{R}_+,X)\quad \text{or}\quad C_b(\mathbb{R}_+,X).$$

If $\mathcal{U}=\{U(t,s)\}_{t\geq s\geq 0}$ is an evolution family on X, then we associate the integral equation

$$(E_{\mathcal{U}})\qquad f(t)=U(t,s)f(s)+\int_s^t U(t,\tau)v(\tau)\ d\tau,\quad \forall t\geq s\geq 0$$

where $f \in W(\mathbb{R}_+, X)$ and $v \in V(\mathbb{R}_+, X)$.

We consider the initial stable subspace:

$$X_{1,W(\mathbb{R}_+,X)} = \{x \in X : U(\cdot, 0)x \in W(\mathbb{R}_+, X)\}.$$

Definition 4.1. The pair $(W(\mathbb{R}_+, X), V(\mathbb{R}_+, X))$ is said to be *admissible* for $\mathcal{U}$ if for every $v \in V(\mathbb{R}_+, X)$ there exists a function $f \in W(\mathbb{R}_+, X)$ such that the pair (f, v) satisfies the equation $(E_{\mathcal{U}})$.

Theorem 4.2. *Let $\mathcal{U} = \{U(t, s)\}_{t \geq s \geq 0}$ be an evolution family on the Banach space X.*

Let $n \in \mathbb{N}^*$, let $p, q_1, ..., q_n \in [1, \infty)$ with $\min\{q_1, ..., q_n\} \leq p$ and let

$$W^{q_1,...,q_n}(\mathbb{R}_+, X) = L^{q_1}(\mathbb{R}_+, X) \cap ... \cap L^{q_n}(\mathbb{R}_+, X) \cap C_{00}(\mathbb{R}_+, X).$$

Let $r_1, \ldots, r_n \in (1, \infty)$ and

$$V^{r_1,...,r_n}(\mathbb{R}_+, X) = L^{r_1}(\mathbb{R}_+, X) \cap \ldots L^{r_n}(\mathbb{R}_+, X) \cap C_{00}(\mathbb{R}_+, X).$$

The following assertions are equivalent:
(i) *$\mathcal{U}$ is uniformly exponential dichotomic;*
(ii) *the pair $(C_0(\mathbb{R}_+, X), C_0(\mathbb{R}_+, X))$ is admissible for $\mathcal{U}$ and the subspace $X_{1,C_0(\mathbb{R}_+,X)}$ is closed and complemented in X;*
(iii) *the pair $(C_0(\mathbb{R}_+, X), C_{00}(\mathbb{R}_+, X))$ is admissible for $\mathcal{U}$ and the subspace $X_{1,C_0(\mathbb{R}_+,X)}$ is closed and complemented in X;*
(iv) *the pair $(L^p(\mathbb{R}_+, X) \cap C_b(\mathbb{R}_+, X),\ L^p(\mathbb{R}_+, X))$ is admissible for $\mathcal{U}$ and the subspace $X_{1,L^p(\mathbb{R}_+,X)}$ is closed and complemented in X;*
(v) *the pair $(L^p(\mathbb{R}_+, X), W^{q_1,...,q_n}(\mathbb{R}_+, X))$ is admissible for $\mathcal{U}$ and $X_{1,L^p(\mathbb{R}_+,X)}$ is closed and complemented in X;*
(vi) *the pair $(C_b(\mathbb{R}_+, X), V^{r_1,...,r_n}(\mathbb{R}_+,\ X))$ is admissible for $\mathcal{U}$ and the subspace $X_{1,C_b(\mathbb{R}_+,X)}$ is closed and complemented in X.*

The equivalence $(i) \Longleftrightarrow (ii)$ was proved by Minh, Räbiger and Schnaubelt in [6]. Their result was generalized in [4], where using discrete techniques we obtained the equivalence $(i) \Longleftrightarrow (iii)$. We note that moreover, the sufficiency holds even when the evolution family is with nonuniform exponential growth (see [3]), in the sense that the admissibility of the pair $(C_0(\mathbb{R}_+, X), C_{00}(\mathbb{R}_+, X))$ for a non-uniform evolution family implies the existence of non-uniform exponential dichotomy, if the subspace $X_{1,C_0(\mathbb{R}_+,X)}$ is closed and complemented in X.

The equivalence $(i) \Longleftrightarrow (iv)$ was deduced in [7]. It was for the first time, when one proposes the admissibility with L^p-spaces.

It is well known, that in the study of the exponential dichotomy via admissibility it is natural to work with input spaces as small as possible and with output spaces as large as possible. That is why in admissibility we consider test functions from the input space and verify the ownership of the solution of the integral equation to the output space. Thus, one notes that the equivalence $(i) \Longleftrightarrow (iv)$ is not quite an optimal characterization since the output space is more restrictive compared to the input one. The situation was improved in [8] and [9]. The equivalence $(i) \Longleftrightarrow (v)$ was deduced in [8], using discrete time techniques and the equivalence $(i) \Longleftrightarrow (vi)$ was proved in [9]. In both cases, we note that the input space may be considered smaller and smaller. Moreover, the class of these kind of input spaces is closed to finite intersections of spaces.

