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A CLASS OF CONTINUOUS ACCURATE IMPLICIT LMMs WITH CHEBYSHEV BASIS FUNCTION

BY

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Abstract. In a recent paper, we reported a class of methods for a continuous formulation of some classical initial values solvers obtained by employing the Chebyshev polynomials as basis functions in a multistep collocation technique. In this paper, we extend the scope of this work to a new class of accurate implicit methods and this is supported with some numerical evidences.

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1. Introduction. Over the years, techniques for derivation of Linear Multistep Methods (LMMs) for the numerical solution of the initial value problems (IVPs) in the first order differential equation:

$$y'(x) = f(x, y(x)), \quad y(a) = y_a \quad a \leq x \leq b < +\infty \tag{1.1}$$

have been reported in the literature and these include, among others, interpolation, collocation and integration of interpolating polynomials. However, lately there has been an upsurge in research on the continuous formulation of these discrete initial values solvers. See for example, [1]–[8],[13],[17]-[19] for some of these methods. This could be attributed to their ability in yielding several solutions/outputs at the off-grid points without requiring additional interpolation when compared with their discrete form equivalents. Many of these methods are based on the collocation of either a perturbed form of 1.1 or its non-perturbed form. These methods also vary according to the type of the basis functions involved in the approximant $Y(x)$ of $y(x)$. Thus, for example, while [13] and [17] employed the monomials $x^r, r \geq 0$, in non-perturbed collocation approach, [2] employed the so-called Canonical polynomials $Q_r(x), r \geq 0$, of the Lanczos tau method in a perturbed collocation approach. Whereas [3] engaged the well-behaved Chebyshev polynomials as basis function in a perturbed collocation method, [4] used the same basis functions in a non-perturbed approach.

For the sake of completeness and readability of this paper, we shall briefly review the above methods in Section 2 of this paper. In section 3 we shall employ the Chebyshev polynomials are basis functions to recover a class of methods reported in [13] as well as develop a new class of methods with smaller error constants than their corresponding Adams-Moulton methods. Section 4 presents numerical examples in support of the new method and the paper finally closes with some concluding remarks in Section 5.

2. A review of some related works.

We briefly review here three methods for the continuous formulation of the solution of 1.1. Without loss of generality, we assume that $a = 0$ in (1.1) since any problem in $[a, b]$ may be transformed into $[0, b]$ by the substitution:

$$x = a + \left(1 - \frac{a}{b}\right)u, \quad 0 \leq u \leq b \tag{2.1}$$

The choice $a = 0$ is made for simplification of the arithmetic that will be involved later. So then, we are concerned with the IVP:

$$y'(x) = f(x, y(x)), y(0) = y_0, 0 \leq x \leq b < +\infty \tag{2.2}$$

We partition $[0, b]$ into the form $0 = x_0 < x_1 < x_2 < x_3 < \cdots < x_n = b$ where $x_k = x_0 + kh$, $h = \frac{(b-a)}{n}$ and then seek an approximant $Y(x)$ of $y(x)$ over the sub-intervals $[x_r, x_{r+p}]$ where p will be specified.

In Onumanyi et al [13], we sought an approximant

$$Y(x) = \sum_{r=0}^{N} a_r x^r, \quad 0 < N < +\infty, \quad x_k \leq x \leq x_{k+p} \tag{2.3}$$

where N and P are positive real integers, and then collocate (2.2) with $Y(x)$ as well as interpolate (2.3) at approximately selected points in $[x_k, x_k + p]$

in order to yield a linear algebraic system in the unknown a_k, $k = 0(1)n$. That is, we collocate:

$$\sum_{r=0}^{N} r a_r x^{r-1} = f(x, Y(x)) \tag{2.4}$$

and interpolate (2.3) at these points. Having solved this system of equations, we substitute the resulting values into (2.3) for a continuous approximant of $y(x)$. When evaluated at an appropriately chosen grid points, the continuous scheme thus obtained, in most cases, yielded some conventionally known discrete initial value solvers and in a few cases, some new classes. The classes of methods include, among others, Adams-Moulton, Adams-Bashforth and Backward Differentiation Formulae.

In Adeniyi [2], we chose

$$Y(x) = \sum_{r=0}^{N} a_r Q_r(x), \quad x_k \leq x \leq x_{k+p}, \quad 0 < N < +\infty \tag{2.5}$$

and then considered the perturbed form of (1.1):

$$\sum_{r=0}^{N} r a_r x^{r-1} = f(x, Y(x)) + \tau P_n(x) \tag{2.6}$$

where $P_n(x)$ is the n-th degree Legendre polynomial valid in $x_k < x < x_{k+p}$ and $Q_r(x)$ is the sequence of canonical polynomials (see [9], [11], [14], [15] and [16]) defined by

$$LQ_r(x) = x^r. \tag{2.7}$$

and L is the linear differential operator

$$L = \frac{d}{dx} + 1 \tag{2.8}$$

These two equations were then collocated and interpolated at approximately chosen points in order to obtain the required linear system for the unique determination of $a_k, k = 0(1)N$, and τ. We proceeded from here and then obtained the continuous formulation of some optimal order methods as well as the GRAGG and STETTER method (see [10] and [12]). ADENIYI et al [2] considered a modification of this approach by seeking

$$Y(x) = \sum_{r=o}^{P} a_r T_r(x) \tag{2.9}$$

By this we were able to recover the same class of methods. In ADENIYI et al [4], on the other hand, we considered (2.9) in a non-perturbed form and then followed ONUMANYI et al [13] to recover virtually the classes of methods in [13]. For purpose of comparison in Section 5 of this paper, we recall here some of these results.

