ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" DIN IAŞI (S.N.)
MATEMATICĂ, Tomul LV, 2009, f.2

REGULARITY OF THE MINIMAL TIME FUNCTION FOR THE HEAT EQUATION

BY

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Abstract. We study the regularity of the minimal time function $\mathcal{T}_p(\cdot)$, $p \in [1, +\infty]$, associated with the linear control system $y' = Ay + Bu$ on a Banach space X. We show the connection between the estimates of $\mathcal{T}_\infty(\cdot)$ and the estimates of $\mathcal{T}_p(\cdot)$, $p \in [1, +\infty)$. In particular, we consider the case where B is the embedding operator from a Banach space $X_0 \subseteq X$ to X and we obtain the Hölder continuity of the associated minimal time function. We apply the abstract results to the controlled heat equation.

Mathematics Subject Classification 2000: 93C05, 93C25, 49L20.
Key words: linear control system, minimal time function, regularity.

1. Introduction. Let us consider a control system described by a differential equation of the type

$$y'(t) = Ay(t) + Bu(t), \ t \geq 0 \quad (1.1)$$
$$y(0) = \xi,$$

where A generates a C_0-semigroup on a Banach space X, B is a linear and bounded operator from a Banach space U to X and the control function u is assumed to be in $L^p(0, \infty; U)$, $p \in [1, \infty]$. For $p \in [1, +\infty]$ we take the admissible set of controls as

$$U_{ad}^{\rho} = \bigcup_{t>0} \left\{ u \in L^p(0, t; U); \ \ \|u\|_p \leq \rho \right\},$$

where $\rho > 0$ is a given constant. For $t \geq 0$, denote by $R(t)$ the set of all initial states in X which can be transferred to zero during $[0, t]$ by admissible controls, that is

$$R(t) = \left\{\xi \in X;\ y(t,\xi,u) = 0 \text{ for some } u \in U_{ad}^{\rho}\right\},$$

where y is the mild solution of (1.1), i.e.

$$y(t,\xi,u) = S(t)\xi + \int_0^t S(t-s)\,Bu(s)\,ds.$$

Let $R = \cup_{t\geq 0} R(t)$ be the set of all states controllable to zero in free time by admissible controls, and, corresponding to the admissible set of controls U_{ad}^{ρ}, define the minimal time function $\mathcal{T}_p : X \to [0,\infty]$ by

$$\begin{aligned} &\mathcal{T}_p(\xi) = \inf\{t;\ \xi \in R(t)\} && \text{for } \xi \in R, \\ &\mathcal{T}_p(\xi) = +\infty && \text{for } \xi \notin R. \end{aligned} \tag{1.2}$$

In case where $U = X$ and B is the identity operator, it is shown in [1] that the minimal time function $\mathcal{T}(\cdot)$ is locally Lipschitz continuous on the reachable set. Moreover, if $S(t)$ is a semigroup of contractions, then $R = X$ and the minimal time function is globally Lipschitz (see also [2]). In [8, 9] it is considered a control system described by the wave equation, which can be rewritten in a product space as (1.1), with B of the form $Bu = \binom{0}{u}$, and it is shown that the minimal time function is Hölder continuous. In [2] and [7] other regularity results of the minimal time function for system (1.1) are proved. See also [3] and [4] for the semilinear case.

The purpose of this paper is to find new regularity results for the minimal time function $\mathcal{T}_p$. The motivation of this study comes from the fact that the properties of the minimal time function provide the basis of the dynamic programming method in optimal control. Also, the minimal time function plays an important role in the study of Hamilton-Jacobi equation (see, e.g., [12, 6, 10]). In [2] and [7] the minimal time functions $\mathcal{T}_p$, $p \in (1,+\infty)$, and $\mathcal{T}_\infty$ are studied separately and the estimates are obtained by making use of the minimal energy function. Here, we show that the regularity properties of $\mathcal{T}_p$, with $p \in [1,+\infty)$, can be deduced from the corresponding properties of $\mathcal{T}_\infty$. As an application we study the regularity of the minimal time function $\mathcal{T}_p$ in case $U = X_0$ with $X_0 \subset X$ continuously, and B is the embedding operator from X_0 to X.

