ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" DIN IAŞI (S.N.)
MATEMATICĂ, Tomul LV, 2009, f.2

STRONG SOLUTIONS FOR BOUSSINESQ EQUATIONS WITH POTENTIALS*

BY

ADRIANA-IOANA LEFTER

Abstract. This paper deals with the Boussinesq system perturbed with a maximal monotone operator, in a bounded domain, in 2D and 3D. Using the theory of quasi-m-accretive operators, existence results for strong solutions are proved. A stabilization theorem in 2D is also given.

Mathematics Subject Classification 2000: 76D05, 76D03, 47H05, 47J05.
Key words: Navier-Stokes equations, Boussinesq equations, strong solutions, monotone operators, stabilization.

1. Introduction. Let $T > 0$ and $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be an open and bounded domain, with a smooth boundary $\partial\Omega$ (of class C^2 for instance). Consider the Boussinesq system ([7])

$$(1)\quad \begin{cases} \dfrac{\partial y}{\partial t} - \nu\Delta y + (y \cdot \nabla)y - \gamma(\theta - \theta^0)e_d + \nabla p = f_0, & \text{in } Q = \Omega \times (0,T), \\ \dfrac{\partial \theta}{\partial t} - k\Delta\theta + y \cdot \nabla\theta = 0, & \text{in } Q = \Omega \times (0,T), \\ \text{div } y = 0, & \text{in } Q, \\ y = 0, \quad \theta = 0, & \text{on } \Sigma = \partial\Omega \times (0,T), \\ y(\cdot,0) = y_0, \quad \theta(\cdot,0) = \theta_0, & \text{in } \Omega, \end{cases}$$

where $y = (y_1, y_2, \ldots, y_d)$ is the velocity field, the scalar functions p and θ are the scalar pressure and the temperature of the fluid, the density of external forces is $f_0 = (f_{01}, f_{02}, \ldots, f_{0d})$. The constants ν, $k > 0$ denote

*This work was supported by CNCSIS Grant 1373/2007.

the kinematic viscosity coefficient and the thermic diffusivity, respectively. Also, $\gamma = g/\theta^0 > 0$, where g is the gravitational constant and $\theta^0 > 0$ is a constant reference temperature, while $e_d = (0, \ldots, 0, 1)$ is the d^{th} unit vector of $\mathbb{R}^d$.

In this section the functional framework is described and the Boussinesq equations are put in an abstract form. We state the main existence and uniqueness results for strong solutions in Section 2. The first of these theorems is proved in Section 3 and the other in Section 4. The last section presents an application concerning the exponential stabilization of the Boussinesq system (in the bidimensional case); the solution remains in a closed convex set.

Denote the scalar products of $L^2(\Omega)$ and $(L^2(\Omega))^d$ by $(\cdot, \cdot)$ and the corresponding norms by $|\cdot|$. Denote by $\|\cdot\| = |\nabla \cdot|$ the norms of $H_0^1(\Omega)$ and $(H_0^1(\Omega))^d$. Let us introduce the standard spaces (see e.g.[5, 9, 10])

$$H = \{y \in (L^2(\Omega))^d; \text{div } y = 0 \text{ in } \Omega, y \cdot n_{\partial\Omega} = 0 \text{ on } \partial\Omega\},$$

$$V = \{y \in (H_0^1(\Omega))^d; \text{div } y = 0 \text{ in } \Omega\}.$$

Let $H_B = H \times L^2(\Omega)$, $V_B = V \times H_0^1(\Omega)$.

Clearly, H and H_B are real Hilbert spaces endowed with L^2-type norms $|\cdot|$, while V and V_B are real Hilbert spaces endowed with H_0^1-type norms $\|\cdot\|$. Moreover, denoting by V' the dual space of V and identifying H with its own dual, we have $V \subset H \subset V'$ algebraically and topologically, with compact injections. By consequence, if V_B' is the dual space of V_B and if we identify H_B with its own dual, we also have $V_B \subset H_B \subset V_B'$ algebraically and topologically, with compact injections.

Denote by $(\cdot, \cdot)$ the scalar products of H and H_B and the pairings between V and V' or between V_B and V_B'.