The continuous formulation of the Adams-Moulton methods of orders three, four and five are respectively

$$\begin{aligned} Y(x) &= Y_{k+1} + \frac{h}{8}\left[-\frac{(x-x_k)^2}{2h^2} + \frac{(x-x_k)^3}{6h^2}\right] f_k \\ &\quad + \frac{h}{8}\left[4\frac{(x-x_k)}{h} - \frac{(x-x_k)^3}{3h^3}\right] f_{k+1} \\ &\quad + \frac{h}{8}\left[\frac{(x-x_k)^2}{2h^2} + \frac{(x-x_k)^3}{6h^2}\right] f_{k+2} \end{aligned} \tag{2.10}$$

$$\begin{aligned} Y(x) &= Y_{k+2} + \frac{h}{384}\left[7 - 12\frac{(x-x_k)}{h} + 2\frac{(x-x_k)^2}{h^2}\right. \\ &\quad \left. +4\frac{(x-x_k)^3}{h^3} - \frac{(x-x_k)^4}{h^4}\right] f_k \\ &\quad + \frac{h}{384}\left[-54 + 108\frac{(x-x_k)}{h} + 54\frac{(x-x_k)^2}{h^2}\right. \\ &\quad \left. -4\frac{(x-x_k)^3}{h^3} - 3\frac{(x-x_k)^4}{h^4}\right] f_{k+1} \\ &\quad + \frac{h}{384}\left[-155 + 108\frac{(x-x_k)}{h} + 54\frac{(x-x_k)^2}{h^2}\right. \\ &\quad \left. -4\frac{(x-x_k)^3}{h^3} - 3\frac{(x-x_k)^4}{h^4}\right] f_{k+2} \\ &\quad + \frac{h}{384}\left[9 - 12\frac{(x-x_k)}{h} - 2\frac{(x-x_k)^2}{h^2}\right. \\ &\quad \left. +4\frac{(x-x_k)^3}{h^3} - \frac{(x-x_k)^4}{h^4}\right] f_{k+3} \end{aligned} \tag{2.11}$$

$$
\begin{aligned}
Y(x) \quad = \quad & Y_{k+3} + \frac{h}{720}\Big[-11 + 15\frac{(x-x_k)^2}{2h^2} - 5\frac{(x-x_k)^3}{4h^3} \\
& -5\frac{(x-x_k)^4}{16h^4} - 3\frac{(x-x_k)^5}{16h^5}\Big] f_k \\
& +\frac{h}{720}\Big[-74 + 60\frac{(x-x_k)^2}{h^2} - 20\frac{(x-x_k)^3}{h^3} \\
& +15\frac{(x-x_k)^4}{h^4} - 3\frac{(x-x_k)^5}{h^5}\Big] f_{k+1} \\
& +\frac{h}{720}\Big[-456 + 360\frac{(x-x_k)^2}{h^2} - 4\frac{(x-x_k)^3}{h^3} \\
& -75\frac{(x-x_k)^4}{h^4} - 9\frac{(x-x_k)^5}{h^5}\Big] f_{k+2} \\
& +\frac{h}{720}\Big[-346 + 60\frac{(x-x_k)^2}{h^2} + 20\frac{(x-x_k)^3}{h^3} \\
& -15\frac{(x-x_k)^4}{h^4} + 3\frac{(x-x_k)^5}{h^5}\Big] f_{k+3} \\
& +\frac{h}{720}\Big[19 - 16\frac{(x-x_k)^2}{h^2} - 5\frac{(x-x_k)^3}{h^3} \\
& -15\frac{(x-x_k)^4}{h^4} + 3\frac{(x-x_k)^5}{h^5}\Big] f_{k+4}.
\end{aligned}
\tag{2.12}
$$

The corresponding discrete forms of the methods (2.11), (2.11) and (2.12) are respectively

$$Y_{k+2} - Y_{k+1} = \frac{h}{12}(5f_{k+2} + 8f_{k+1} - f_k) \tag{2.13}$$

$$Y_{k+3} - Y_{k+4} = \frac{h}{24}(9f_{k+3} + 19f_{k+2} - 5f_{k+1} + f_k) \tag{2.14}$$

$$Y_{k+4} - Y_{k+3} = \frac{h}{720}(251f_{k+4} + 646f_{k+3} - 264f_{k+2} + 106f_{k+1} - 19f_k) \tag{2.15}$$

3. A class of accurate implicit methods. We present here two classes of methods based on Chebyshev polynomials in a multistep collocation. These polynomials are defined by

$$T_r(x) = \cos[r\cos^{-1}((2x-b-a)/(b-a))], \qquad a \le x \le b \tag{3.1}$$

and satisfy in $a \leq x \leq b$, the recurrence relation:

$$T_{r+1}(x) - 2\{(2x - b - a)/(b - a)\}T_r(x) + T_{r-1}(x) = 0, \quad r \geq 1. \tag{3.2}$$

Corresponding to (3.2), for interval $[x_k, x_{k+n}]$, we have

$$T_{r+1}(x) - 2(2x/nh - 2k/n - 1)T_r(x) + T_{r-1}(x) = 0 \tag{3.3}$$

The representation of any continuous function in terms of these polynomials ensures even distribution of the error in the resulting approximant throughout the entire range of consideration. Hence our choice in this paper.

In section 3.1 we shall recover a class of methods earlier obtained in [13] with the monomial $\{x^r\}$ as basis function. Section 3.2 will develop a new class of methods, which compares more favorably with the Adams-Moulton methods.