2. The minimal time function. The following result establishes a connection between the estimates around the target of $\mathcal{T}_\infty(\cdot)$ and the estimates of $\mathcal{T}_p(\cdot)$, with $p \in [1,+\infty)$.

Theorem 2.1. *Suppose there exist $a > 0$ and a continuous function $\beta : [0, a] \to [0, +\infty)$ with $\beta(0) = 0$, strictly increasing, such that*

$$\mathcal{T}_\infty(\xi) \le \beta(\|\xi\|) \text{ for } \|\xi\| \le a.$$

Let $p \in [1, +\infty)$ be such that the function $\alpha(t) := t^{-\frac{1}{p}}\beta^{-1}(t)$, $t \in (0, \beta(a)]$ is strictly increasing and satisfies $\lim_{t\downarrow 0}\alpha(t) = 0$. Then, we have that

$$\mathcal{T}_p(\xi) \le \alpha^{-1}(\|\xi\|) \text{ for } \|\xi\| \le \alpha(\beta(a)).$$

Proof. Let $\xi \in X$ with $\|\xi\| \le \alpha(\beta(a))$. There exists $t \le \beta(a)$ such that $\|\xi\| = \alpha(t)$. Moreover, there exists $b \le a$ such that $\beta(b) = t$. For $\varepsilon \in (0, 1)$ define $z = \varepsilon b\frac{\xi}{\|\xi\|}$, so $\|z\| < b$. We have that $\mathcal{T}_\infty(z) \le \beta(\|z\|) < t$. Hence, there exists $u \in L^\infty(0, t; U)$ with $\|u(s)\|_U \le \rho$ a.e. on $[0, t]$ such that $y(t, z, u) = 0$. Then, we get that $y(t, \xi, \overline{u}) = 0$, where $\overline{u} = \frac{\|\xi\|}{\varepsilon b}u$ and

$$\|\overline{u}\|_{L^p(0,t;U)} \le t^{\frac{1}{p}}\frac{\|\xi\|\,\rho}{\varepsilon b} \le \frac{\rho}{\varepsilon}.$$

Taking $\varepsilon \to 1$ we get $\|\overline{u}\|_{L^p(0,t;U)} \le \rho$. We thus obtain

$$\mathcal{T}_p(\xi) \le t = \alpha^{-1}(\|\xi\|).$$

□

Suppose now that $U = X_0$ with $X_0 \subset X$ continuously and $B : X_0 \to X$ is the embedding operator and make the following assumption over the semigroup: there exist $c > 0$ and $a > 0$ such that

$$\|S(t)\xi\|_{X_0} \le ct^{-a}\|\xi\|_X, \text{ for } t > 0,\ \xi \in X. \tag{2.1}$$

The following theorem gives an estimate of the minimal time function $\mathcal{T}_\infty$ on X.

Theorem 2.2. *Suppose that* (2.1) *holds for some $c > 0$ and $a > 0$. Then $R = X$ and there exists $k > 0$ such that*

$$\mathcal{T}_\infty(\xi) \le k\|\xi\|^{\frac{1}{a+1}}, \tag{2.2}$$

for any $\xi \in X$.

Proof. Let $\xi \in X$, $\xi \neq 0$. Consider the control

$$u(t) = -\frac{\rho S(t)\xi}{\|S(t)\xi\|_{X_0}}, \ t > 0.$$

Obviously, $\|u(t)\|_{X_0} = \rho$ for any $t > 0$, so $u(\cdot)$ is an admissible control. We have that

$$\begin{aligned} y(t,\xi,u) &= S(t)\xi - \int_0^t \rho S(t-s) \frac{S(s)\xi}{\|S(s)\xi\|_{X_0}} ds \\ &= \left(1 - \int_0^t \frac{\rho}{\|S(s)\xi\|_{X_0}} ds\right) S(t)\xi. \end{aligned}$$