Let $A \in L(V, V')$ -the space of linear continuous operators from V in V', be defined as $(Ay, w) = \sum_{i=1}^d \int_\Omega \nabla y_i \cdot \nabla w_i dx, \ \forall y, w \in V$.

We have $(Ay, y) = \|y\|^2, \ \forall y \in V$. We set $D(A) = \{y \in V; Ay \in H\}$ and we denote again by A the restriction of A to H.

Let $A_1 \in L(H_0^1(\Omega), H^{-1}(\Omega))$, $A_1 = -\Delta$. Then $(A_1\theta, \theta) = \|\theta\|^2, \ \forall \theta \in H_0^1(\Omega)$. We set $D(A_1) = \{\theta \in H_0^1(\Omega); A_1\theta \in L^2(\Omega)\}$ and we denote again by A_1 the restriction of A_1 to $L^2(\Omega)$.

Let $\mathcal{A} \in L(V_B, V_B')$ be defined by

$$\mathcal{A}z = \begin{pmatrix} \nu A y \\ k A_1 \theta \end{pmatrix}, \ \forall \ z = \begin{pmatrix} y \\ \theta \end{pmatrix} \in V_B. \tag{2}$$

Let $D(\mathcal{A}) = D(A) \times D(A_1)$. The operator $\mathcal{A}$ maps $D(\mathcal{A})$ into H_B. Denote by $m_{\nu k} = \min\{\nu, k\}$.

Let $b : V \times V \times V \to \mathbb{R}$ be the trilinear continuous functional defined by

$$b(y,v,w) = \sum_{i,j=1}^{d} \int_\Omega y_i \frac{\partial v_j}{\partial x_i} w_j dx, \ \forall y,v,w \in V.$$

Denoting by $\|\cdot\|_s$ the norm of the Sobolev space $(H^s(\Omega))^d$, the functional b satisfies (see e.g. [5, 9, 10])

$$b(y,w,w) = 0, \ b(y,v,w) = -b(y,w,v), \ \forall y,v,w \in V, \tag{3}$$

$$|b(y,v,w)| \le C_b |y|^{\frac{1}{2}} \|y\|^{\frac{1}{2}} \|v\|^{\frac{1}{2}} |Av|^{\frac{1}{2}} |w|, \ \forall y,w \in V, v \in D(A) \ (d=2), \tag{4}$$

$$|b(y,v,w)| \le C_b |y|^{\frac{1}{2}} \|y\|^{\frac{1}{2}} |v|^{\frac{1}{2}} \|v\|^{\frac{1}{2}} \|w\|, \ \forall y,v,w \in V \ (\text{for } d=2), \tag{5}$$

$$|b(y,v,w)| \le C_b \|y\| \|v\|^{\frac{1}{2}} |Av|^{\frac{1}{2}} |w|, \ \forall y,w \in V, v \in D(A) \ (\text{for } d=3), \tag{6}$$

$$|b(y,v,w)| \le C_b \|y\| \|v\|_{3/2} |w|, \ \forall y,w \in V, v \in D(A) \ (\text{for } d=3), \tag{7}$$

Let $\bar{b} : V \times H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ be the trilinear continuous functional defined by

$$\bar{b}(y,\theta,\tilde{\theta}) = \sum_{i=1}^{d} \int_\Omega y_i \frac{\partial \theta}{\partial x_i} \tilde{\theta} dx, \ \forall \ y \in V, \ \forall \ \theta, \tilde{\theta} \in H_0^1(\Omega).$$

Remark 1.1. The functional b can be extended to $V \times (H_0^1(\Omega))^d \times (H_0^1(\Omega))^d$. Taking $y \in V$, $\theta, \tilde{\theta} \in H_0^1(\Omega)$ and

$$\tilde{v} = \begin{pmatrix} \theta \\ \theta \\ \vdots \\ \theta \end{pmatrix}, \quad \tilde{w} = \begin{pmatrix} \tilde{\theta} \\ \tilde{\theta} \\ \vdots \\ \tilde{\theta} \end{pmatrix} \in (H_0^1(\Omega))^d,$$

we deduce that $\bar{b}(y,\theta,\tilde{\theta}) = (1/d) b(y,\tilde{v},\tilde{w})$ and thus $\bar{b}$ will verify inequalities analogous to (3) - (7).

