ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" DIN IAŞI (S.N.)
MATEMATICĂ, Tomul LIV, 2008, f.2

ASYMPTOTIC BEHAVIOUR FOR A NONLINEAR AGE-STRUCTURED POPULATION MODEL WITH DIFFUSION*

BY

OANA CARMEN TARNICERIU

Abstract. The internal zero-stabilization of the nonnegative solution to a nonlinear age-dependent population model with diffusion is investigated in this paper. It is provided here a necessary condition and a sufficient condition for the nonnegative zero-stabilizability in terms of the value of the principal eigenvalue to a certain elliptic operator. This principal eigenvalue is related to the rate of the convergence of the solution.

Mathematics Subject Classification 2000: 35B37.

Key words: age-structured population models with diffusion, asymptotic behavior, stabilizati.

1. Introduction and setting of the problem. In this paper we are concerned in the large time behaviour of population dynamics with age-dependence and spatial diffusion.

Let $\Omega \subset \mathbf{R}^N$, $N \geq 2$, be a bounded domain and $\omega \subset\subset \Omega$ be a nonempty open subset, both with smooth enough boundaries $\partial\Omega$ and $\partial\omega$, respectively, and such that $\Omega \backslash \omega$ is a domain.

Our starting point is the following system

$$(1.1) \quad \begin{cases} y_t + y_a + \mu(a)y - k\Delta y + \Phi(Y(t,x))y = 0, & (a,t,x) \in Q, \\ Y(t,x) = \int_0^A y(a,t,x)\,\mathrm{d}a, & (t,x) \in (0,+\infty) \times \Omega, \\ y(0,t,x) = \int_0^A \beta(a)y(a,t,x)\,\mathrm{d}a, & (t,x) \in (0,+\infty) \times \Omega, \\ \frac{\partial y}{\partial \nu}(a,t,x) = 0, & (a,t,x) \in \Sigma, \\ y(a,0,x) = y_0(a,x), & (a,x) \in (0,A) \times \Omega, \end{cases}$$

*This research was supported by the grant ID 381/2007

which models the dynamics of an age-structured population over the habitat Ω.

Here $Q = (0, A) \times (0, +\infty) \times \Omega$, $\Sigma = (0, A) \times (0, +\infty) \times \partial\Omega$, where $A \in (0, +\infty)$ represents the maximal age for the population species, and $k \in (0, +\infty)$ is the diffusivity constant of the population.

$y(a, t, x)$ is the population density of age $a \in [0, A]$ at time $t \in [0, +\infty)$ in the location $x \in \overline{\Omega}$ and by $y_0(a, x)$ we have denoted the initial distribution of densities.

The third equation in (1.1) describes the birth process and it is known as the *renewal equation*; here $y(0, t, x)$ gives the density of the newborn population at time t in the location x, and β is the fertility rate which depends in this case only on age a. Therefore $\beta(a)y(a, t, x)$ stands for the density of newborns at time t, with parents of age a in the location x.

We have prescribed homogeneous Neumann conditions on the boundary $\partial\Omega$, corresponding to the case of isolated populations.

The model includes a logistic term $\Phi(Y(t, x))$ which represents an additional mortality rate due to the overpopulation, depending on the total population density $Y(t, x)$.
For basic results concerning (1.1) we refer to the monographs [4] and [9].

Throughout this paper we consider the following standing assumptions, except for the cases when we shall specify otherwise.

A1) $\beta \in L^\infty(0, A)$, $\beta(a) \geq 0$ a.e. $a \in (0, A)$;

A2) $\mu \in L^1_{loc}([0, A))$, $\mu(a) \geq 0$ a.e. $a \in (0, A)$;

A3) $y_0 \in L^\infty((0, A) \times \Omega)$, $y_0(a, x) \geq 0$ a.e. $(a, x) \in (0, A) \times \Omega$;

A4) $\Phi : [0, +\infty) \mapsto [0, +\infty)$ is an increasing continuously differentiable function which satisfies $\Phi(0) = 0$ and $\lim_{r\to+\infty} \Phi(r) = +\infty$.