3.1. A Chebyshev-based collocation method. Let us assume an approximation of the form

$$Y(x) = \sum_{r=0}^{n+1} a_r T_r(u) \tag{3.4}$$

whence $u = 2x/nh - 2k/n - 1$, $x_k \leq x \leq x_{k+n}$, which satisfies

$$Y'(x) = f(x, Y(x)), \quad x_k \leq x \leq x_{k+n}. \tag{3.5}$$

We shall interpolate (3.4) at x_k, x_{k+1} as well as collocate (3.5) at $x_{k+1}, x_{k+2}, \ldots, x_{k+n}$ for an $(n+1)-$step approximant of $y(x)$ over the range of $x_k \leq x \leq x_{k+n}$.

For a two-step approximation we have from (3.4) and (3.5) respectively:

$$\begin{aligned} Y(x) &= a_0 + a_1\left(\frac{x}{h} - k - 1\right) + a_2\left[2\left(\frac{x}{h} - k - 1\right)^2 - 1\right] \\ &\quad + a_3\left[4\left(\frac{x}{h} - k - 1\right)^3 - 3\left(\frac{x}{h} - k - 1\right)\right] \end{aligned} \tag{3.6}$$

and

$$\frac{a_1}{h} + \frac{4a_2}{h}\left(\frac{x}{h} - k - 1\right) + \frac{3a_3}{h}\left[4\left(\frac{x}{h} - k - 1\right)^3 - 1\right] = f(x, Y(x)). \tag{3.7}$$

We interpolate (3.6) at x_k and x_{k+1} to obtain

$$a_0 - a_1 + a_2 - a_3 = Y_k, \quad a_0 - a_1 = Y_{k+1} \tag{3.8}$$

and then collocate (3.7) at x_{k+1} and x_{k+2} to have

$$a_1 - 3a_3 = hf_{k+1}, \quad a_1 + 4a_2 + 9a_3 = hf_{k+2}, \tag{3.9}$$

where $f(x_r, y_r) \equiv f_r$, $r = k+1, k+2$. The solution of (3.8)-(3.9) when inserted back into (3.6) yield the continuous approximant of $y(x)$, over $[x_k, x_{k+2}]$:

$$\begin{aligned} Y(x) &= \frac{1}{20}\left[3\frac{(x-x_k)^2}{h^2} - \frac{(x-x_k)^3}{h^3}\right]Y_k \\ &+\frac{1}{20}\left[20 - 3\frac{(x-x_k)^2}{h^2} - \frac{(x-x_k)^3}{h^3}\right]Y_{k+1} \\ &+\frac{h}{20}\left[10\frac{(x-x_k)}{h} + 2\frac{(x-x_k)^2}{h^2} - 3\frac{(x-x_k)^3}{2h^3}\right]f_{k+1} \\ &+\frac{h}{20}\left[\frac{(x-x_k)^2}{h^2} + \frac{(x-x_k)^3}{2h^3}\right]f_{k+2} \end{aligned} \tag{3.10}$$

and which, at the grid point x_{k+2}, produces the two-step implicit scheme:

$$Y_{k+2} - \frac{4}{5}Y_{k+1} + \frac{1}{5}Y_k = \frac{2h}{5}(f_{k+2} + 2f_{k+1}). \tag{3.11}$$

From (3.11) we obtain f_{k+2} for our proposed continuous scheme (3.10).

For a three-step method we similarly obtain the continuous formulation:

$$\begin{aligned} Y(x) &= \frac{1}{128}\left[96 + 128\frac{(x-x_k)}{h} - 64\frac{(x-x_k)^2}{3h^2}\right. \\ &\left.-128\frac{(x-x_k)^3}{3h^3} + 32\frac{(x-x_k)^4}{4h^4}\right]Y_k \\ &+\frac{1}{1536}\left[1440 + 128\frac{(x-x_k)}{h} - 64\frac{(x-x_k)^2}{3h^2}\right. \\ &\left.+128\frac{(x-x_k)^3}{3h^3} - 32\frac{(x-x_k)^4}{4h^4}\right]Y_{k+1} \end{aligned} \tag{3.12}$$

$$
\begin{aligned}
&+\frac{h}{128}\Big[696+1600\frac{(x-x_k)}{3h}-2096\frac{(x-x_k)^2}{9h^2}\\
&-448\frac{(x-x_k)^3}{9h^3}+184\frac{(x-x_k)^4}{9h^4}\Big]f_{k+1}\\
&+\frac{h}{1536}\Big[192+1216\frac{(x-x_k)}{3h}-1984\frac{(x-x_k)^2}{9h^2}\\
&-64\frac{(x-x_k)^3}{9h^3}-128\frac{(x-x_k)^4}{9h^4}\Big]f_{k+2}\\
&+\frac{h}{1536}\Big[-24-128\frac{(x-x_k)}{3h}-80\frac{(x-x_k)^2}{9h^2}\\
&+128\frac{(x-x_k)^3}{9h^3}+40\frac{(x-x_k)^4}{9h^4}\Big]f_{k+3}
\end{aligned}
$$