We note that all the characterizations for exponential dichotomy on the half-line in terms of the admissibility are developed in the generic background that the space $X_{1,W(\mathbb{R}_+,X)}$ is closed and complemented in X, i.e. there exists a closed linear subspace $X_{2,W(\mathbb{R}_+,X)}$ such that

$$X = X_{1,W(\mathbb{R}_+,X)} \oplus X_{2,W(\mathbb{R}_+,X)}.$$

For every $(x,t) \in X \times \mathbb{R}_+$ let

$$\varphi_{x,t} : \mathbb{R}_+ \to X, \quad \varphi_{x,t}(\tau) = \begin{cases} U(\tau,t)x & , \quad \tau \geq t \\ 0 & , \quad \tau < t. \end{cases}$$

For $t \in \mathbb{R}_+$, we set

$$X_{1,W(\mathbb{R}_+,X)}(t) = \{x \in X : \varphi_{x,t} \in W(\mathbb{R}_+, X)\}$$

and respectively

$$X_{2,W(\mathbb{R}_+,X)}(t) = U(t,0)X_{2,W(\mathbb{R}_+,X)}.$$

Theorem 4.3. *If an evolution family $\mathcal{U} = \{U(t,s)\}_{t \geq s \geq 0}$ is uniformly exponentially dichotomic with respect to a family of projections $\{P(t)\}_{t \in \mathbb{R}_+}$, then:*

(i) $\sup_{t \in \mathbb{R}_+} ||P(t)|| < \infty$;

(ii) $Im\ P(0) = X_{1,W(\mathbb{R}_+,X)}$.

It follows that if an evolution family is uniformly exponentially dichotomic, then all the initial stable subspaces coincide and are equal with the range of the projection at the zero moment. But, that does not implies the uniqueness of the projection family, as the following example shows:

Example. Let $X = \mathbb{R}^2$, which is a Banach space with respect to the norm $||(x_1, x_2)|| = max\{|x_1|, |x_2|\}$.

We consider the evolution family $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$, given by

$$U(t,s)(x_1,x_2) = (e^{-(t-s)}x_1, e^{t-s}x_2), \quad \forall (x_1,x_2) \in X.$$

Let $h \geq 1$. For every $t_0 \geq 0$, consider the subspaces

$$X_1(t_0) = \mathbb{R} \times \{0\}$$

and

$$X_2^h(t_0) = \{(e^{-t_0}r, he^{t_0}r) : r \in \mathbb{R}\}.$$

It is obvious that

$$X_1(t_0) \oplus X_2^h(t_0) = X, \quad \forall t_0 \geq 0.$$

For every $t_0 \geq 0$, let $P_h(t_0)$ be the projection with $Im\ P_h(t_0) = X_1(t_0)$ and $Ker\ P_h(t_0) = X_2^h(t_0)$. Then a simple computation shows that

$$U(t,t_0)P_h(t_0) = P_h(t_0)U(t,t_0), \quad \forall t \geq t_0 \geq 0.$$

Moreover, it is easily checked that $U(t,t_0)_| : X_2^h(t_0) \to X_2^h(t)$ is an isomorphism, for all $t \geq t_0 \geq 0$.

We observe that

$$||U(t,t_0)(x_1,0)|| = e^{-(t-t_0)}|x_1|$$

$$= e^{-(t-t_0)}||(x_1,0)||, \quad \forall (x_1,0) \in Im\ P_h(t_0), \forall t \geq t_0 \geq 0.$$

Let $t \geq t_0 \geq 0$ and $w \in X_2^h(t_0)$. Then there is $r \in \mathbb{R}$ such that $w = (e^{-t_0}r, he^{t_0}r)$. We note that

$$||w|| = max\{e^{-t_0}|r|, he^{t_0}|r|\} = he^{t_0}|r|$$

and

$$||U(t,t_0)w|| = max\{e^{-t}|r|, he^{t}|r|\} = he^{t}|r| = e^{t-t_0}||w||.$$

It follows that

$$||U(t,t_0)w|| \geq e^{t-t_0}||w||, \quad \forall w \in X_2^h(t_0), \forall t \geq t_0 \geq 0.$$

In conclusion, for every $h \geq 1$, the family $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ is uniformly exponentially dichotomic with respect to the family of projections $\{P_h(t)\}_{t\geq 0}$.

It follows that on the half-line an evolution family may be uniformly exponentially dichotomic with respect to an infinite number of projections family.

Another interesting question arises: how important is the choice of the initial unstable complement? It is clear that the estimations done depend on its choice. But, we'll see that from qualitative point of view the choice may be arbitrary.