The last condition is natural because of the biological meaning of the function Φ.

The main goal of this paper is the study of the following control problem:

$$
(1.2)\qquad \begin{cases}
y_t + y_a + \mu(a)y - k\Delta y + \Phi(Y(t,x))y & \\
\qquad = m(x)u(a,t,x), & (a,t,x) \in Q,\\
Y(t,x) = \displaystyle\int_0^A y(a,t,x)\,\mathrm{d}a, & (t,x) \in (0,+\infty)\times\Omega,\\
y(0,t,x) = \displaystyle\int_0^A \beta(a)y(a,t,x)\,\mathrm{d}a, & (t,x) \in (0,+\infty)\times\Omega,\\
\dfrac{\partial y}{\partial \nu}(a,t,x) = 0, & (a,t,x) \in \Sigma,\\
y(a,0,x) = y_0(a,x), & (a,x) \in (0,A)\times\Omega,
\end{cases}
$$

where $u(a,t,x)$ is a control function (which depends on a, t and x) and $m(x)$ is the characteristic function of the subset ω, meaning

$$
(1.3)\qquad m(x) = \begin{cases} 1, & \text{if } x \in \omega,\\ 0, & \text{if } x \in \Omega\setminus\bar{\omega}. \end{cases}
$$

Namely, we are looking for the answer to the the following question:

"Is there any control $u \in L^\infty_{loc}([0,A]\times[0,+\infty)\times\bar{\omega})$ *such that the solution to* (1.2) *satisfies*

$$
(1.4)\qquad y(a,t,x) \geq 0, \qquad a.e. \qquad (a,t,x) \in Q,
$$

and

$$
(1.5)\qquad \lim_{t\to\infty} y(t) = 0 \quad in \quad L^\infty((0,A)\times\Omega)?\text{"}
$$

Definition. If the answer to the above mentioned question is affirmative for any y_0 satisfying **A3**, then we say that the system (1.2) is *nonnegatively zero-stabilizable*, and the property is called *nonnegative zero-stabilizability.*

This is in fact null stabilizability under state constraints. In this case it is also desirable to find a feedback control u that realizes (1.4) and (1.5).

In the case $\Phi \equiv 0$, it has been proved in [1] that there is a deep relationship between the nonnegative zero-stabilizability of (1.2) and the value of the principal eigenvalue of the following problem:

$$
(1.6)\qquad \begin{cases}
-k\Delta\varphi = \lambda\varphi, & \text{in } \Omega\setminus\bar{\omega},\\
\varphi = 0, & \text{on } \partial\omega,\\
\dfrac{\partial\varphi}{\partial\nu} = 0 & \text{on } \partial\Omega.
\end{cases}
$$

In [8] some large time behaviour results for a nonlinear age-dependent system with spatial diffusion have been obtained.

The large time behaviour of age-dependent logistic population models (without diffusion) has been investigated in [6].

For stabilizability results for parabolic equations we refer to [2] and [3]. For basic results concerning the elliptic and parabolic equations in L^k frame, we refer to [5].

The next results and notations will be needed in our paper. Considering the linear model

$$\begin{cases} y_t + y_a + \mu(a)y - k\Delta y = 0, & (a,t,x) \in Q, \\ y(0,t,x) = \displaystyle\int_0^A \beta(a)y(a,t,x)\,\mathrm{d}a, & (t,x) \in (0,+\infty) \times \Omega, \\ \dfrac{\partial y}{\partial \nu}(a,t,x) = 0, & (a,t,x) \in \Sigma, \\ y(a,0,x) = y_0(a,x), & (a,x) \in (0,A) \times \Omega, \end{cases} \tag{1.7}$$

we shall denote by

$$\pi(a) = \exp\left(-\int_0^a \mu(s)\,\mathrm{d}s\right)$$

and by α^* the root of the characteristic equation

$$1 = \int_0^A \beta(a)\pi(a)\exp(-\alpha a)\,\mathrm{d}a.$$

Then, denoting by λ_1 the principal eigenvalue for the operator $-k\Delta$ on Ω, subject to Neumann boundary conditions, the large-time behaviour of the solution to (1.7) depends on α^* and λ_1 in the following way:

1. If $\alpha^* < \lambda_1$, then $\|y(\cdot,t,\cdot)\|_{L^2((0,A)\times\Omega)} \longrightarrow 0$ as $t \to +\infty$;
2. If $\alpha^* > \lambda_1$, then $\|y(\cdot,t,\cdot)\|_{L^2((0,A)\times\Omega)} \longrightarrow +\infty$ as $t \to +\infty$;
3. If $\alpha^* = \lambda_1$, then $\|y(\cdot,t,\cdot) - c(y_0)\pi(a)\|_{L^2((0,A)\times\Omega)} \longrightarrow 0$ as $t \to +\infty$, where $c(y_0) \in \mathbf{R}^+$.

For details see [7] for the system without diffusion and [8] for the system with diffusion.

The paper is organized as follows.

In Section 2 the notion of a *separable solution* to the problem (1.1) is introduced and the asymptotic behaviour of such a solution is investigated.

In Section 3 the main result of our paper, concerning the nonnegative zero-stabilizability of the solution to (1.2), is stated.

2. Separable solutions. Let us recall first a result that would be helpful in what will follow. Considering the linear age-structured system

$$
\begin{cases}
\overline{h}_t(a,t) + \overline{h}_a(a,t) + \mu(a)\overline{h}(a,t) = 0, & (a,t) \in (0,A) \times (0,+\infty), \\
\overline{h}(0,t) = \displaystyle\int_0^A \beta(a)\overline{h}(a,t)\,\mathrm{d}a, & t \in (0,+\infty), \\
\overline{h}(a,0) = h_0(a), & a \in (0,A),
\end{cases} \tag{2.1}
$$

with $h_0 \in L^\infty(0,A)$, $h_0(a) > 0$ a.e. in $(0,A)$, the solution $\overline{h}$ is given by (see [4], Chapter 2 for details)

$$
\overline{h}(a,t) = \exp\{\alpha^*(t-a)\}b_0(t-a)\exp\{-\int_0^a \mu(s)\,\mathrm{d}s\}, \tag{2.2}
$$

for almost any $(a,t) \in (0,A)\times(0,+\infty)$, $a < t$, where $\lim_{t\to\infty} b_0(t) = \bar{b}_0 \geq 0$, and α^* is the root of the characteristic equation introduced in the first section.

Then, for any nontrivial datum h_0, the asymptotic behaviour of $\overline{h}$ is given by:

$$
\lim_{t\to\infty} \|\overline{h}(t)\|_{L^\infty(0,A)} = 0, \quad if \quad \alpha^* < 0;
$$

$$
\lim_{t\to\infty} \|\overline{h}(t)\|_{L^1(0,A)} = +\infty, \quad if \quad \alpha^* > 0;
$$

$$
\lim_{t\to\infty} \|\overline{h}(t) - h^0\|_{L^\infty(0,A)} = 0, \quad if \quad \alpha^* = 0.
$$

where h^0 is a stationary solution to $(2.1)_{1-2}$.

Remark 1. *If h_0 is a trivial datum, then*

$$
\lim_{t\longrightarrow\infty} \|\overline{h}(t)\|_{L^\infty(0,A)} = 0.
$$

Next notice that the problem (1.1) can be written in an equivalent form as:

$$
\begin{cases}
y_t + y_a + \mu(a)y - k\Delta y + \Phi(Y(t,x))y & \\
\qquad -\alpha^* y + \alpha^* y = 0, & (a,t,x) \in Q, \\
Y(t,x) = \displaystyle\int_0^A y(t,a,x)\,\mathrm{d}a, & (t,x) \in (0,+\infty) \times \Omega, \\
y(0,t,x) = \displaystyle\int_0^A \beta(a)y(a,t,x)\,\mathrm{d}a, & (t,x) \in (0,+\infty) \times \Omega, \\
\dfrac{\partial y}{\partial \nu}(a,t,x) = 0, & (a,t,x) \in \Sigma, \\
y(a,0,x) = y_0(a,x), & (a,x) \in (0,A) \times \Omega.
\end{cases} \tag{2.3}
$$