of the implicit finite difference scheme

$$Y_{k+3}-Y_{k+1}=\frac{h}{3}(f_{k+2}+4f_{k+2}+f_{k+1}). \tag{3.13}$$

For a four-step scheme the resulting continuous scheme is

$$
\begin{aligned}
Y(x) \;=\; &\frac{1}{1506}\Big[114-45\frac{(x-x_k)^2}{h^2}+30\frac{(x-x_k)^3}{4h^3}\\
&+90\frac{(x-x_k)^4}{16h^4}-18\frac{(x-x_k)^5}{16h^5}\Big]Y_k\\
&+\frac{1}{1506}\Big[1392+45\frac{(x-x_k)^2}{h^2}-30\frac{(x-x_k)^3}{4h^3}v\\
&-90\frac{(x-x_k)^4}{16h^4}+18\frac{(x-x_k)^5}{16h^5}\Big]Y_{k+1}\\
&+\frac{h}{1506}\Big[826-2654\frac{(x-x_k)^2}{16h^2}+777\frac{(x-x_k)^3}{16h^3}\\
&+574\frac{(x-x_k)^4}{64h^4}-1651\frac{(x-x_k)^5}{64h^5}\Big]f_{k+1}\\
&+\frac{h}{1506}\Big[192-3012\frac{(x-x_k)}{4h}+66\frac{(x-x_k)^2}{4h^2}\\
&-1299\frac{(x-x_k)^3}{16h^3}-33\frac{(x-x_k)^4}{16h^4}+177\frac{(x-x_k)^5}{64h^5}\Big]f_{k+2}
\end{aligned} \tag{3.14}
$$

$$+\frac{h}{1506}\left[138-1902\frac{(x-x_k)^2}{16h^2}-687\frac{(x-x_k)^3}{16h^3}\right.$$
$$\left.+198\frac{(x-x_k)^4}{64h^4}+111\frac{(x-x_k)^5}{64h^5}\right]f_{k+3}$$
$$+\frac{h}{1506}\left[20-58\frac{(x-x_k)^2}{4h^2}-45\frac{(x-x_k)^3}{16h^3}\right.$$
$$\left.+29\frac{(x-x_k)^4}{16h^4}+27\frac{(x-x_k)^5}{64h^5}\right]f_{k+4}$$

with the corresponding finite difference scheme

(3.15)

$$Y_{k+1}-\frac{1}{125}(224Y_{k+1}+27Y_k)=\frac{h}{251}(156f_{k+1}+216f_{k+2}+324f_{k+3}+84f_{k+4}),$$

which is a four-step implicit method.

3.2. A new accurate class of methods. The procedure adopted here is not strikingly different from that of Section 3.1 above as, in both cases, the set of collocation points are the same. However, the interpolation points are now chosen as x_k and x_{k+p-l} for a p-step approximant of $y(x)$.

We remark here that even with this choice for a two-step formulation, the set of collocation points as well as that of interpolation points are exactly the same and hence the continuous scheme (3.10) with its corresponding finite difference equivalent (3.11) are also members of this new class.

Now, for methods of higher step number $p(p>3)$, the schemes obtained differ, thus for a three-step method we have from (3.4):

$$\begin{aligned}Y(x) &= a_0+a_1\left(\frac{2x-2kh-3h}{3h}\right)+a_2\left[2\left(\frac{2x-2kh-3h}{3h}\right)^2-1\right]\\ &+ a_3\left[4\left(\frac{2x-2kh-3h}{3h}\right)^3-3\left(\frac{2x-2kh-3h}{3h}\right)^2\right]\\ &+ a_4\left[8\left(\frac{2x-2kh-3h}{3h}\right)^4-8\left(\frac{2x-2kh-3h}{3h}\right)^2+1\right]\end{aligned}\tag{3.16}$$

and from (3.5):

$$\frac{2a_1}{3h}+\frac{8a_2}{3h}\left(\frac{2x-2kh-3h}{3h}\right)^4+\frac{6a_3}{3h}\left[4\left(\frac{2x-2kh-3h}{3h}\right)^2-1\right]\tag{3.17}$$

$$+\frac{32a_4}{3h}\left[2\left(\frac{2x-2kh-3h}{3h}\right)^3-\left(\frac{2x-2kh-3h}{3h}\right)\right]=f(x,Y(x)).$$

We interpolate (3.16) at x_k and x_{k+2} to have $a_0-a_1+a_2-a_3+a_4=Y_k$

$$81a_0+27a_1-63a_2-69a_3+17a_4=81Y_{k+2} \tag{3.18}$$

and collocate (3.17) at $x_{k+1}, x_{k+2}, x_{k+3}$ to have, respectively

$$\begin{array}{rcl} 54a_1+72a_2-90a_3+224a_4 & = & 81hf_{k+1} \\ 54a_1-72a_2-90a_3+224a_4 & = & 81hf_{k+2} \\ 2a_1+8a_2+18a_3+32a_4 & = & 3hf_{k+3} \end{array} \tag{3.19}$$

The solution (3.18) (3.19) together with (3.16) yields the continuous scheme:

$$\begin{aligned} Y(x) &= \frac{1}{128}\Bigg[-7+12\frac{(x-x_k)}{h}-2\frac{(x-x_k)^2}{h^2} \\ &\quad -4\frac{(x-x_k)^3}{h^3}+\frac{(x-x_k)^4}{h^4}\Bigg]Y_k \\ &\quad +\frac{1}{128}\Bigg[135-12\frac{(x-x_k)}{h}-2\frac{(x-x_k)^2}{h^2} \\ &\quad +4\frac{(x-x_k)^3}{h^3}+\frac{(x-x_k)^4}{h^4}\Bigg]Y_{k+2} \\ &\quad +\frac{h}{128}\Bigg[-27+52\frac{(x-x_k)}{h}-62\frac{(x-x_k)^2}{3h^2} \\ &\quad -20\frac{(x-x_k)^3}{3h^3}+\frac{(x-x_k)^4}{3h^4}\Bigg]f_{k+1} \\ &\quad +\frac{h}{128}\Bigg[-54+40\frac{(x-x_k)}{h}+52\frac{(x-x_k)^2}{3h^2} \\ &\quad -8\frac{(x-x_k)^3}{3h^3}-2\frac{(x-x_k)^4}{3h^4}\Bigg]f_{k+2} \\ &\quad +\frac{h}{128}\Bigg[3-4\frac{(x-x_k)}{h}-2\frac{(x-x_k)^2}{3h^2} \\ &\quad +4\frac{(x-x_k)^3}{3h^3}+\frac{(x-x_k)^4}{3h^4}\Bigg]f_{k+3} \end{aligned} \tag{3.20}$$