Indeed suppose that an evolution family $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ is uniformly exponentially dichotomic with respect to a family of projections $\{P(t)\}_{t\geq 0}$. Then, according to Theorem 4.3 we have that

$$X_{1,C_0(\mathbb{R}_+,X)} = Im\ P(0).$$

Let $\tilde{X}_{2,C_0(\mathbb{R}_+,X)}$ be such that $X = X_{1,C_0(\mathbb{R}_+,X)} \oplus \tilde{X}_{2,C_0(\mathbb{R}_+,X)}$. For every $t \in \mathbb{R}_+$, we set as always

$$X_{1,C_0(\mathbb{R}_+,X)}(t) = \{x \in X : \varphi_{x,t} \in C_0(\mathbb{R}_+, X)\}$$

and respectively

$$\tilde{X}_{2,C_0(\mathbb{R}_+,X)}(t) = U(t,0)\tilde{X}_{2,C_0(\mathbb{R}_+,X)}.$$

Since $\mathcal{U}$ is uniformly exponentially dichotomic, the pair $(C_0(\mathbb{R}_+, X), C_{00}(\mathbb{R}_+, X))$ is admissible for $\mathcal{U}$. Then (see e.g. [5]) there are $K, \nu > 0$ such that:
(i) $||U(t,t_0)x|| \leq Ke^{-\nu(t-t_0)}||x||$, for all $x \in X_{1,C_0(\mathbb{R}_+,X)}(t_0)$ and all $t \geq t_0 \geq 0$;
(ii) $||U(t,t_0)y|| \geq \frac{1}{K}e^{\nu(t-t_0)}||y||$, for all $y \in \tilde{X}_{2,C_0(\mathbb{R}_+,X)}(t_0)$ and all $t \geq t_0 \geq 0$;
where from we immediately deduce that for every $t_0 \geq 0$, $X_{1,C_0(\mathbb{R}_+,X)}(t_0)$ is a closed linear subspace as a consequence of (i) and that $\tilde{X}_{2,C_0(\mathbb{R}_+,X)}(t_0)$

is a closed linear subspace as a consequence of (ii). Also, a consequence of (i) and (ii) is that $X_{1,C_0(\mathbf{R}_+,X)}(t_0) \cap \tilde{X}_{2,C_0(\mathbf{R}_+,X)}(t_0) = \{0\}$.

Let $t_0 > 0$ and $x \in X$. Let $\alpha : \mathbf{R}_+ \to \mathbf{R}$ be a function with compact support such that supp $\alpha \subset (t_0, \infty)$ and $\int_{t_0}^{\infty} \alpha(s)\, ds = 1$. Let

$$f_1 : [t_0, \infty) \to X, \quad f_1(t) = \int_t^{\infty} \alpha(s)\, ds\; U(t, t_0)x$$

and

$$u : \mathbf{R}_+ \to X, \quad u(t) = -\alpha(t)U(t, t_0)x.$$

From the admissibility there is $f_2 \in C_0(\mathbb{R}_+, X)$ such that (f_2, u) satisfies the equation $(E_{\mathcal{U}})$. It follows that

$$f_1(t) - f_2(t) = U(t, t_0)(f_1(t_0) - f_2(t_0)), \quad \forall t \geq t_0$$

so $\lim_{t\to\infty} U(t, t_0)(f_1(t_0) - f_2(t_0)) = 0$. Thus, $f_1(t_0) - f_2(t_0) = x - f_2(t_0) \in X_{1,C_0(\mathbf{R}_+,X)}(t_0)$. On the other hand, $f_2(t_0) = U(t_0, 0)f_2(0)$. Moreover, there are $x_1 \in X_{1,C_0(\mathbf{R}_+,X)}$ and $x_2 \in \tilde{X}_{2,C_0(\mathbf{R}_+,X)}$ such that $f_2(0) = x_1 + x_2$. It follows that $x = [f_1(t_0) - f_2(t_0) + U(t_0, 0)x_1] + U(t_0, 0)x_2 X_{1,C_0(R_+,X)}(t_0) + \widetilde{X}_{2,C_0(R_+,X)}(t_0)$. So,

$$X = X_{1,C_0(\mathbf{R}_+,X)}(t_0) \oplus \tilde{X}_{2,C_0(\mathbf{R}_+,X)}(t_0), \quad \forall t_0 \geq 0.$$

It follows that there is a family of projections $\{\tilde{P}(t)\}_{t\geq 0}$ such that

$$Im\ \tilde{P}(t_0) = X_{1,C_0(\mathbf{R}_+,X)}(t_0) \text{ and } Ker\ \tilde{P}(t_0) = \tilde{X}_{2,C_0(\mathbf{R}_+,X)}(t_0), \quad \forall t_0 \geq 0.$$

An easy computation shows that property (i) from Definition 3.2 is also fulfilled, so the evolution family $\mathcal{U}$ is uniformly exponentially dichotomic with the family of projections determined by the choice of $\tilde{X}_{2,C_0(\mathbf{R}_+,X)}$.

In conclusion, if an evolution family on the half-line is uniformly exponentially dichotomic then it remains uniformly exponentially dichotomic with respect to a family of projections determined by an arbitrary choice of the initial unstable subspace.

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