Let us consider the "separable" problem, i.e. we consider the initial data as

$$y_0(a,x) = h_0(a)g_0(x), \quad (a,x) \in (0,A) \times \Omega,$$

with $g_0 \in L^\infty(\Omega)$, $g_0(x) > 0$ a.e. in Ω, and we look for a solution to the system in the following form

$$y(a,t,x) = h(a,t)g(t,x), \quad (a,t,x) \in Q,$$

where h satisfies

$$\begin{cases} h_t + h_a + \mu(a)h + \alpha^* h = 0, & (a,t) \in (0,A) \times (0,+\infty) \\ h(0,t) = \displaystyle\int_0^A \beta(a)h(a,t)\,\mathrm{d}a, & t \in (0,+\infty), \\ h(a,0) = h_0(a), & a \in (0,A), \end{cases} \tag{2.4}$$

and g is the solution to the following system:

$$\begin{cases} g_t - k\Delta g + \Phi(H(t)g)g - \alpha^* g = 0, & (t,x) \in (0,+\infty) \times \Omega, \\ H(t) = \displaystyle\int_0^A h(a,t)\,\mathrm{d}a, & t \in (0,+\infty), \\ \dfrac{\partial g}{\partial \nu}(t,x) = 0, & (t,x) \in (0,+\infty) \times \partial\Omega, \\ g(0,x) = g_0(x), & x \in \Omega. \end{cases} \tag{2.5}$$

It is easy to verify that $y(a,t,x) = h(a,t)g(t,x)$ is a solution to (2.3). Let us prove that the solution h to (2.4), and the solution g to (2.5) satisfy:

$$\lim_{t \longrightarrow \infty} \|h(t) - \tilde{h}\|_{L^\infty(0,A)} = 0, \tag{2.6}$$

where $\tilde{h}$ is a stationary solution to $(2.4)_{1-2}$, respectively

$$\lim_{t \longrightarrow \infty} \|g(t) - \tilde{g}\|_{L^\infty(0,A)} = 0, \tag{2.7}$$

where, $\tilde{g}$ is a stationary solution to $(2.5)_{1,3}$. Actually the solution to (2.4) can be written as

$$h(a,t) = e^{-\alpha^* t}\overline{h}(a,t), \quad (a,t) \in (0,A) \times (0,+\infty),$$

where $\overline{h}$ is the solution to (2.1). Then, by (2.2), we obtain that h satisfies (2.6), where

$$\tilde{h}(a) = \exp\{-\alpha^* a\} \exp\{-\int_0^a \mu(s)\,\mathrm{d}s\}\bar{b}_0, \quad \forall a \in (0,A),$$

which is obviously a stationary solution to $(2.4)_{1-2}$. The set of all steady states to $(2.4)_{1-2}$ is a linear space of dimension one.

Therefore, it remains to analyze the large time behaviour of (2.5). Due to (2.6), we obtain

$$\lim_{t\longrightarrow\infty} H(t) = H_0,$$

where $H_0 = \int_0^A \tilde{h}(a)\,\mathrm{d}a > 0$. Equivalently, we get that for any $\delta > 0$, there exists $t_\delta \geq 0$, such that, for any $t \geq t_\delta$:

$$H_0 - \delta \leq H(t) \leq H_0 + \delta.$$

Due to the assumption **A4**, we have that

$$\Phi((H_0-\delta)g(t,x)) \leq \Phi(H(t)g(t,x)) \leq \Phi((H_0+\delta)g(t,x)),\ \forall t \geq t_\delta,\ a.e.\ x \in \Omega.$$

