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TANGENT SETS IN THE VIABILITY OF DIFFERENTIAL EQUATIONS DRIVEN BY YOUNG INTEGRAL*

BY

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Abstract. In this paper we introduce a $g-$contingent set and a weak $g-$contingent con for the differential equations driven by a control function g which is Hölder continuous of order $\mu > 1/2$ and we study their properties. It is also established the relation between the two $g-$contingent sets and the viability property for the differential equations.

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1. Introduction. Suppose that f and g are Hölder continuous functions on the interval $[a, b]$, of order λ and μ, respectively, with $\lambda+\mu > 1$. Then the Riemann-Stieltjes integral $\int_a^b f dg$ can be expressed as a Lebesgue integral using fractional derivatives (see ZAHLE [7], [9]). This fact has been exploited by NUALART and RĂŞCANU in [4]. They analyze dynamic systems driven by a control function g which is Hölder continuous of order $\mu > 1/2$ and proved a global existence and uniqueness result of the solution for a time dependent differentiae equation with the form

$$X_t^{0,x_0} = x_0 + \int_0^t b\left(s, X_s^{0,x_0}\right) ds + \int_0^t \sigma\left(s, X_s^{0,x_0}\right) dg\left(s\right) \quad (1)$$

where

- $W_T^{1-\alpha,\infty}\left(0, T; \mathbb{R}^m\right)$ is the space of measurable function $g : [0, T] \to \mathbb{R}$ satisfying (5)

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- $g \in W_T^{1-\alpha,\infty}(0,T;\mathbb{R}^m)$
- $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ are measurable functions.

It is also proved that such a solution is $(1-\alpha)-$ Hölder continuous.

The same equation was used by Răşcanu and Ciotir in [3] to prove a viability result for stochastic differential equations driven by fractional Brownian motion. See also [2] and [3].

They introduced the notion of $(1-\alpha)$ fractional $g-$contingence to K like it follows.

Definition 1. *Let $(t,x) \in [0,T]\times K$. We say that the pair $(b(t,x),\sigma(t,x))$ is $(1-\alpha)$ fractional g contingent to K in (t,x) if there exist $\bar{h} = \bar{h}^{t,x,\theta} > 0$, a function $Q = Q^{t,x,\theta} : [t, t+\bar{h}] \to \mathbb{R}^d$ and three constants $c, L > 0$, $\delta, \mu \in (0,1)$ depending only on $\theta = (\alpha, T, b, \sigma, g, d, k)$ such that:*

(d_1) $|Q(s)| \le c(1+|x|)(s-t)^{\delta}$, *for all* $s \in \left[t, t+\bar{h}\right]$;

(d_2) $\tilde{Q} : \left[t, t+\bar{h}\right] \to \mathbb{R}^d$ *defined by* $\tilde{Q}(s) = (s-t)Q(s)$ *is* $L(1+|x|)-\mu$ *Hölder continuous with the Hölder constant* $L(1+|x|)$.*i.e. i.e.* $|\tilde{Q}(s) - \tilde{Q}(\tau)| \le L(1+|x|)|s-\tau|^{\mu}$, *for all* $s,\tau \in \left[t, t+\bar{h}\right]$ *and satisfying*

(d_3) $x + hb(t,x) + \sigma(t,x)[g(t+h) - g(t)] + hQ(t+h) \in K$, *for all* $h \in [0,\bar{h}]$.

Recall that a function $x : [0,T] \to \mathbb{R}^d$ is said to be viable in a given subset K on $[0,T]$ if, for any $t \in [0,T]$, the state $x(t)$ remains in K.

Definition 2. *A subset $K \subset \mathbb{R}^d$ is viable for the equation* (1) *if, starting at any time $t \in [0,T]$ and from any point x of K, at least one solution to the differential equation evolves in K.*

Definition 3. *The subset $K \subset \mathbb{R}^d$ is said to be invariant for the equation* (1) *if, for any $t \in [0,T]$ and for any starting point x in K, all solutions to the differential equation* (1) *evolve in K.*

Remark that, in the case the equation has a unique solution (which is the case of the equation (1)), viability is equivalent with invariance.

The stochastic viability result from [3] is based on a deterministic theorem which prove that under some Hölder and Lipschitz continuity hypothesis on b and σ following assertions are equivalent:

- K is viable for the fractional differential equation (1), i.e. for all $(t,x) \in [0,T] \times K$ there exists a solution $X^{t,x}(\cdot) \in C^{1-\alpha}\left(t,T;\ \mathbb{R}^d\right)$ of the equation

$$X_s^{t,x} = x + \int_t^{s \vee t} b(r, X_r^{t,x})dr + \int_t^{s \vee t} \sigma(r, X_r^{t,x})dg\left(r\right),\ s \in [t,T],$$

such that $X_s^{t,x} \in K$, for all $s \in [t,T]$.

- For all $t \in [0,T]$ and all $x \in K$ the pair $(b(t,x), \sigma(t,x))$ is $(1-\alpha)$ fractional g contingent to K in (t,x).

Remark that the theorem is a Nagumo type result, but the necessary condition does not refer to a contingent con, independent of the functions b and σ, like in the classical theory.

The purpose of this paper is to introduce such sets (a $g-$contingent set and a weak $g-$contingent con for a nonempty closed set K) and to study their properties and the relation between them and viability of K for the equation (1).