Using the inequality (2.1) we obtain

$$\begin{aligned} 1 - \int_0^t \frac{\rho}{\|S(s)\xi\|_{X_0}} ds &\leq 1 - \int_0^t c^{-1} s^a \frac{\rho}{\|\xi\|_X} ds \\ &= 1 - \frac{1}{c(a+1)} \frac{\rho}{\|\xi\|_X} t^{a+1}. \end{aligned}$$

Let $\bar{t} = [c(a+1)]^{\frac{1}{a+1}} \rho^{-\frac{1}{a+1}} \|\xi\|_X^{\frac{1}{a+1}}$. As $\int_0^{\bar{t}} \frac{\rho}{\|S(s)\xi\|_{X_0}} ds \geq 1$, $\int_0^t \frac{\rho}{\|S(s)\xi\|_{X_0}} ds$ is a continuous function and $\lim_{t\downarrow 0} \int_0^t \frac{\rho}{\|S(s)\xi\|_{X_0}} ds = 0$ we obtain that there exists $\tau \in (0, \bar{t})$ such that $\int_0^\tau \frac{\rho}{\|S(s)\xi\|_{X_0}} ds = 1$. Hence, $y(\tau,\xi,u) = 0$, with $u \in U_{ad}^{\rho}$. In conclusion,

$$\mathcal{T}_\infty(\xi) \leq k \|\xi\|^{\frac{1}{a+1}},$$

where $k = [c(a+1)]^{\frac{1}{a+1}} \rho^{-\frac{1}{a+1}}$. □

Remark 2.1. In the previous proof we considered that $\|S(s)\xi\|_{X_0} \neq 0$ for any $s > 0$. Otherwise, if there exists $t_0 > 0$ such that $\|S(t_0)\xi\|_{X_0} = 0$ then $\xi \in R(t_0)$, so $\mathcal{T}_\infty(\xi) \leq t_0$. Hence, if $t_0 \leq \bar{t}$ the assertion of the theorem clearly holds, and if $t_0 > \bar{t}$ we can follow the previous proof to obtain the conclusion.

By Theorem 2.2 we obtain a result regarding the null controllability of system (1.1). Recall that system (1.1) is said to be null controllable by L^p-controls, $p \in [1,+\infty]$ if for any $t > 0$ and $\xi \in X$ there exists a control function $u \in L^p(0,t;X_0)$ such that the corresponding mild solution $y(\cdot)$ of (1.1) satisfies $y(t,\xi,u) = 0$.

Corollary 2.1. *Under the hypothesis of Theorem* 2.2, *the system* (1.1) *is null controllable by* L^{∞}*-controls.*

In [2] (see also [7]) it is proved that the regularity properties of $\mathcal{T}_{\infty}(\cdot)$ around the target provide the same regularity properties of $\mathcal{T}_{\infty}(\cdot)$ on the whole reachable set, using the Bellman optimality principle ([2, Lemma 2.1]). Moreover, if $S(t)$ is a contraction semigroup, then the regularity properties are global. In our case, we obtain the following consequence.

Proposition 2.1. *Under the hypothesis of Theorem* 2.2, *the minimal time function* $\mathcal{T}_{\infty}(\cdot)$ *is Hölder continuous on* X, *i.e. there exists* $\overline{k} > 0$ *such that*

$$|\mathcal{T}_{\infty}(\xi_1) - \mathcal{T}_{\infty}(\xi_2)| \leq \overline{k} \, \|\xi_1 - \xi_2\|^{\frac{1}{a+1}},$$

for any $\xi_1, \xi_2 \in X$.

Now, let $p \in [1, +\infty)$. Applying Theorem 2.1 with $\beta(r) = kr^{\frac{1}{a+1}}$ we obtain estimates for $\mathcal{T}_p$.