Let us consider now the following problem:

$$\begin{cases} l_t - k\Delta l + \Phi(Cl)l - \alpha^* l = 0, & (t,x) \in (t_\delta, +\infty) \times \Omega \\ \dfrac{\partial l}{\partial \nu}(t,x) = 0, & (t,x) \in (t_\delta, +\infty) \times \partial\Omega \\ l(t_\delta, x) = g(t_\delta, x), & x \in \Omega, \end{cases}$$

and denote by g^1 and g^2 the corresponding solutions to $C := H_0 - \delta$ and $C := H_0 + \delta$ respectively. Then, by the comparison principle we obtain that

$$g^2(t,x) \leq g(t,x) \leq g^1(t,x), \quad \forall t \geq t_\delta, \quad a.e. \quad x \in \Omega.$$

Now let $\theta_0 > 0$ be a small enough constant and $\theta_1 > 0$ a large enough constant, such that

$$\theta_0 \leq g(t_\delta, x) \leq \theta_1, \quad a.e. \quad x \in \Omega,$$

and let l^1 and l^2 the solutions to the following problems, respectively:

$$\begin{cases} l^1_t - k\Delta l^1 + \Phi((H_0 - \delta)l^1)l^1 - \alpha^* l^1 = 0, & (t,x) \in (t_\delta, +\infty) \times \Omega \\ \dfrac{\partial l^1}{\partial \nu}(t,x) = 0, & (t,x) \in (t_\delta, +\infty) \times \partial\Omega \\ l^1(t_\delta, x) = \theta_1, & x \in \Omega, \end{cases}$$

and

$$\begin{cases} l^2_t - k\Delta l^2 + \Phi((H_0 + \delta)l^2)l^2 - \alpha^* l^2 = 0, & (t,x) \in (t_\delta, +\infty) \times \Omega \\ \dfrac{\partial l^2}{\partial \nu}(t,x) = 0, & (t,x) \in (t_\delta, +\infty) \times \partial\Omega \\ l^2(t_\delta, x) = \theta_0, & x \in \Omega. \end{cases}$$

Applying once again the comparison principle, we get that

$$l^2(t,x) \le g^2(t,x) \le g(t,x) \le g^1(t,x) \le l^1(t,x),\ \forall t \ge t_\delta,\ a.e.\ x \in \Omega. \tag{2.8}$$

The solutions l^1 and l^2 do not depend explicitly on x; actually they satisfy:

$$\begin{cases} (l^1)' + \Phi((H_0-\delta)l^1)l^1 - \alpha^* l^1 = 0, & t \in (t_\delta, +\infty) \\ l^1(t_\delta) = \theta_1, \end{cases} \tag{2.9}$$

and

$$\begin{cases} (l^2)' + \Phi((H_0+\delta)l^2)l^2 - \alpha^* l^2 = 0, & t \in (t_\delta, +\infty) \\ l^2(t_\delta) = \theta_0, \end{cases} \tag{2.10}$$

respectively.

In the case $\alpha^* \le 0$, we obtain

$$\lim_{t\longrightarrow\infty} l^j(t) = 0, \qquad j = \overline{1,2}, \tag{2.11}$$

and consequently

$$\lim_{t\longrightarrow\infty} y(t) = 0, \qquad \text{in} \quad L^\infty((0,A)\times\Omega).$$

Therefore it remains to analyze only the case $\alpha^* > 0$.
Actually, in this case, the solutions l^1, l^2 satisfy

$$\lim_{t\longrightarrow\infty} l^j(t) = \frac{1}{C}\Phi^{-1}(\alpha^*), \quad j = \overline{1,2}, \tag{2.12}$$

with $C := H_0 - \delta$ and $C := H_0 + \delta$, respectively. Then, using (2.12) and making $t \to +\infty$ in (2.8), we get that

$$\frac{1}{H_0+\delta}\Phi^{-1}(\alpha^*) \le \lim_{t\to\infty} \operatorname{ess\,inf} g(t,x) \le \lim_{t\to\infty} \operatorname{ess\,sup} g(t,x) \le \frac{1}{H_0-\delta}\Phi^{-1}(\alpha^*),$$