The organization of the paper is as follows.In section 2 we recall some classical definitions. In section 3 we consider the assumptions on the coefficients supposed to hold. and we state and prove the main results.

2. Preliminaries.

Let a, b with $a < b$ and

$$\|f\|_{L^p(a,b)} = \begin{cases} \left(\displaystyle\int_a^b |f(t)|^p\, dt\right)^{1/p}, & \text{if } 1 \leq p < \infty \\ ess\sup\left\{|f(t)| : t \in [a,b]\right\}, & \text{if } p = \infty. \end{cases}$$

The space of Lebeque measurable functions $f : [a,b] \to \mathbb{R}$ for which $\|f\|_{L^p(a,b)} < \infty$, is denoted by $L^p(a,b)$, $p \geq 1$.

Recall from [5] some basic notions of fractional integral and derivatives. For a function $f \in L^1(a,b)$ and $\alpha > 0$. can be defined the left-sided and right-side fractional Riemann-Liouville integrals of f of order α as follows

$$I_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\, ds$$

and

$$I_{b-}^{\alpha} f(t) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_a^t (s-t)^{\alpha-1} f(s)\, ds$$

for almost all $x \in (a,b)$, $(-1)^{-\alpha} = e^{-i\pi\alpha}$ and $\Gamma(\alpha) = \int_0^\infty r^{\alpha-1} e^{-r} dr$ the Euler function. The image of $L^p(a,b)$ by the operator I_{a+}^{α} (resp. I_{b-}^{α}) is denoted by $I_{a+}^{\alpha}(L^p)$ (resp. $I_{b-}^{\alpha}(L^p)$). For a function $f \in I_{a+}^{\alpha}(L^p)$ (resp. $f \in I_{b-}^{\alpha}(L^p)$) and $0 < \alpha < 1$ the Weyl derivatives are defined as

$$D_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(t)}{(t-a)^{\alpha}} + \alpha \int_a^t \frac{f(t) - f(s)}{(t-s)^{\alpha+1}} ds \right)$$

and

$$D_{b-}^{\alpha} f(t) = \frac{(-1)^{-\alpha}}{\Gamma(1-\alpha)} \left(\frac{f(t)}{(b-t)^{\alpha}} + \alpha \int_t^b \frac{f(t) - f(s)}{(s-t)^{\alpha+1}} ds \right),$$

where $a \leq t \leq b$ (the convergence of the integrals at the singularity $s = t$ holds point-wise for almost all $t \in (a,b)$ if $p = 1$ and moreover in L^p -sense if $1 < p < \infty$). For any $f, g \in L^1(a,b)$ we have

$$\int_a^b I_{a+}^{\alpha} f(t)\, g(t)\, dt = (-1)^{\alpha} \int_a^b f(t)\, I_{b-}^{\alpha} g(t)\, dt, \tag{2}$$

and for $f \in I_{a+}^{\alpha}(L^p)$ and $g \in I_{b-}^{\alpha}(L^p)$ we have

$$\int_a^b D_{a+}^{\alpha} f(t)\, g(t)\, dt = (-1)^{\alpha} \int_a^b f(t)\, D_{b-}^{\alpha} g(t)\, dt. \tag{3}$$

We also have from [5] the following inversion formulas:

$$I_{a+}^{\alpha}\left(D_{a+}^{\alpha} f\right) = f, \qquad \forall f \in I_{a+}^{\alpha}(L^p)$$
$$I_{b-}^{\alpha}\left(D_{b-}^{\alpha} f\right) = f, \qquad \forall f \in I_{b-}^{\alpha}(L^p)$$

and

$$D_{a+}^{\alpha}\left(I_{a+}^{\alpha} f\right) = f, \qquad D_{b-}^{\alpha}\left(I_{b-}^{\alpha} f\right) = f, \qquad \forall f \in L^1(a,b).$$

Denote by $C^{\lambda}(a,b)$ the space of $\lambda-$ Hölder continuous functions on the interval $[a,b]$ for any $\lambda \in (0,1)$. Suppose that $f \in C^{\lambda}(a,b)$ and $g \in C^{\mu}(a,b)$ with $\lambda + \mu > 1$. Then, from the classical paper of YOUNG [6], the Riemann-Stieltjes integral $\int_a^b f dg$ exists. The following proposition can be regarded as a fractional integration by parts formula, and provides an explicit expression for the integral $\int_a^b f dg$ in terms of fractional derivatives (see [7]). Let $\lambda > \alpha$ and $\mu > 1 - \alpha$. Then the Riemann-Stieltjes integral $\int_a^b f dg$ exists and it can be expressed as

$$\int_a^b f dg = (-1)^{\alpha} \int_a^b \left(D_{a+}^{\alpha} f\right)(t) \left(D_{b-}^{1-\alpha} g_{b-}\right)(t)\, dt \tag{4}$$

where $g_{b-}(t) = g(t) - g(b)$ .Using (2), (3) and (4), it is easy to see that, for a constant C , we have $\int_a^b C dg = C(g(b) - g(a))$. Fix a parameter $0 < \alpha < 1/2$. Denote by $W_T^{1-\alpha,\infty}(0,T;\mathbb{R}^m)$ the space of measutable function $g : [0,T] \to \mathbb{R}$ such that

$$(5) \quad \|g\|_{1-\alpha,\infty,T} = \sup_{0<s<t<T} \left(\frac{|g(t) - g(s)|}{(t-s)^{1-\alpha}} + \int_s^t \frac{|g(y) - g(s)|}{(y-s)^{1-\alpha}} dy \right) < \infty.$$