Corollary 2.2. *Assume the hypothesis of Theorem* 2.2 *and let* $p \in [1, \infty)$. *Then* $R = X$ *and there exists* $K > 0$ *such that*

$$\mathcal{T}_p(\xi) \leq K \, \|\xi\|^{\frac{1}{a+1-\frac{1}{p}}},$$

for any $\xi \in X$. *Moreover, the system* (1.1) *is null controllable by* L^p*-controls.*

In [7, Theorem 5.16] it is proved the Bellman optimality principle for $p \in (1, +\infty)$. The authors added a new variable ρ, the radius of the admissible controls, for the minimal time function, denoted now by $\mathcal{T}_p(\rho, \xi)$. We rewrite the conclusion of Corollary 2.2, giving the explicit form of K,

$$\mathcal{T}_p(\rho, \xi) \leq \widetilde{k} \rho^{-\frac{1}{a+1-\frac{1}{p}}} \|\xi\|^{\frac{1}{a+1-\frac{1}{p}}}, \tag{2.3}$$

for any $\xi \in X$, with $\widetilde{k} = [c(a+1)]^{\frac{1}{a+1-\frac{1}{p}}}$. Moreover, using the same line of the proof of [7, Theorem 4.8], we obtain the following result regarding the Hölder continuity of the minimal time function $\mathcal{T}_p$.

Theorem 2.3. *Assume the hypothesis of Theorem* 2.2. *Let* $p \in (1, +\infty)$ *and* $\rho_0 > 0$. *Then,* $\mathcal{T}_p(\rho_0, \cdot)$ *is locally Hölder continuous on* X, *i.e. for every* $\xi \in X$ *there exist a neighborhood* $\mathcal{W}$ *of* ξ *and* $K_{\rho_0} > 0$ *such that*

$$|\mathcal{T}_p(\rho_0, \xi_1) - \mathcal{T}_p(\rho_0, \xi_2)| \leq K_{\rho_0} \|\xi_1 - \xi_2\|^{\frac{1-\frac{1}{p}}{a+1-\frac{1}{p}}},$$

for any $\xi_1, \xi_2 \in \mathcal{W}$.

Proof. Let $\xi \in X$ and $\mathcal{W} = B\left(\xi, \frac{1}{2}\right)$. Let $\xi_1, \xi_2 \in \mathcal{W}$ and denote by $\lambda = 1 - \|\xi_1 - \xi_2\|$. Suppose that $\mathcal{T}_p(\rho_0, \xi_1) > \mathcal{T}_p(\rho_0, \xi_2)$. Let $t > 0$ and $v \in U_{ad}^{\rho_0}$ be such that $y(t, \xi_2, v) = 0$. By the Bellman principle [7, Theorem 4.8] we obtain that

$$\mathcal{T}_p(\rho_0, \xi_1) \leq t + \mathcal{T}_p(\rho(t, \rho_0, \lambda v), y(t, \xi_1, \lambda v)),$$

where $\rho(t, \rho_0, u)$ satisfies

$$\rho^p(t, \rho_0, u) = \rho_0^p - \int_0^t \|u(s)\|^p\, ds. \tag{2.4}$$

Using (2.3) we get

$$\mathcal{T}_p(\rho(t, \rho_0, \lambda v), y(t, \xi_1, \lambda v)) \leq \widetilde{k}\,[\rho(t, \rho_0, \lambda v)]^{-\frac{1}{a+1-\frac{1}{p}}} \|y(t, \xi_1, \lambda v)\|^{\frac{1}{a+1-\frac{1}{p}}}.$$

Since $y(t, \xi_2, v) = 0$ we have that

$$y(t, \xi_1, \lambda v) = S(t)(\xi_1 - \xi_2) + (1 - \lambda) S(t)\, \xi_2.$$

By (2.1) we obtain that the semigroup $S(t)$ is bounded, i.e. there exists $M > 0$ such that $\|S(t)\| \leq M$ for any $t \geq 0$. Hence,

$$\|y(t, \xi_1, \lambda v)\| \leq M\|\xi_1 - \xi_2\|(1 + \|\xi_2\|) \leq M\|\xi_1 - \xi_2\|\left(\frac{3}{2} + \|\xi\|\right).$$

On the other hand, by (2.4),

$$\rho^p(t, \rho_0, \lambda v) = \rho_0^p - \lambda^p \int_0^t \|v(s)\|^p\, ds \geq \rho_0^p(1 - \lambda^p) > \rho_0^p(1 - \lambda).$$