and making $\delta \to 0$ we obtain

$$g(t) \to \frac{1}{H_0}\Phi^{-1}(\alpha^*) \quad in \quad L^\infty(\Omega).$$

Therefore we have obtained that

$$\lim_{t\to\infty} y(t) = \tilde{y} \quad in \quad L^\infty((0,A)\times\Omega),$$

where

$$\tilde{y}(a,x) = \exp\{-\alpha^* a\}\bar{b}_0 \exp\{-\int_0^a \mu(s)\,\mathrm{d}s\}\frac{1}{H_0}\Phi^{-1}(\alpha^*)$$

is a stationary solution to $(2.3)_{1-4}$.

3. The nonnegative zero-stabilizability of (1.2). In the following, we shall denote by λ_1^ω the first eigenvalue of (1.6). We are ready now to state the main result of the paper:

Theorem 1. *If system* (1.2) *is nonnegatively zero-stabilizable, then* $\lambda_1^\omega \geq \alpha^*$. *Moreover, if* $\lambda_1^\omega > \alpha^*$, *then system* (1.2) *is nonnegatively zero-stabilizable, and for* $\gamma > 0$ *large enough,* $u := -\gamma y$ *is a control which stabilizes the system.*

Proof. In order to prove the second assertion, let us notice first that, due to the nonnegativity of Φ, we obtain that the solution y to (1.2) satisfies:

$$0 \leq y(a,t,x) \leq z(a,t,x), \ \ a.e.\ (a,t,x) \in Q, \tag{3.1}$$

where z is the solution to the following problem:

$$\begin{cases} z_t + z_a + \mu(a)z - k\Delta z = m(x)u(a,t,x), & (a,t,x) \in Q, \\ z(0,t,x) = \displaystyle\int_0^A \beta(a)z(a,t,x)\,\mathrm{d}a, & (t,x) \in (0,+\infty) \times \Omega, \\ \dfrac{\partial z}{\partial \nu}(a,t,x) = 0, & (a,t,x) \in \Sigma, \\ z(a,0,x) = z_0(a,x), & (a,x) \in (0,A) \times \Omega. \end{cases} \tag{3.2}$$

For this case (the case $\Phi \equiv 0$ that we reminded in the first section), the stabilization of the system to zero has been proved in [1].
Namely, it has been proved that for $\lambda_1^\omega > \alpha^*$, the system (3.2) is nonnegatively zero-stabilizable, and more, that for $\gamma > 0$ large enough, $u := -\gamma z$ is a control which stabilizes to zero the solution. Therefore, due to (3.1), we obtain that the same conclusion holds also for (1.2).

We shall prove now the first assertion. Knowing that the system (1.2) is nonnegatively zero-stabilizable and using the definition of the stabilizability of the system, we get that, for every $y_0 \in L^\infty((0,A) \times \Omega)$, the solution to (1.2) satisfies (1.4) and (1.5). In particular, for

$$y_0(a,x) = h_0(a)g_0(x), \quad (a,x) \in (0,A) \times \Omega,$$

(with $h_0 \in L^\infty(0,A)$, $h_0(a) > 0$ a.e in $(0,A)$), the same conclusion holds. Now, since $y \geq 0$ on $[0,A) \times [0,+\infty) \times \partial\omega$, we obtain via a comparison principle (see [4]) that

$$y(a,t,x) \geq \overline{y}(a,t,x), \quad a.e. \quad (a,t,x) \in [0,A) \times [0,+\infty) \times (\Omega \setminus \bar{\omega}), \tag{3.3}$$

where $\overline{y}$ is the solution to

(3.4)