From the relation (4) we have

$$(6) \quad \begin{aligned} \left| \int_s^t f(r)\, dg(r) \right| &= \left| \int_s^t \left(D_{s+}^{\alpha} f\right)(r) \left(D_{t-}^{1-\alpha} g_{t-}\right)(r)\, dr \right| \\ &\le \Lambda_\alpha(g) \left(\int_s^t \frac{|f(r)|}{(r-s)^\alpha} dr + \int_s^t \int_s^r \frac{|f(r) - f(y)|}{(r-y)^{\alpha+1}} dy dr \right) \end{aligned}$$

where

$$\begin{aligned} \Lambda_\alpha(g) &= \frac{1}{\Gamma(1-\alpha)} \sup_{0<s<t<T} \left| \left(D_{t-}^{1-\alpha} g_{t-}\right)(s) \right| \\ &\le \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} \|g\|_{1-\alpha,\infty,T} < \infty \end{aligned}$$

since $g \in W_T^{1-\alpha,\infty}(0,T;\mathbb{R}^m)$.

3. Main results. Let $\mathcal{K} = \{K(t); t \ge 0\}$ where $K(t) = \overline{K(t)} \subset \mathbb{R}^d$ are the constraints on the state at the moment t. Recall that a function $x : [0,T] \to \mathbb{R}$ is said to be viable in $\mathcal{K}$ if, for all $t \in [0,T]$ we have $x(t) \in K(t)$ .Consider the differential equation on $\mathbb{R}^d$

$$(7) \quad X_t^{0,x_0} = x_0 + \int_0^t b\left(s, X_s^{0,x_0}\right) ds + \int_0^t \sigma\left(s, X_s^{0,x_0}\right) dg(s)$$

where

- $g \in W_T^{1-\alpha,\infty}(0,T;\mathbb{R}^m)$
- $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ are measurable functions satisfying the assumptions

($\mathbf{H}_b$) There exist some constants $N > 0$ and $0 < \mu < 1$ such that the following properties hold

$$\text{Lipschitz continuity and Hölder continuity in time}$$
$$|b(t,x) - b(t,y)| + |b(t,x) - b(s,x)| \leq N\left(|x-y| + |t-s|^{\mu}\right),$$
$$\forall x, y \in \mathbb{R}^d, \forall t \in [0,T].$$

($\mathbf{H}_\sigma$)There exist some constants $M > 0$ and $0 < \beta < 1$ such that the following properties hold

$$\text{Lipschitz continuity and Hölder continuity in time}$$
$$|\sigma(t,x) - \sigma(t,y)| + |\sigma(t,x) - \sigma(s,x)| \leq M\left(|x-y| + |t-s|^{\beta}\right),$$
$$\forall x, y \in \mathbb{R}^d, \forall t \in [0,T].$$

Fix $0 < \alpha < \min\{1/2, \beta\}$. We define a firsts kind a contingence. In this case the contingent set is not a closed con.

Definition 4. *For any $t \in [0,T]$ and any $x \in K(t)$ we say that a pair $(u_t, V_t) \in \mathbb{R}^d \times \mathbb{R}^{d\times m}$ is g- contingent to $K(t)$ in (t,x) if there exists $\bar{h} > 0$ and a function $Q : \left[t, t+\bar{h}\right] \to \mathbb{R}^d$ such that $\lim_{s\to t}|Q(s)| = 0$ and*

$$x + hu_t + V_t\left(g(t+h) - g(t)\right) + hQ(t+h) \in K(t+h), \text{ for all } h \in \left[0, \bar{h}\right].$$

Definition 5. *The set of pairs $(u_t, V_t) \in \mathbb{R}^d \times \mathbb{R}^{d\times m}$ g- contingent to $K(t)$ in (t,x) forme the g-contingent set to $K(t)$ in (t,x). We denote this set $\mathcal{T}^g_{K(t)}(t,x)$.*

Proposition 6. *For any $t \in [0,T]$, any $x \in K(t)$ and a pair $(u_t, V_t) \in \mathbb{R}^d \times \mathbb{R}^{d\times m}$, the following assertions are equivalent:*

- $\lim\limits_{h\to 0}\dfrac{1}{h} d_{K(t+h)}\left(x + hu_t + V_t\left(g(t+h) - g(t)\right)\right) = 0$
- *There exist $\bar{h} > 0$ and a function $Q : \left[t, t+\bar{h}\right] \to \mathbb{R}^d$ such that $\lim\limits_{s\to t}|Q(s)| = 0$ and*

  $$x + hu_t + V_t\left(g(t+h) - g(t)\right) + hQ(t+h) \in K(t+h), \text{ for all } h \in \left[0, \bar{h}\right].$$

Proof. i)→ii) Let $P_t(h)$ be the projection of $x + hu_t + V_t\left(g(t+h) - g(t)\right)$ on $K(t+h)$. We define

$$Q(t+h) = \frac{1}{h}\left[P_t(h) - x - hu_t - V_t\left(g(t+h) - g(t)\right)\right].$$

ii)$\rightarrow$i) We have

$$d_{K(t+h)}\left(x+hu_t+V_t\left(g\left(t+h\right)-g\left(t\right)\right)\right)\leq\left|hQ\left(t+h\right)\right|.$$