In the rest of the paper, we will denote by C_b the positive constants arising in estimations of b or $\bar{b}$ of the type (4) - (7). Other various positive constants will be simply denoted by the symbol C.

Let $B : V_B \to V_B'$ be defined by

$$(Bz, \tilde{z}) = b(y, y, \tilde{y}) + \bar{b}(y, \theta, \tilde{\theta}),\ \forall\ z = \begin{pmatrix} y \\ \theta \end{pmatrix},\ \tilde{z} = \begin{pmatrix} \tilde{y} \\ \tilde{\theta} \end{pmatrix} \in V_B. \tag{8}$$

Let us introduce $\mathcal{R} \in L(H_B, H_B)$ by

$$\mathcal{R}z = \begin{pmatrix} P(-\gamma\theta e_d) \\ 0 \end{pmatrix},\ \forall z = \begin{pmatrix} y \\ \theta \end{pmatrix} \in H_B \tag{9}$$

($P : (L^2(\Omega))^d \to H$ is the Leray projection). Using the notations (2), (8), (9) and denoting also by

$$f = P(f_0 - \gamma\theta^0 e_d), \quad F = \begin{pmatrix} f \\ 0 \end{pmatrix}, \quad z_0 = \begin{pmatrix} y_0 \\ \theta_0 \end{pmatrix},\ \text{and } z = \begin{pmatrix} y \\ \theta \end{pmatrix}, \tag{10}$$

the equations (1) may be rewritten

$$\begin{cases} \dfrac{dz}{dt}(t) + \mathcal{A}z(t) + Bz(t) + \mathcal{R}z(t) = F(t), \quad t \in (0, T) \\ z(0) = z_0. \end{cases} \tag{11}$$

We intend to obtain exponential stability results for the problem (11). To this aim, we need existence results for slightly more complicated systems.

Let $\Phi \subset H_B \times H_B$ satisfy the following hypotheses,

- (h_1) $\Phi = \partial\varphi$, where $\varphi : H_B \to \overline{\mathbb{R}}$ is a lower semicontinuous proper convex function (hence Φ is a maximal monotone operator in $H_B \times H_B$);
- (h_2) $0 \in D(\Phi)$;
- (h_3) there exist two constants $\alpha_0 \geq 0,\ \alpha \in (0, 1)$ such that

$$(\mathcal{A}h, \Phi_\lambda(h)) \geq -\alpha_0(1 + \|h\|^2) - \alpha|\Phi_\lambda(h)|^2,\ \forall\lambda > 0, \forall h \in D(\mathcal{A}), \tag{12}$$

where $\Phi_\lambda = \frac{1}{\lambda}(I - (I + \lambda\Phi)^{-1}) : H_B \to H_B$ is the Yosida approximation of Φ.

We consider the classical definition of the maximal monotone operator. For $h \in D(\Phi)$, we will denote by $|\Phi(h)| = \inf\{|\eta|; \eta \in \Phi(h)\}$.

Let $z_e = \begin{pmatrix} y_e \\ \theta_e \end{pmatrix} \in D(\mathcal{A})$ be fixed and define the linear continuous operator

$$\mathcal{A}_0 : V_B \to V_B', \quad (\mathcal{A}_0 z, \tilde{z}) = b(y, y_e, \tilde{y}) + b(y_e, y, \tilde{y}) \tag{13}$$
$$+\bar{b}(y, \theta_e, \tilde{\theta}) + \bar{b}(y_e, \theta, \tilde{\theta}), \ \forall z = \begin{pmatrix} y \\ \theta \end{pmatrix}, \ \tilde{z} = \begin{pmatrix} \tilde{y} \\ \tilde{\theta} \end{pmatrix} \in V_B.$$

Hence $\mathcal{A}_0$ maps $D(\mathcal{A})$ into H_B.