At the grid point x_{k+3} this produces the three-step implicit scheme:

$$Y_{k+3}-\frac{9}{8}Y_{k+2}+\frac{Y_k}{8}=\frac{h}{8}(3f_{k+1}+6f_{k+2}-3f_{k+1}) \tag{3.21}$$

From (3.21) we obtain f_{k+3} for our proposed scheme (3.21).

When now adopted for four-step formulation, the corresponding continuous scheme obtained is

$$
\begin{aligned}
Y(x) \;=\; & \frac{1}{486}\left[22 - 15\frac{(x-x_k)^2}{h^2} - 10\frac{(x-x_k)^3}{4h^3}\right.\\
& \left.+30\frac{(x-x_k)^4}{16h^4} - 6\frac{(x-x_k)^5}{16h^5}\right]Y_k\\
& +\frac{1}{486}\left[464 + 15\frac{(x-x_k)^2}{h^2} - 10\frac{(x-x_k)^3}{4h^3}\right.\\
& \left.-30\frac{(x-x_k)^4}{16h^4} + 6\frac{(x-x_k)^5}{16h^5}\right]Y_{k+3}\\
& +\frac{h}{162}\left[26 - 318\frac{(x-x_k)^2}{3h^2} + 89\frac{(x-x_k)^3}{16h^3}\right.\\
& \left.+78\frac{(x-x_k)^4}{64h^4} - 21\frac{(x-x_k)^5}{64h^5}\right]f_{k+1}\\
& +\frac{h}{162}\left[-96 + 324\frac{(x-x_k)}{4h} - 18\frac{(x-x_k)^2}{4h^2}\right.\\
& \left.-123\frac{(x-x_k)^3}{16h^3} + 9\frac{(x-x_k)^4}{16h^4} + 9\frac{(x-x_k)^5}{64h^5}\right]f_{k+2}\\
& +\frac{h}{162}\left[74 - 174\frac{(x-x_k)^2}{16h^2} - 79\frac{(x-x_k)^3}{16h^3}\right.\\
& \left.+6\frac{(x-x_k)^4}{64h^4} + 15\frac{(x-x_k)^5}{64h^5}\right]f_{k+3}\\
& +\frac{h}{162}\left[4 - 6\frac{(x-x_k)^2}{4h^2} - 5\frac{(x-x_k)^3}{16h^3}\right.\\
& \left.+3\frac{(x-x_k)^4}{16h^4} + 3\frac{(x-x_k)^5}{64h^5}\right]f_{k+4}
\end{aligned} \tag{3.22}
$$

with the associated implicit finite difference equivalent

$$
\begin{aligned}
& Y_{k+4} - \frac{1}{243}(224Y_{k+3} + 19Y_k)\\
& = \frac{h}{243}(84f_{k+4} + 228f_{k+3} - 72f_{k+2} + 60f_{k+1}).
\end{aligned} \tag{3.23}
$$

4. Numerical examples. We consider here two members of differential equations $\underline{y}'(x) = A(x)\underline{y}(x) + \underline{b}(x)$, $0 \le x \le 1$ together with given associated conditions, for illustration.

Example 4.1.

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \quad \underline{b} = \underline{0}, \quad \underline{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix},$$

$$\underline{y}(0) = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \quad \underline{y}(x) = \begin{pmatrix} 1 + \frac{1}{2}e^{-x} + 1 + \frac{1}{2}e^{-3x} \\ 1 - \frac{1}{2}e^{-3x} \\ 1 - \frac{1}{2}e^{-x} + \frac{1}{2}e^{-3x} \end{pmatrix}.$$

See Table 5.2 for the numerical results.

Example 4.2.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 \end{pmatrix}, \underline{b} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ e^{-x} \end{pmatrix}, \underline{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix},$$

$$\underline{y}(0) = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \underline{y}(x) = \begin{pmatrix} e^{-x} \\ -e^{-x} \\ e^{-x} \\ -e^{-x} \end{pmatrix}.$$

Numerical results for this example are presented in Table 5.2

5. Concluding remarks. For purpose of comparison of our class of implicit methods with the Adams-Moulton methods of corresponding order, we present below in Table 5.1 the error constants of the schemes

Table 5.1: Order and Error Constants of Methods.

Methods	Order	Error Constant
Adams-Moulton (2.13)	3	-1/24
Method (3.11)	3	-1/30
Adams-Moulton (2.14)	4	-19/20
Method (3.13)	4	-18/720
Method (3.21)	4	-1/90
Adams-Moulton (2.15)	5	- 27/1140
Method (3.15)	5	- 7/405
Method (3.23)	5	- 21/1255

Table 5.2: Errors of Methods for Example 4.1.