$\square$

Remark 7. Take K independent of t. Any pair $\left(b\left(t,x\right),\sigma\left(t,x\right)\right)\in\mathbb{R}^d\times\mathbb{R}^{d\times m}$ that is $(1-\alpha)$ fractional $g-$contingent to K in (t,x) in the sense of the Definition 1 satisfy

$$\left(b\left(t,x\right),\sigma\left(t,x\right)\right)\in\mathcal{T}_K^g\left(t,x\right).$$

The following proposition prove that the viability of K for the fractional differential equation (7) assure that $\left(b\left(t,x\right),\sigma\left(t,x\right)\right)\in\mathcal{T}_K^g\left(t,x\right)$ under some weaker hypothesis then those used to prove that $\left(b\left(t,x\right),\sigma\left(t,x\right)\right)$ is $(1-\alpha)$ fractional $g-$contingent to K in $\left(t,x\right)$.

Proposition 8. *Let K be a nonempty subset of $\mathbb{R}^d$and assume (H_b) and (H_σ) are satisfied. If K is viable for the fractional differential equation then for all $t\in[0,T]$ and all $x\in K$ the pair $\left(b\left(t,x\right),\sigma\left(t,x\right)\right)\in\mathcal{T}_K^g\left(t,x\right)$*

Proof. Suppose K is viable for the fractional differential equation. Let $(t,x)\in[0,T]\times K$ be arbitrar fixed and $X^{t,x}\in C^{1-\alpha}\left(t,T;\,\mathbb{R}^d\right)$ a solution of the equation (7) such that $X_s^{t,x}\in K$, for all $s\in[t,T]$. Let $\bar{h}=\min\left\{T-t,1\right\}$. Then

$$X_{t+h}^{t,x}=x+\int_t^{t+h}b(r,X_r^{t,x})dr+\int_t^{t+h}\sigma(r,X_r^{t,x})dg\left(r\right)\in K,\quad\forall\ h\in\left[0,\bar{h}\right].$$

We clearly have

$$X_{t+h}^{t,x}=x+hb\left(t,x\right)+\sigma\left(t,x\right)\left(g_{t+h}-g_t\right)+hQ\left(t+h\right)\in K,\quad\forall\ h\in\left[0,\bar{h}\right],$$

where $Q:\left[t,t+\bar{h}\right]\rightarrow\mathbb{R}^d$

$$\begin{aligned}Q\left(s\right)&\overset{def}{=}\frac{1}{s-t}\int_t^s\left[b(r,X_r^{t,x})-b\left(t,x\right)\right]dr\\&\quad+\frac{1}{s-t}\int_t^s\left[\sigma(r,X_r^{t,x})-\sigma\left(t,x\right)\right]dg\left(r\right)\\&=p(s)+q(s).\end{aligned}$$

We denoted

$$p(s)=\begin{cases}0, & \text{if } t=s\\ \dfrac{1}{s-t}\displaystyle\int_t^s [b(r,X_r^{t,x})-b(t,x)]dr, & \text{if } t<s\le T\end{cases}$$

and

$$q(s)=\begin{cases}0, & \text{for } t=s\\ \dfrac{1}{s-t}\displaystyle\int_t^s [\sigma(r,X_r^{t,x})-\sigma(t,x)]dg(r), & \end{cases}.$$

We have

$$\begin{aligned}|p(s)| &= \left|\frac{1}{s-t}\int_t^s \left[b(r,X_r^{t,x})-b\left(t,x\right)\right]dr\right|\\ &\le \sup_{r\in[t,s]}\left|b(r,X_r^{t,x})-b\left(t,x\right)\right|\\ &\le \sup_{r\in[t,s]}|N_1((r-t)^{1-\alpha}+N(r-t)^{\mu})|\\ &\le C_1(s-t)^{\min\{\mu,1-\alpha\}}.\end{aligned}$$

By the assumptions (H_σ) we have

$$\begin{aligned}|\sigma(r,X_r^{t,x})-\sigma(\theta,X_\theta^{t,x})| &\le M[|r-\theta|^{\beta}+|X_r^{t,x}-X_\theta^{t,x}|]\\ &\le M_1[|r-\theta|^{\beta}+|r-\theta|^{1-\alpha}]\end{aligned}$$

and

$$\begin{aligned}|q(s)| &= \left|\frac{1}{s-t}\int_t^s \left[\sigma(r,X_r^{t,x})-\sigma\left(t,x\right)\right]dg\left(r\right)\right|\\ &\le \frac{\Lambda_\alpha(g)}{s-t}\left(\int_t^s \frac{|\sigma(r,X_r^{t,x})-\sigma(t,x)|}{(r-t)^{\alpha}}dr\right.\\ &\quad\left.+\int_t^s\int_t^r \frac{|\sigma(r,X_r^{t,x})-\sigma(\theta,X_\theta^{t,x})|}{(r-\theta)^{1+\alpha}}d\theta dr\right)\\ &\le \left|\frac{\Lambda_\alpha(g)M}{s-t}\left(\int_t^s [|r-t|^{\beta-\alpha}+|r-t|^{1-2\alpha}]dr\right.\right.\\ &\quad\left.\left.+\int_t^s\int_t^r [|r-\theta|^{\beta-1-\alpha}+|r-\theta|^{-2\alpha}]d\theta dr\right)\right|\\ &\le \frac{C_2}{s-t}\{(s-t)^{\beta-\alpha+1}+(s-t)^{2-2\alpha}+(s-t)^{\beta-\alpha+1}+(s-t)^{2-2\alpha}\}\\ &\le C_3(s-t)^{\min\{\beta-\alpha,1-2\alpha\}}.\end{aligned}$$