So, we have

$$\begin{aligned}
&\mathcal{T}_p(\rho(t, \rho_0, \lambda v), y(t, \xi_1, \lambda v)) \\
&\leq \widetilde{k}\rho_0^{-\frac{1}{a+1-\frac{1}{p}}}(1 - \lambda)^{-\frac{1}{p}\frac{1}{a+1-\frac{1}{p}}}\left(M\|\xi_1 - \xi_2\|\left(\frac{3}{2} + \|\xi\|\right)\right)^{\frac{1}{a+1-\frac{1}{p}}} \\
&\leq \widetilde{k}\rho_0^{-\frac{1}{a+1-\frac{1}{p}}} M^{\frac{1}{a+1-\frac{1}{p}}}\left(\frac{3}{2} + \|\xi\|\right)^{\frac{1}{a+1-\frac{1}{p}}}\|\xi_1 - \xi_2\|^{\frac{1-\frac{1}{p}}{a+1-\frac{1}{p}}}.
\end{aligned}$$

Letting $t \downarrow \mathcal{T}_p(\rho_0, \xi_2)$ we obtain

$$\mathcal{T}_p(\rho_0, \xi_1) \leq \mathcal{T}_p(\rho_0, \xi_2) + K \left\| \xi_1 - \xi_2 \right\|^{\frac{1-\frac{1}{p}}{a+1-\frac{1}{p}}},$$

with $K = \widetilde{k} M^{\frac{1}{a+1-\frac{1}{p}}} \left(\frac{3}{2} + \|\xi\|\right)^{\frac{1}{a+1-\frac{1}{p}}} \rho_0^{-\frac{1}{a+1-\frac{1}{p}}}$. By interchanging the roles of ξ_1 and ξ_2 we obtain the conclusion. □

3. Application. A typical example for the case considered in this paper is the following:

Let Ω be a domain in $\mathbb{R}^n$ and consider the control system described by the heat equation

$$\begin{array}{ll} y_t = \Delta y + u, & (t,x) \in \mathbb{R}_+ \times \Omega \\ y = 0, & (t,x) \in \mathbb{R}_+ \times \partial\Omega \\ y(0,x) = \xi(x), & x \in \Omega. \end{array} \tag{3.1}$$

Let $1 \leq p_1 \leq p_2 < \infty$. We are interested in transferring the initial state $\xi \in L^{p_1}(\Omega)$ into the origin, in minimum time, by an admissible control $u(t,x)$ satisfying

$$\int_\Omega |u(t,x)|^{p_2}\, dx \leq 1.$$

We rewrite equation (3.1) as an ordinary differential equation in $L^{p_1}(\Omega)$ of the form

$$\begin{array}{l} y'(t) = A_{p_1} y(t) + Bu(t), \quad t \geq 0 \\ y(0) = \xi, \end{array} \tag{3.2}$$

where A_{p_1} is the Laplace operator subjected to the Dirichlet boundary condition on $L^{p_1}(\Omega)$, B is the embedding operator from $L^{p_2}(\Omega)$ to $L^{p_1}(\Omega)$ and $u(t) \in L^{p_2}(\Omega)$ for $t \geq 0$. It is known that A_{p_1} generates a C_0-semigroup of contractions $S(t)$, $t \geq 0$, on $L^{p_1}(\Omega)$ (see [11]). Moreover, if Ω is a nonempty, bounded and open subset in $\mathbb{R}^n$, whose boundary is of class C^1, then the semigroup $S(t)$ generated by the Laplace operator on $L^{p_1}(\Omega)$ satisfies a condition of type (2.1). More exactly,

$$\|S(t)\xi\|_{L^{p_2}(\Omega)} \leq (4\pi t)^{-\frac{n}{2}\left(\frac{1}{p_1} - \frac{1}{p_2}\right)} \|\xi\|_{L^{p_1}(\Omega)},$$

for any $\xi \in L^{p_1}(\Omega)$ and any $t > 0$ (see [11, Theorem 7.2.6]).