Methods	X	Continuous y_1	Discrete y_1	Continuous y_2	Discrete y_2	Continuous y_3	Discrete y_3
AMM 2.15	0.05	5.624765D-5		5.708435D-5		5.3423651 D-5	
Method 3.23	0.05	5.321454D-5		5.423874D-5		5.123433 D-5	
Method 3.15	0.05	5.134976D-5		5.032457D-5		5.0213457 D-5	
AMM 2.15	0.10	5.309234D-5	5.309234D-5	5.409128 D-5	5.409128D-5	5.301242 D-5	5.301242 D-5
Method 3.23	0.10	5.108753D-5	5.108753D-5	5.300234 D-5	5.300234 D-5	5.102363 D-5	5.102363 D-5
Method 3.15	0.10	5.011234D-5	5.011234 D-5	5.210985 D-5	5.210985 D-5	5.0202351 D-5	5.0202351 D-5
AMM 2.15	0.15	5.079858 D-5		5.390205D-5		5.194582D-5	
Method 3.23	0.15	5.003487D-5		5.110985D-5		5.086709D-5	
Method 3.12	0.15	4.998745D-5		5.093543D-5		5.012782D-5	
AMM 2.15	0.20	5.008732 D-5	5.008732 D-5	5.109234D-5	5.109234D-5	5.127825D-5	5.127825D-5
Method 3.23	0.20	4.987453 D-5	4.987453 D-5	5.094266D-5	5.094266D-5	5.034583D-5	5.034583D-5
Method 3.15	0.20	4.689235 D-5	4.689235 D-5	5.076435D-5	5.076435D-5	5.008754D-5	5.008754D-5
AMM 2.15	0.25	4.903417D-5		5.098437D-5		5.115763D-5	
Method 3.23	0.25	4.884572D-5		5.033473D-5		5.012764D-5	
Method 3.15	0.25	4.599387D-5		5.012145D-5		5.002239D-5	
AMM 2.15	0.30	4.890478D-5	4.890478D-5	5.069435D-5	5.069435D-5	5.109873D-5	5.109873D-5
Method 3.23	0.30	4.823054D-5	4.823054D-5	5.015769D-5	5.015769D-5	5.004658D-5	5.004658D-5
Method 3.15	0.30	4.487238D-5	4.487238D-5	5.009763D-5	5.009763D-5	5.001987D-5	5.001987D-5
AMM 2.15	0.35		4.790354D-5		5.012298D-5		5.104594 D-5
Method 3.23	0.35		4.698734D-5		5.008302D-5		5.002983D-5
Method 3.15	0.35		4.437549D-5		5.034975D-5		5.001341D-5
AMM 2.15	0.40	4.773473D-5	4.773473D-5	5.012983D-5	5.012983D-5	5.012983D-5	5.100982D-5
Method 3.23	0.40	4.589343D-5	4.589343D-5	5.009824D-5	5.009824D-5	5.009824D-5	5.001287D-5
Method 3.15	0.40	4.410348D-5	4.410348D-5	5.000582D-5	5.000582D-5	5.000582D-5	4.938738D-5
AMM 2.15	0.45	4.567987D-5		5.009873D-5		5.098528D-5	
Method 3.23	0.45	4.329793D-5		4.978351D-5		4.997326D-5	
Method 3.15	0.45	4.239879D-5		4.887359D-5		4.909217D-5	
AMM 2.15	0.50	4.349752D-5	4.349752D-5	4.978575D-5	4.978575D-5	5.074398D-5	5.074398D-5
Method 3.23	0.50	4.298723D-5	4.298723D-5	4.935176D-5	4.935176D-5	4.930477D-5	4.930477D-5
Method 3.15	0.50	4.178459D-5	4.178459D-5	5.821973D-5	5.821973D-5	4.879457D-5	4.879457D-5
AMM 2.15	0.55	4.198759D-5		4.867209D-5		5.029605D-5	
Method 3.23	0.55	4.098304D-5		4.810387D-5		4.956036D-5	
Method 3.15	0.55	4.040981D-5		4.794287D-5		4.823092D-5	

AMM 2.15	0.60	4.097854 D-5	4.097854 D-5	4.590746D-5	4.590746D-5	5.017836D-5	5.017836D-5
Method 3.23	0.60	4.094503D-5	4.094503D-5	4.320981D-5	4.320981D-5	4.901754D-5	4.901754D-5
Method 3.15	0.60	4.029055D-5	4.029055D-5	5.290387D-5	5.290387D-5	4.790275D-5	4.790275D-5
AMM 2.15	0.65	4.094376D-5		4.498035D-5		5.003789D-5	
Method 3.23	0.65	4.065738D-5		4.309872D-5		4.798328D-5	
Method 3.15	0.65	4.019475D-5		4.109475D-5		4.509215D-5	
AMM 2.15	0.70	4.057937D-5	4.057937D-5	4.198705D-5	4.198705D-5	4.907758D-5	4.907758D-5
Method 3.23	0.70	4.032983D-5	4.032983D-5	4.190372D-5	4.190372D-5	4.678302D-5	4.678302D-5
Method 3.15	0.70	4.009873D-5	4.009873D-5	5.097235D-5	5.097235D-5	4.498703D-5	4.498703D-5
AMM 2.15	0.75	4.029075D-5		5.708435D-5		5.342365D-5	
Method 3.23	0.75	4.029783D-5		5.423874D-5		5.123433D-5	
Method 3.15	0.75	3.997837D-5		5.032457D-5		5.021457D-5	
AMM 2.15	0.80	4.019836D-5	4.019836D-5	4.057693D-5	4.057693D-5	4.398154D-5	4.398154D-5
Method 3.20	0.80	4.013987D-5	4.013987D-5	4.039714D-5	4.039714D-5	4.290237D-5	4.290237D-5
Method 3.15	0.80	3.965738D-5	3.965738D-5	3.897491D-5	3.897491D-5	4.199736D-5	4.199736D-5
AMM 2.15	0.85	4.009817D-5		4.038975D-5		4.190867D-5	
Method 3.23	0.85	4.011985D-5		4.017893D-5		4.130928D-5	
Method 3.15	0.85	3.879047D-5		3.867901D-5		4.104926D-5	
AMM 2.15	0.90	3.997826D-5	3.997826D-5	4.019835D-5	4.019835D-5	4.128739D-5	4.128739D-5
Method 3.23	0.90	3.987398D-5	3.987398D-5	3.990582D-5	3.990582D-5	4.093757D-5	4.093757D-5
Method 3.15	0.90	3.678093D-5	3.678093D-5	3.798356D-5	3.798356D-5	4.056983D-5	4.056983D-5
AMM 2.15	0.95	3.874092D-5		3.987204D-5		4.102946D-5	
Method 3.23	0.95	3.785904D-5		3.870925D-5		4.078278D-5	
Method 3.15	0.95	3.598027D-5		3.509838D-5		4.023901D-5	
AMM 2.15	1.00	3.590375D-5	3.590375D-5	3.798602D-5	3.798602D-5	4.098157D-5	4.098157D-5
Method 3.23	1.00	3.339872D-5	3.339872D-5	3.48712 7D-5	3.48712 7D-5	3.995815D-5	3.995815D-5
Method 3.15	1.00	3.190854D-5	3.190854D-5	3.309471D-5	3.309471D-5	3.876391D-5	3.876391D-5