Where we denoted by $C_1, C_2, C_3, ...$ positive constants independent of t, x and h, that can change from a line to an other. We now have $|Q(s)| \leq c(s-t)^{\delta}$, where $c = C_3 + C_1$ and $\delta = \min\{\beta - \alpha, 1 - 2\alpha, \mu\}$ .It is now easy to see that $(b(t,x), \sigma(t,x)) \in \mathcal{T}_K^g(t,x)$ since $\lim_{s\to t} |Q(s)| = 0$. □

Proposition 9. *If $K(t)$ is a convex set for all $t \in [0,T]$, then $\mathcal{T}_{K(t)}^g(t,x)$ is also a convex set.*

Proof. Let (u_t, V_t) and $(\tilde{u}_t, \tilde{V}_t)$ be two pairs from $\mathcal{T}_{K(t)}^g(t,x)$. Form the Definition 4 we have

$$x + h\ u_t + V_t(g(t+h) - g(t)) + hQ(t+h) \in K(t+h),$$

for all $h \in [0, \bar{h}_1]$ and

$$x + h\ \tilde{u}_t + \tilde{V}_t(g(t+h) - g(t)) + h\ \tilde{Q}(t+h) \in K(t+h),$$

for all $h \in [0, \bar{h}_2]$ with $\lim_{s\to t} Q(s) = \lim_{s\to t} \tilde{Q}(s) = 0$. Set $\bar{h} = \min\{\bar{h}_1, \bar{h}_2\}$. For all $\lambda \in (0,1)$ we have

$$(x + h\ u_t + V_t(g(t+h) - g(t)) + hQ(t+h))\lambda$$

$$+(x + h\ \tilde{u}_t + \tilde{V}_t(g(t+h) - g(t)) + h\ \tilde{Q}(t+h))(1-\lambda) \in K(t+h),$$

for all $h \in [0, \bar{h}]$. It follows that

$$x + h\ [u_t\lambda + \tilde{u}_t(1-\lambda)] + [V_t\lambda + \tilde{V}_t(1-\lambda)](g(t+h) - g(t))$$

$$+h[Q(t+h)\lambda + \tilde{Q}(t+h)(1-\lambda)] \in K(t+h),$$

for all $h \in [0, \bar{h}]$. Since $\lim_{s\to t}[Q(s)\lambda + \tilde{Q}(s)(1-\lambda)] = 0$ it is clear that $\lambda(u_t, V_t) + (1-\lambda)(\tilde{u}_t, \tilde{V}_t) \in \mathcal{T}_{K(t)}^g(t,x)$. □

Proposition 10. *We assume that the maps b and σ from* (7) *are satisfying* ($\mathbf{H}_b$) *and* ($\mathbf{H}_\sigma$).

The following assertions are equivalent:

i) For any $t \in [0,T]$ and any $x \in K(t)$ we have $(b(t,x), \sigma(t,x)) \in \mathcal{T}_{K(t)}^g(t,x)$

ii) For any $t \in [0,T]$ and any $x \in K(t)$ and for any $(1-\alpha)-$Hölder continuous function $u : [t,T] \to \mathbb{R}$, which has the Hölder constant C, independent of t and x and satisfy $u(t) = x$, we have:

$$\lim_{h\to 0} \frac{1}{h} d_{K(t+h)}\left(x + \int_t^{t+h} b\left(s, u_s^{t,x}\right) ds + \int_t^{t+h} \sigma\left(s, u_s^{t,x}\right) dg(s)\right) = 0.$$

Proof. i) $\rightarrow$ ii) We take $u(s) = x$ for all $s \in [t, T]$.

ii) $\rightarrow$ i) Sence $d_{K(t)}(y+z) \leq d_{K(t)}(y) + |z|$ then we have

$$
\begin{aligned}
& \frac{1}{h} d_{K(t+h)} \left(x + \int_t^{t+h} b\left(s, u_s^{t,x}\right) ds + \int_t^{t+h} \sigma\left(s, u_s^{t,x}\right) dg(s) \right) \\
& = \frac{1}{h} d_{K(t+h)} [x + hb(t,x) + \sigma(t,x)(g(t+h) - g(t)) \\
& + \int_t^{t+h} b\left(s, u_s^{t,x}\right) ds + \int_t^{t+h} \sigma\left(s, u_s^{t,x}\right) dg(s) - hb(t,x) \\
& -\sigma(t,x)(g(t+h) - g(t))] \\
& \leq \frac{1}{h} d_{K(t+h)} [x + hb(t,x) + \sigma(t,x)(g(t+h) - g(t))] \\
& + \frac{1}{h} \left| \int_t^{t+h} \left(b\left(s, u_s^{t,x}\right) - b(t,x) \right) ds \right| \\
& + \frac{1}{h} \left| \int_t^{t+h} \left(\sigma\left(s, u_s^{t,x}\right) - \sigma(t,x) \right) dg(s) \right|.
\end{aligned} \tag{8}
$$

For the sake of simplicity denote

$$I_{t+h} = \frac{1}{h} \left| \int_t^{t+h} (b(s, u_s^{t,x}) - b(t,x)) ds \right|$$

and

$$J_{t+h} = \frac{1}{h} \left| \int_t^{t+h} (\sigma(s, u_s^{t,x}) - \sigma(t,x)) dg(s) \right|.$$