In this section we are going to apply the abstract results previously obtained to the system (3.2).

First, by Theorem 2.2 and Corollary 2.2 we obtain the following estimates for $\mathcal{T}_p$, with $p \in [1, +\infty]$.

Theorem 3.1. *Let Ω be a nonempty, bounded and open subset in $\mathbb{R}^n$, whose boundary is of class C^1 and $1 \leq p_1 \leq p_2 \leq \infty$. Then, for any $\xi \in X$, we have*

$$\mathcal{T}_\infty(\xi) \leq k \|\xi\|^{\frac{1}{\frac{n}{2}\left(\frac{1}{p_1} - \frac{1}{p_2}\right)+1}}, \tag{3.3}$$

for some $k > 0$. In case $p \in [1, +\infty)$, for any $\xi \in X$ and for some $K > 0$, we have

$$\mathcal{T}_p(\xi) \leq K \|\xi\|^{\frac{1}{\frac{n}{2}\left(\frac{1}{p_1} - \frac{1}{p_2}\right)+1-\frac{1}{p}}}. \tag{3.4}$$

In [5], we obtained estimates for the minimal time function $\mathcal{T}_\infty$ and we showed that $\mathcal{T}_\infty$ is globally Hölder continuous, in the case where $1 < p_2 < \infty$. We point out that the approach used in [5] requires that $L^{p_2}(\Omega)$ should be reflexive. Here, by a completely different approach, we extend this results to the case where $p_2 \in [1, +\infty]$. Moreover, we obtain regularity results for $\mathcal{T}_p$, with $p \in (1, +\infty)$.

Proposition 3.1. *Let Ω be a nonempty, bounded and open subset in $\mathbb{R}^n$, whose boundary is of class C^1. Let $1 \leq p_1 \leq p_2 \leq \infty$. Then, $\mathcal{T}_\infty(\cdot)$ is globally Hölder continuous, i.e. there exists $k > 0$ such that*

$$|\mathcal{T}_\infty(\xi_1) - \mathcal{T}_\infty(\xi_2)| \leq k \|\xi_1 - \xi_2\|^{\frac{1}{\frac{n}{2}\left(\frac{1}{p_1} - \frac{1}{p_2}\right)+1}}, \tag{3.5}$$

for any $\xi_1, \xi_2 \in L^{p_1}(\Omega)$.

Remark 3.1. In the particular case when $p_1 = p_2$ we get the known result that $\mathcal{T}_\infty(\cdot)$ is Lipschitz continuous.

Using Theorem 2.3 and (3.4), we get the local Hölder continuity for $\mathcal{T}_p(\cdot)$, $p \in (1, +\infty)$.

Corollary 3.1. *Let Ω be a nonempty, bounded and open subset in $\mathbb{R}^n$, whose boundary is of class C^1. Let $p \in (1, \infty)$ and $1 \leq p_1 \leq p_2 \leq \infty$. Then,*

$\mathcal{T}_p(\cdot)$ is locally Hölder continuous, i.e. for every $\xi \in L^{p_1}(\Omega)$ there exists a neighborhood $\mathcal{W}$ of ξ and a constant $K > 0$ such that

$$|\mathcal{T}_p(\xi_1) - \mathcal{T}_p(\xi_2)| \leq K \|\xi_1 - \xi_2\|^{\frac{1-\frac{1}{p}}{\frac{n}{2}\left(\frac{1}{p_1}-\frac{1}{p_2}\right)+1-\frac{1}{p}}}, \tag{3.6}$$

for any $\xi_1, \xi_2 \in \mathcal{W}$.

Remark 3.2. Let us observe that in case $p = +\infty$ the exponent from the norm of ξ in (3.3) is the same as in (3.5) for the Hölder continuity. In case $p \in (1, +\infty)$ the exponents are different, more exactly, the exponent from (3.4) multiplied by $1 - \frac{1}{p}$ gives the exponent for Hölder continuity in (3.6).

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Received: 8.XII.2008
Revised: 2.II.2009

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