Table 5.2: Errors of Methods for Example 4.2.

Methods	X	Continuous y_1	Discrete y_1	Continuous y_2	Discrete y_2	Continuous y_3	Discrete y_3	Continuous y_4	Discrete y_4
AMM 2.15	0.05	6.679037D-5		6.723853D-5		7.695783D-5		6.985034D-5	
Method 3.23	0.05	6.658492D-5		6.701393D-5		7.674901D-5		6.780548D-5	
Method 3.15	0.05	6.634793D-5		6.701197D-5		7.642097D-5		6.758902D-5	
AMM 2.15	0.10	6.643905D-5	6.643905D-5	6.701947D-5	6.701947D-5	7.678041D-5	7.678041D-5	6.870491D-5	6.870491D-5
Method 3.23	0.10	6.634091D-5	6.634091D-5	6.690573D-5	6.690573D-5	7.650437D-5	7.650437D-5	6.760931D-5	6.760931D-5
Method 3.15	0.10	6.610487D-5	6.610487D-5	6.678031D-5	6.678031D-5	7.623971D-5	7.623971D-5	6.731093D-5	6.731093D-5
AMM 2.15	0.15	6.612094D-5		6.698023D-5		7.621094 D-5		6.830914D-5	
Method 3.23	0.15	6.298037D-5		6.635182D-5		7.590821D-5		6.710984D-5	
Method 3.15	0.15	6.067104D-5		6.540915D-5		7.541903D-5		6.701982D-5	
AMM 2.15	0.20	6.598047D-5	6.598047D-5	6.625093D-5	6.625093D-5	7.602985D-5	7.602985D-5	6.801487D-5	6.801487D-5
Method 3.23	0.20	6.093713D-5	6.093713D-5	6.603071D-5	6.603071D-5	7.570391D-5	7.570391D-5	6.609027D-5	6.609027D-5
Method 3.15	0.20	6.021047D-5	6.021047D-5	6.540915D-5	6.540915D-5	7.531093D-5	7.531093D-5	6.658734D-5	6.658734D-5
AMM 2.15	0.25	6.560932D-5		6.602817D-5		7.598031D-5		6.790254D-5	
Method 3.23	0.25	6.054903D-5		6.598702D-5		7.534095D-5		6.678021D-5	
Method 3.15	0.25	6.987017D-5		6.520971D-5		7.511096D-5		6.658734D-5	
AMM 2.15	0.30	6.530275D-5	6.530275D-5	6.598702D-5	6.598702D-5	7.560915D-5	7.560915D-5	6.745727D-5	6.745727 D-5
Method 3.23	0.30	6.021937D-5	6.021937D-5	6.573204D-5	6.573204D-5	7.509253D-5	7.509253D-5	6.643091D-5	6.643091 D-5
Method 3.15	0.30	6.870921D-5	6.870921D-5	6.501279D-5	6.501279D-5	7.498702D-5	7.498702D-5	6.612094D-5	6.612094 D-5
AMM 2.15	0.35	6.501964D-5		6.509372D-5		7.499341D-5		6.720193D-5	
Method 3.23	0.35	6.980912D-5		5.980615D-5		7.463017D-5		6.601285D-5	
Method 3.15	0.35	5.834019D-5		5.670921D-5		7.309318D-5		6.598073D-5	
AMM 2.15	0.40	6.487093D-5	6.487093D-5	6.498027D-5	6.498027D-5	7.450916D-5	7.450916D-5	6.697811D-5	6.697811D-5
Method 3.23	0.40	5.967092D-5	5.967092D-5	5.789031D-5	5.789031D-5	7.420971D-5	7.420971D-5	6.580293D-5	6.580293D-5
Method 3.15	0.40	5.834091D-5	5.834091D-5	5.390782D-5	5.390782D-5	7.109536D-5	7.109536D-5	6.561014D-5	6.561014D-5
AMM 2.15	0.45	6.380791D-5		6.476091D-5		7.420811D-5		6.674501D-5	
Method 3.23	0.45	5.939102D-5		5.759022D-5		7.390154D-5		6.540971D-5	
Method 3.15	0.45	5.792851D-5		5.340917D-5		7.097103D-5		6.524795D-5	
AMM 2.15	0.50	6.109232D-5	6.109232D-5	6.410937D-5	6.410937D-5	7.409712D-5	7.409712D-5	6.631092D-5	6.631092D-5
Method 3.23	0.50	5.864011D-5	5.864011D-5	5.720913D-5	5.720913D-5	7.350117D-5	7.350117D-5	6.521096D-5	6.521096D-5
Method 3.15	0.50	5.730127D-5	5.730127D-5	5.291054D-5	5.291054D-5	7.045231D-5	7.045231D-5	6.501124D-5	6.501124D-5