We estimate

$$
\begin{aligned}
I_{t+h} &= \frac{1}{h} \left| \int_t^{t+h} (b(s, u_s^{t,x}) - b(t,x)) ds \right| \\
&\leq \frac{N}{h} \left\{ \int_t^{t+h} |u_s^{t,x} - u_t^{t,x}| ds + \int_t^{t+h} |s-t|^{\mu} ds \right\} \\
&\leq \frac{N(C+1)}{2-\alpha} h^{\min\{1-\alpha, \mu\}}
\end{aligned} \tag{9}
$$

and

$$J_{t+h} = \frac{1}{h} \left| \int_t^{t+h} (\sigma(s, u_s^{t,x}) - \sigma(t,x)) dg(s) \right|$$

$$\leq \frac{1}{h}\left|\int_t^{t+h}(\sigma(s,u_s^{t,x})-\sigma(s,x)+\sigma(s,x)-\sigma(t,x))dg(s)\right|$$
$$\leq \frac{1}{h}\left|\int_t^{t+h}(\sigma(s,u_s^{t,x})-\sigma(s,x))dg(s)\right|$$
$$+\frac{1}{h}\left|\int_t^{t+h}(\sigma(s,x)-\sigma(t,x))dg(s)\right|.$$

We denote by $C_0, C_1, ...$ a constant independent of t, x and h that may change from a line to another. Using (6) we have

$$\frac{1}{h}\left|\int_t^{t+h}(\sigma(s,u_s^{t,x})-\sigma(s,x))dg(s)\right|$$
$$\leq \frac{\Lambda_\alpha(g)}{h}\left(\int_t^{t+h}\frac{|\sigma(s,u_s^{t,x})-\sigma(s,u_t^{t,x})|}{(s-t)^\alpha}ds\right.$$
$$\left.+\int_t^{t+h}\int_t^s\frac{|\sigma(s,u_s^{t,x})-\sigma(s,x)-\sigma(y,u_y^{t,x})+\sigma(y,x)|}{(s-y)^{\alpha+1}}dyds\right)$$
$$\leq \frac{\Lambda_\alpha(g)}{h}\left(\int_t^{t+h}\frac{M|u_s^{t,x}-u_t^{t,x}|}{(s-t)^\alpha}ds\right.$$
$$+\int_t^{t+h}\int_t^s\frac{|\sigma(s,u_s^{t,x})-\sigma(y,u_y^{t,x})|}{(s-y)^{\alpha+1}}dyds$$
$$\left.+\int_t^{t+h}\int_t^s\frac{|\sigma(s,x)-\sigma(y,x)|}{(s-y)^{\alpha+1}}dyds\right) \tag{10}$$
$$\leq \frac{\Lambda_\alpha(g)}{h}\left(\int_t^{t+h}\frac{M|u_s^{t,x}-u_t^{t,x}|}{(s-t)^\alpha}ds\right.$$
$$+\int_t^{t+h}\int_t^s\frac{|\sigma(s,u_s^{t,x})-\sigma(s,u_y^{t,x})|}{(s-y)^{\alpha+1}}dyds$$
$$+\int_t^{t+h}\int_t^s\frac{|\sigma(s,u_y^{t,x})-\sigma(y,u_y^{t,x})|}{(s-y)^{\alpha+1}}dyds$$
$$+\int_t^{t+h}\int_t^s\frac{|\sigma(s,x)-\sigma(y,x)|}{(s-y)^{\alpha+1}}dyds)$$
$$\leq \frac{\Lambda_\alpha(g)}{h}C_0\left(\int_t^{t+h}(s-t)^{1-2\alpha}ds\right.$$

$$+\int_t^{t+h}\int_t^s (s-y)^{-2\alpha} dydr + \int_t^{t+h}\int_t^s (s-y)^{\beta-\alpha-1} dydr\Big)$$
$$\leq \Lambda_\alpha(g)\, C_1 h^{\min\{1-2\alpha,\beta-\alpha\}}$$

and

$$\begin{aligned}
&\frac{1}{h}\left|\int_t^{t+h} (\sigma(s,x)-\sigma(t,x))\, dg(s)\right| \\
&\leq \frac{\Lambda_\alpha(g)}{h}\left(\int_t^{t+h} \frac{|\sigma(s,x)-\sigma(t,x)|}{(s-t)^\alpha} ds\right. \\
&\left. + \int_t^{t+h}\int_t^s \frac{|\sigma(s,x)-\sigma(y,x)|}{(s-y)^{\alpha+1}} dyds\right) \\
&\leq \frac{\Lambda_\alpha(g)}{h} C_2 \left(\int_t^{t+h} (s-t)^{\beta-\alpha}\, ds\right. \\
&\left. + \int_t^{t+h}\int_t^s (s-y)^{\beta-1-\alpha}\, dydr\right) \\
&\leq \Lambda_\alpha(g)\, C_3 h^{\min\{1-2\alpha,\beta-\alpha\}}.
\end{aligned} \tag{11}$$

Substituting (9), (10)and (11) in (8) yields

$$\lim_{h\to 0}\frac{1}{h} d_{K(t+h)}\left(x+\int_t^{t+h} b(s,u_s^{t,x})ds + \int_t^{t+h}\sigma(s,u_s^{t,x})dg(s)\right)$$
$$\leq \lim_{h\to 0}\left(\frac{N(C+1)}{2-\alpha} h^{\min\{1-\alpha,\mu\}} + \Lambda_\alpha(g) C_4 h^{\min\{1-2\alpha,\beta-\alpha\}}\right) = 0.$$

The proof is now complete. □

Remark that $\mathcal{T}^g_{K(t)}(t,x)$ is not a closed con.