$$\begin{cases} \overline{y}_t + \overline{y}_a + \mu(a)\overline{y} - k\Delta\overline{y} & \\ \quad +\Phi(Y(t,x))\overline{y} = 0, & (a,t,x) \in (0,A)\times(0,+\infty)\times(\Omega\setminus\bar{\omega}) \\ Y(t,x) = \displaystyle\int_0^A \overline{y}(a,t,x)\,\mathrm{d}a, & (t,x)\in(0,+\infty)\times(\Omega\setminus\bar{\omega}), \\ \overline{y}(0,t,x) = \displaystyle\int_0^A \beta(a)\overline{y}(a,t,x)\,\mathrm{d}a, & (t,x)\in(0,+\infty)\times(\Omega\setminus\bar{\omega}), \\ \dfrac{\partial\overline{y}}{\partial\nu}(a,t,x) = 0, & (a,t,x)\in\Sigma, \\ \overline{y}(a,t,x) = 0, & (a,t,x)\in(0,A)\times(0,+\infty)\times\partial\omega, \\ \overline{y}(a,0,x) = h_0(a)g_0(x), & (a,x)\in(0,A)\times(\Omega\setminus\bar{\omega}). \end{cases}$$

Then there exists a unique separable solution to (3.4) given by

$$\overline{y}(a,t,x) = \overline{h}(a,t)\overline{g}(t,x), \quad (a,t,x)\in(0,A)\times(0,+\infty)\times(\Omega\setminus\bar{\omega}), \tag{3.5}$$

where $\overline{h}$ is the solution for the system

$$\begin{cases} \overline{h}_t(a,t) + \overline{h}_a(a,t) + \mu(a)\overline{h}(a,t) = 0, & (a,t)\in(0,A)\times(0,+\infty), \\ \overline{h}(0,t) = \displaystyle\int_0^A \beta(a)\overline{h}(a,t)\,\mathrm{d}a, & t\in(0,+\infty), \\ \overline{h}(a,0) = h_0(a), & a\in(0,A), \end{cases} \tag{3.6}$$

and $\overline{g}$ satisfies

$$\begin{cases} \overline{g}_t(t,x) - k\Delta\overline{g}(t,x) & \\ \qquad +\Phi(H(t)\overline{g}(t,x))\overline{g}(t,x) = 0, & (t,x)\in(0,+\infty)\times(\Omega\setminus\bar{\omega}), \\ H(t) = \displaystyle\int_0^A \overline{h}(a,t)\,\mathrm{d}a, & t\in(0,+\infty) \\ \dfrac{\partial\overline{g}}{\partial\nu}(t,x) = 0, & (t,x)\in(0,+\infty)\times\partial\Omega, \\ \overline{g}(t,x) = 0, & (t,x)\in(0,+\infty)\times\partial\omega, \\ \overline{g}(0,x) = g_0(x), & x\in\Omega\setminus\bar{\omega}. \end{cases} \tag{3.7}$$

Having in mind the asymptotic behaviour of $\overline{h}$ and of $\overline{g}$, respectively, in the second section, we get that, for the case $\alpha^* \leq 0$ the solution $\overline{y}$ goes to zero without any control, therefore we shall consider in what follows only the case $\alpha^* > 0$. We shall prove that, in this case, $\lambda_1^\omega \geq \alpha^*$. Due to the nonnegative zero-stabilizability and to (2.2), it follows that $\overline{g}(t) \longrightarrow 0$ in $L^\infty(\Omega \setminus \bar{\omega})$ as $t \to +\infty$. Since Φ is a continuous differentiable function, and $H(t) \longrightarrow H_0$, as $t \to +\infty$, we get that

$$\Phi(H(t)\overline{g}(t)) \longrightarrow \Phi(0) = 0 \quad \text{in} \quad L^\infty(\Omega \setminus \bar{\omega}) \quad \text{as} \quad t \to +\infty,$$

or, equivalently, for any $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that

$$-\varepsilon < \Phi(H(t)\overline{g}(t,x)) < \varepsilon, \quad \text{a.e.} \quad x \in \Omega \setminus \bar{\omega}, \quad \forall t \geq t_\varepsilon.$$