Consider the perturbed Boussinesq problem

$$\begin{cases} \dfrac{dz}{dt}(t) + (\mathcal{A} + B + \mathcal{A}_0 + \mathcal{R})z(t) + \Phi(z(t)) \ni F(t), \ t \in (0,T) \\ z(0) = z_0 \end{cases} \tag{14}$$

A nonlinear term Φ arises usually as a nonlinear feedback controller.

In the sequel the symbol $\rightharpoonup$ will be used to denote the convergence in the weak topology, while the strong convergence will be indicated by $\to$.

2. Existence results for strong solutions

Definition 2.1. *Let $z_0 \in V_B$ and $F \in L^2(0,T; H_B)$. By a strong solution for the problem* (14) *we mean a function $z \in L^2(0,T; D(\mathcal{A})) \cap L^\infty(0,T; V_B) \cap C([0,T]; H_B)$, satisfying $\frac{dz}{dt} \in L^2(0,T; H_B)$, for which there exists a selection $\eta \in L^2(0,T; H_B)$, $\eta(t) \in \Phi(z(t))$ a.e. $t \in (0,T)$, such that*

$$\begin{cases} \dfrac{dz}{dt}(t) + (\mathcal{A} + B + \mathcal{A}_0 + \mathcal{R})z(t) + \eta(t) = F(t), \ \textit{a.e.} \ t \in (0,T) \\ z(0) = z_0. \end{cases} \tag{15}$$

Theorem 2.1. *Let $T > 0$ and let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be an open and bounded domain, with a smooth boundary. Assume that $\Phi \subset H_B \times H_B$ satisfies the hypotheses $(h_1) - (h_3)$. Let $z_0 \in D(\mathcal{A}) \cap D(\Phi)$ and $F \in W^{1,1}(0,T; H_B)$.*

If $d = 2$, then there exists a unique $z \in W^{1,\infty}(0,T; H_B) \cap L^\infty(0,T; D(\mathcal{A})) \cap C([0,T]; V_B)$ such that

$$\begin{cases} \dfrac{dz}{dt}(t) + (\mathcal{A} + B + \mathcal{A}_0 + \mathcal{R})z(t) + \Phi(z(t)) \ni F(t), \ \textit{a.e.} \ t \in (0,T) \\ z(0) = z_0. \end{cases} \tag{16}$$

Moreover, z is right differentiable, $\frac{d^+}{dt}z$ is right continuous, and

$$\frac{d^+}{dt}z(t) + ((\mathcal{A} + B + \mathcal{A}_0 + \mathcal{R})z(t) + \Phi(z(t)) - F(t))^0 = 0, \ \forall t \in [0, T).$$

If $d = 3$, then the solution z exists on some interval $[0, T_0)$, where

$$T_0 = T_0(\|F\|^2_{L^2(0,T;H_B)}, \|z_0\|^2) \leq T.$$

We have denoted by $z \to ((\mathcal{A}+B+\mathcal{A}_0+\mathcal{R})z+\Phi(z)-F(t))^0$ the minimal section of the multivalued mapping $z \to (\mathcal{A}+B+\mathcal{A}_0+\mathcal{R})z+\Phi(z)-F(t)$.

If we assume lower regularity on the initial data, we obtain the following result:

Theorem 2.2. *Let $T > 0$ and $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be an open and bounded domain, with a smooth boundary. Assume that $\Phi \subset H_B \times H_B$ satisfies the hypotheses $(h_1) - (h_3)$ and that φ is bounded on the sets of $D(\mathcal{A}) \cap D(\Phi)$ which are bounded in V_B. Let $z_0 \in \overline{D(\mathcal{A}) \cap D(\Phi)}^{V_B}$, $F \in L^2(0, T; H_B)$. If $d = 2$, then there exists a unique $z \in C([0, T]; H_B) \cap L^2(0, T; D(\mathcal{A})) \cap L^\infty(0, T; V_B)$ with $\frac{dz}{dt} \in L^2(0, T; H_B)$, $Bz \in L^2(0, T; H_B)$, solution of the problem (16). In the dimension $d = 3$, the solution z exists on some interval $[0, T_0)$, where*

$$T_0 = T_0(\|F\|^2_{L^2(0,T;H_B)}, \|z_0\|^2) \leq T.$$

Remark 2.1. Particularizing the proofs of Theorems 2.1, 2.2 we get similar existence results for the simpler system

$$\begin{cases} \dfrac{dz}{dt}(t) + \mathcal{A}z(t) + Bz(t) + \mathcal{R}z(t) + \Phi(z(t)) \ni F(t), \ t \in (0, T) \\ z(0) = z_0. \end{cases}$$