We will now define a another type of contingence. In this case the contingent set is a closed con .

Consider K a subset of $\mathbb{R}^d$.

Definition 11. *For any $t\in[0,T]$ and any $x\in K$ we say that a pair $(u_t,V_t)\in\mathbb{R}^d\times\mathbb{R}^{d\times m}$ is called weak g- contingent to K in (t,x) if there exist $\bar{h}>0$ and two functions $p:[t,t+\bar{h}]\to\mathbb{R}^d$ and $q:[t,t+\bar{h}]\to\mathbb{R}^d$such that $\lim_{s\to t}|p(s)|=\lim_{s\to t}|q(s)|=0$ and*

$$x+hu_t+V_t\,(g(t+h)-g(t))+hp(t+h)+h^\alpha q(t+h)\in K, \textit{for all } h\in\left[0,\bar{h}\right].$$

Definition 12. *The set of pairs* $(u_t, V_t) \in \mathbb{R}^d \times \mathbb{R}^{d\times m}$ *weak* g- *contingent to* K *in* (t,x) *forme the weak* $g-$*contingent set to* K *in* (t,x). *We denote this set* $\mathcal{T}_K^{(w)g}(t,x)$.

Remark that

$$\mathcal{T}_K^{g}(t,x) \subseteq \mathcal{T}_K^{(w)g}(t,x). \tag{12}$$

Proposition 13. *Let* K *be a nonempty closed subset of* $\mathbb{R}^d$ *and assume* (H_b) *and* (H_σ) *are satisfied. If* K *is viable for the fractional differential equation then for all* $t \in [0,T]$ *and all* $x \in K$ *the pair* $(b(t,x), \sigma(t,x)) \in \mathcal{T}_K^{(w)g}(t,x)$ *and the functions* p *from the Definition* 11 *is* ρ *Hölder continuous with* $\rho = \min\{1-\alpha, \mu\}$.

Proof. Using Proposition 8 and (12) it is easy to see that $(b(t,x), \sigma(t,x)) \in \mathcal{T}_K^{(w)g}(t,x)$. It now remains to prove the properties of p.

Let $\bar{h} = \min\{T-t, 1\}$. Then

$$X_{t+h}^{t,x} = x + \int_t^{t+h} b(r, X_r^{t,x})dr + \int_t^{t+h} \sigma(r, X_r^{t,x})dg(r) \in K, \quad \forall\, h \in \left[0, \bar{h}\right].$$

We clearly have

$$X_{t+h}^{t,x} = x + hb(t,x) + \sigma(t,x)(g_{t+h} - g_t) + hp(t+h) + h^\alpha q(t+h) \in K,$$

for all $h \in [0, \bar{h}]$, where $p : [t, t+\bar{h}] \to \mathbb{R}^d$, $q : [t, t+\bar{h}] \to \mathbb{R}^d$

$$p(s) = \begin{cases} 0, & \text{if } s = t \\ \dfrac{1}{s-t}\displaystyle\int_t^s \left[b(r, X_r^{t,x}) - b(t,x)\right] dr, & \text{if } t < s \leq T \end{cases}$$

and

$$q(s) = \begin{cases} 0, & \text{if } s = t \\ \dfrac{1}{(s-t)^\alpha}\displaystyle\int_t^s \left[\sigma(r, X_r^{t,x}) - \sigma(t,x)\right] dg(r), & \text{if } t < s \leq T \end{cases}.$$

Clearly

$$p(s) = \int_0^1 (b(t+u(s-t), X_{t+u(s-t)}^{t,x})du - b(t,x))du,$$

for all $s \in [t, t+\bar{h}]$.

Let $t < \tau < s < t + \bar{h}$. We have

$$
\begin{aligned}
|p(s) - p(\tau)| &\leq \int_0^1 |b(t + u(s-t), X^{t,x}_{t+u(s-t)})du \\
&- b(t + u(\tau - t), X^{t,x}_{t+u(\tau-t)})|du \\
&\leq N \int_0^1 [u^\mu |s-\tau|^\mu + |X^{t,x}_{t+u(s-t)} - X^{t,x}_{t+u(\tau-t)}|]du \\
&\leq N_1 \int_0^1 [u^\mu |s-\tau|^\mu + u^{1-\alpha}|s-\tau|^{1-\alpha}]du \\
&= N_1 \left[\frac{1}{1+\mu} |s-\tau|^\mu + \frac{1}{2-\alpha} |s-\tau|^{1-\alpha} \right] \\
&\leq C|s-\tau|^{\min\{\mu, 1-\alpha\}}.
\end{aligned}
$$

□

Proposition 14. *If $K(t)$ is a convex set for all $t \in [0,T]$, then $\mathcal{T}^{(w)g}_{K(t)}(t,x)$ is also a convex set.*

Proof. The proof is similar to the proof of Proposition 9. □

Proposition 15. *For any $t \in [0,T]$ and any $x \in K$, the weak $g-$contingent set to K in (t,x), $\mathcal{T}^{(w)g}_K(t,x)$ is a con.*