AMM 2.15	0.55	6.034911D-5		6.398104 D-5		7.029605D-5		6.476901D-5	
Method 3.23	0.55	5.834093D-5		5.543733D-5		7.112096D-5		6.323011D-5	
Method 3.15	0.55	5.687451D-5		5.039821D-5		6.789105D-5		6.211067D-5	
AMM 2.15	0.60	5.896021D-5	5.896021D-5	6.093512D-5	6.093512D-5	7.011283D-5	7.011283D-5	6.201134D-5	6.201134D-5
Method 3.23	0.60	5.655489D-5	5.655489D-5	5.454791D-5	5.454791D-5	6.987021D-5	6.987021D-5	6.098147D-5	6.098147D-5
Method 3.15	0.60	5.321992D-5	5.321992D-5	4.998212D-5	4.998212D-5	6.340143D-5	6.340143D-5	6.012644D-5	6.012644D-5
AMM 2.15	0.65	5.657241D-5		6.048122D-5		6.970457D-5		6.091254D-5	
Method 3.23	0.65	5.329082D-5		5.201947D-5		6.769024D-5		6.054713D-5	
Method 3.15	0.65	5.199806D-5		4.750103D-5		6.109873D-5		5.958701D-5	
AMM 2.15	0.70	5.309554D-5	5.309554D-5	6.012842D-5	6.012842D-5	6.759803D-5	6.759803D-5	6.038742D-5	6.038742D-5
Method 3.23	0.70	5.109876D-5	5.109876D-5	6.039112D-5	6.039112D-5	6.309112D-5	6.309112D-5	5.879024D-5	5.879024D-5
Method 3.15	0.70	5.034191D-5	5.034191D-5	6.021334D-5	6.021334D-5	6.021334D-5	6.021334D-5	5.340247D-5	5.340247D-5
AMM 2.15	0.75	5.039844D-5		5.970225D-5		6.598021D-5		5.690225D-5	
Method 3.23	0.75	4.987041D-5		4.989012D-5		6.059835D-5		5.340911D-5	
Method 3.15	0.75	4.798012D-5		4.321105D-5		6.011309D-5		5.095882D-5	
AMM 2.15	0.80	5.011209D-5	5.011209D-5	5.931023D-5	5.931023D-5	6.322901D-5	6.322901D-5	5.390122D-5	5.390122D-5
Method 3.23	0.80	4.509347D-5	4.509347D-5	4.650591D-5	4.650591D-5	6.032114D-5	6.032114D-5	5.048723D-5	5.048723D-5
Method 3.15	0.80	4.232685D-5	4.232685D-5	4.133507D-5	4.133507D-5	5.780921D-5	5.780921D-5	5.012094D-5	5.012094D-5
AMM 2.15	0.85	4.709663D-5		5.789012D-5		6.083985D-5		5.093855D-5	
Method 3.23	0.85	4.390865D-5		4.680114D-5		5.982091D-5		5.012157D-5	
Method 3.15	0.85	4.079511D-5		4.015745D-5		5.439033D-5		4.850331D-5	
AMM 2.15	0.90	4.422174D-5	4.422174D-5	5.409224D-5	5.409224D-5	5.890331D-5	5.890331D-5	5.035711D-5	5.035711D-5
Method 3.23	0.90	4.110934D-5	4.110934D-5	4.389021D-5	4.389021D-5	5.578922D-5	5.578922D-5	4.790332D-5	4.790332D-5
Method 3.15	0.90	4.011283D-5	4.011283D-5	3.908244D-5	3.908244D-5	5.230117D-5	5.230117D-5	4.503411D-5	4.503411D-5
AMM 2.15	0.95	4.209475D-5		5.109325D-5		5.640921D-5		4.890722D-5	
Method 3.23	0.95	4.019831D-5		4.102968D-5		5.209575D-5		4.320913D-5	
Method 3.15	0.95	3.890112D-5		3.689021D-5		5.012773D-5		4.011298D-5	
AMM 2.15	1.00	4.012874D-5	4.012874D-5	4.980227D-5	4.980227D-5	5.209432D-5	5.209432D-5	4.570924D-5	4.570924D-5
Method 3.20	1.00	3.894031D-5	3.894031D-5	3.810452D-5	3.810452D-5	5.010988D-5	5.010988D-5	4.012985D-5	4.012985D-5
Method 3.15	1.00	3.409225D-5	3.409225D-5	3.257091D-5	3.257091D-5	4.809237D-5	4.809237D-5	3.780912D-5	3.780912D-5

We note from this table the effectiveness of the new class in terms of higher accuracy. This is further confirmed by the results in Tables 5.2 and 5.2 of Section 4. The class of methods (3.11), (3.13) and (3.15) cannot be discountenanced as it performs better than the class of Adams-Moulton methods, even though the third class of methods (3.11), (3.21) and (3.23) is the choicest of the three.

The need for the continuous forms of solution is justified as they are characterized by the ability to yield several output/values/solutions at the off-grid points without requiring additional interpolation and also at no extra vis-à-vis their discrete form equivalents.

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