Then, applying the comparison principle for parabolic equations, we obtain that:

$$\overline{g}(t,x) \geq f(t,x), \quad \forall t \geq t_\varepsilon, \quad \text{a.e.} \quad x \in \Omega \setminus \bar{\omega},$$

where f is the solution to the next problem:

$$(3.8) \quad \begin{cases} f_t(t,x) - k\Delta f(t,x) + \varepsilon f(t,x) = 0, & (t,x) \in (t_\varepsilon, +\infty) \times (\Omega \setminus \bar{\omega}), \\ \dfrac{\partial f}{\partial \nu}(t,x) = 0, & (t,x) \in (t_\varepsilon, +\infty) \times \partial\Omega, \\ f(t,x) = 0, & (t,x) \in (t_\varepsilon, +\infty) \times \partial\omega, \\ f(t_\varepsilon, x) = \overline{g}(t_\varepsilon, x), & x \in \Omega \setminus \bar{\omega}. \end{cases}$$

Now, multiplying the first equation in (3.8) with φ_1 ($\varphi_1 \geq 0$ in $\Omega \setminus \bar{\omega}$, φ_1 is not identically zero), where φ_1 denotes the eigenfunction corresponding to λ_1^ω we obtain:

$$< f(t), \varphi_1 >' + (\lambda_1^\omega + \varepsilon) < f(t), \varphi_1 >= 0, \quad \forall t \geq t_\varepsilon,$$

where $<,>$ is the inner product in $L^2(\Omega \setminus \bar{\omega})$.
The latest implies that
(3.9)
$$\overline{g}(t,x) \geq \exp\{-(\lambda_1^\omega + \varepsilon)(t - t_\varepsilon)\} < \overline{g}(t_\varepsilon), \varphi_1 >, \quad \text{a.e.} \quad x \in \Omega \setminus \bar{\omega}, \quad \forall t \geq t_\varepsilon.$$

Having in mind the expression of the solution $\overline{h}$ given by (2.2) (for $t \geq t_\varepsilon$), and using (3.5) and (3.9), we obtain that the asymptotic behaviour of $\overline{y}(t)$ is given by

$$\exp\{-(\lambda_1^\omega - \alpha^* + \varepsilon)(t - t_\varepsilon)\}.$$

Since we have assumed that the nonnegative zero-stabilizability holds, it follows by (3.3) that

$$\lim_{t \to \infty} \overline{y}(t) = 0 \quad \text{in} \quad L^\infty((0, A) \times (\Omega \setminus \bar{\omega})),$$

which implies that

$$\lambda_1^{\omega} \geq \alpha^*$$

and the proof is complete. □

REFERENCES

1. Ainseba, B.; Anita, S.; Langlais, M. – *Internal Stabilizability of Some Diffusive Models* J. Math. Anal. Appl., 265(1) (2002), 91–102.
2. Ainseba, B.; Anita, S. – *Internal Nonnegative Stabilization for Some Parabolic Equations*, Comm. Pure Appl. Anal., to appear.
3. Aniţa, L.-I.; Aniţa, S. – *Characterization of the internal stabilizability of the diffusion equation*, Nonlinear Studies, 8 (2001), 193–202.
4. Aniţa, S. – *Analysis and Control of Age-Dependent Population Dynamics*, Kluwer Acad. Publ., Dodrecht, 2000.
5. Barbu, V. – *Partial Differential Equations and Boundary Value Problems*, Kluwer Acad. Publ., Dodrecht, 1998.
6. Chan, W.-L.; Guo, B.-Z. – *Global Behaviour of Age-Dependent Logistic Population Models*, J. Math. Biol. 28 (1990), 225–235.
7. Iannelli, M. – *Mathematical Theory of Age-Structured Population Dynamics*, Appl. Math. Monographs – C.N.R., Giardini Editori e Stampatori, Pisa, 1995.
8. Langlais, M. – *Large time behavior in a nonlinear age-dependent population dynamics with spatial diffusion*, J. Math. Biol. 26 (1988), 319–346.
9. Webb, G.F. – *Theory of Nonlinear Age-Dependent Population Dynamics*, Marcel Dekker, New York, 1985.

Received: 3.IX.2007

Faculty of Mathematics,
"AL.I. Cuza" University,
Iaşi, 700506,
ROMÂNIA
tarniceriuoana@yahoo.co.uk