Proof. Let $a > 0$ and $(u_t, V_t) \in \mathcal{T}^{(w)g}_K(t,x)$.
Since

$$
x + hu_t + V_t(g(t+h) - g(t)) + hp(t+h) + h^\alpha q(t+h) \in K,
$$

for all $h \in [0, \bar{h}]$. We can write

$$
\begin{aligned}
& x + hu_t + V_t(g(t+h) - g(t)) + hp(t+h) + h^\alpha q(t+h) \\
& = x + \frac{h}{a} a\ u_t + aV_t \frac{1}{a} \left(g(t+h) - ag\left(t + \frac{h}{a}\right) \right. \\
& \left. + ag\left(t + \frac{h}{a}\right) - ag(t) + ag(t) - g(t) \right) \\
& + hp(t+h) + h^\alpha q(t+h) \\
& = x + \frac{h}{a} a\ u_t + aV_t \left(g\left(t + \frac{h}{a}\right) - g(t) \right) + V_t(g(t+h) - g(t)) \\
& - aV_t \left(g\left(t + \frac{h}{a}\right) + g(t) \right) + hp(t+h) + h^\alpha q(t+h).
\end{aligned}
$$

We set $\tilde{h} = \dfrac{h}{a}$ and

$$\tilde{p}(t+\tilde{h}) = a \cdot p(t+a\tilde{h})$$

$$\tilde{q}(t+\tilde{h}) = V_t \left(\frac{g(t+a\ \tilde{h})-g(t)}{\tilde{h}^\alpha}) - aV_t(\frac{g(t+\tilde{h})+g(t)}{\tilde{h}^\alpha} \right) + a^\alpha q(t+a\ \tilde{h}).$$

It is easy to see that $\lim_{\tilde{h}\to 0}|\tilde{p}(t+\tilde{h})| = 0$ and $\lim_{\tilde{h}\to 0}|\tilde{q}(t+\tilde{h})| = 0$ since

$$\begin{aligned}|\tilde{q}(t+\tilde{h})| &= \left| V_t \left(\frac{g(t+a\ \tilde{h})-g(t)}{\tilde{h}^\alpha} \right) - aV_t \left(\frac{g(t+\tilde{h})+g(t)}{\tilde{h}^\alpha} \right) + a^\alpha q(t+a\ \tilde{h}) \right| \\ &\leq\ C_{(5)}(|V_t(\tilde{h}^{1-2\alpha})| + a^\alpha |q(t+a\ \tilde{h})|).\end{aligned}$$

It is now clear that

$$x + \tilde{h}(au_t) + (aV_t)(g(t+\tilde{h}) - g(t)) + \tilde{h}\ \tilde{p}(t+\tilde{h}) + \tilde{h}^\alpha \tilde{q}(t+\tilde{h}) \in K,$$

for all $\tilde{h} \in [0, \frac{\bar{h}}{a}]$. We have $(au_t, aV_t) \in \mathcal{T}_K^{(w)g}(t,x)$ and the proof is complete.□

Proposition 16. *For any $t \in [0,T]$ and any $x \in K$, the weak $g-$contingent set to K in (t,x), $\mathcal{T}_K^{(w)g}(t,x)$ is a closed set.*

Proof. Let $(u_n^{(t)}, V_n^{(t)})_{n\in\mathbb{N}}$ be a sequences of pair in $\mathcal{T}_K^{(w)g}(t,x)$, convergent to $(u^{(t)}, V^{(t)})$.

Since

$$\begin{aligned}&x + hu_n^{(t)} + V_n^{(t)}(g(t+h) - g(t)) + hp_n(t+h) + h^\alpha q_n(t+h) \in K,\\ &\text{for all } h \in \left[0, \bar{h}\right] \text{ and all } n \in \mathbb{N}\end{aligned}$$

we can write

$$\begin{aligned}&x + hu_n^{(t)} + V_n^{(t)}(g(t+h) - g(t)) + hp_n(t+h) + h^\alpha q_n(t+h)\\ &= x + hu^{(t)} + V^{(t)}(g(t+h) - g(t))\\ &+ h(u_n^{(t)} - u^{(t)}) + (V_n^{(t)} - V^{(t)})(g(t+h) - g(t)) + hp_n(t+h) + h^\alpha q_n(t+h)\\ &= x + hu^{(t)} + V^{(t)}(g(t+h) - g(t))\\ &+ h(u_n^{(t)} - u^{(t)} + p_n(t+h)) + h^\alpha((V_n^{(t)} - V^{(t)})\frac{g(t+h)-g(t)}{h^\alpha} + q_n(t+h)).\end{aligned}$$

Set

$$\tilde{p}(t+h) = u_n^{(t)} - u^{(t)} + p_n(t+h)$$

and

$$\tilde{q}(t+h) = (V_n^{(t)} - V^{(t)})\frac{g(t+h)-g(t)}{h^\alpha} + q_n(t+h).$$

We have

$$\lim_{h\to 0}|\tilde{p}(t+h)| = |u_n^{(t)} - u^{(t)}| \qquad \text{and} \qquad \lim_{h\to 0}|\tilde{q}(t+h)| = 0.$$

Since $\lim_{n\to\infty}|u_n^{(t)} - u^{(t)}| = 0$, it follows that

$$(u^{(t)}, V^{(t)}) \in \mathcal{T}_K^{(w)g}(t,x).$$

□

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