3. Proof of Theorem 2.1. The proof involves the theory of nonlinear differential equations of accretive type in Banach spaces ([1, 3, 4, 8]). We intend to obtain a quasi-m-accretive operator on the left hand side of the Boussinesq equation and for that purpose we substitute the nonlinearity B with a truncation B_N, $N \in \mathbb{N}^*$ (Proposition 3.2). We may then state existence and uniqueness results for the approximate problems (17), (38) containing B_N, Φ and B_N, Φ_λ, $\lambda > 0$ instead of B, Φ (Propositions 3.3, 3.4).

Our aim is to prove that for N large enough the solution of the truncated problem involving B_N, Φ coincides with the solution of the initial problem.

In order to do that, we need some estimates on the solution z_N of problem (17). As relation (12) does not extend in a suitable way to arbitrary elements of $\Phi(z_N(t))$, we are obliged to deduce the convenient estimates first on problem (38) (the one involving Φ_λ). Passing to the limit with $\lambda \searrow 0$ in (38), we return to the problem in B_N, Φ and conclude the proof.

3.1. Approximate problems. Existence and uniqueness. For $N \in \mathbb{N}^*$, define the modified nonlinearity $B_N : V_B \to V_B'$,

$$B_N y = \begin{cases} Bz, & \text{if } \|z\| \le N \\ \left(\dfrac{N}{\|z\|}\right)^2 Bz, & \text{if } \|z\| > N \end{cases}$$

and consider the approximate problem

$$(17) \quad \begin{cases} \dfrac{dz_N}{dt}(t) + (\mathcal{A}+B_N+\mathcal{A}_0 + \mathcal{R})z_N(t) + \Phi(z_N(t)) \ni F(t), \ t \in (0,T) \\ z_N(0) = z_0. \end{cases}$$

It may be shown ([6, 11]) that $D(\mathcal{A} + B_N + \mathcal{R}) = \{h \in V_B;\ (\mathcal{A} + B_N + \mathcal{R})(h) \in H_B\} = D(\mathcal{A})$ and that

Proposition 3.1. *For constants $\alpha_N > 0$ big enough, the operator $\mathcal{A} + B_N + \mathcal{R} + \alpha_N I$ is maximal monotone in $H_B \times H_B$.*

Actually, the proof of Proposition 3.1 can be easily adapted to obtain the maximal monotony in $H_B \times H_B$ of the operator $\Upsilon_N = \mathcal{A}+B_N+\mathcal{A}_0+\mathcal{R}+\alpha_N I$ with domain $D(\Upsilon_N) = D(\mathcal{A})$ (for α_N big enough).

We may assume, without loss of generality, that α_N is fixed such that

$$(h_{\alpha_N}) \quad \begin{cases} \text{in the case } d = 2,\ \alpha_N > \gamma/2 + C_b^2\|z_e\|^2/(2m_{\nu k}), \text{ where } C_b \text{ is the} \\ \text{constant arising in (5);} \\ \text{in the case } d = 3,\ \alpha_N > \gamma/2 + C_b^2\|z_e\|_{3/2}^2/(2m_{\nu k}), \text{ where } C_b \text{ is} \\ \text{the constant arising in (7).} \end{cases}$$

Proposition 3.2 below is one of the main ingredients of the proof of Theorem 2.1.

Proposition 3.2. *Let $N \in \mathbb{N}^*$ be fixed. Let $\Phi \subset H_B \times H_B$ be a maximal monotone operator satisfying the hypotheses (h_2), (h_3).*

Define the operator $\Lambda_N : D(\Lambda_N) \to H_B$*,* $\Lambda_N = \mathcal{A}+B_N+\mathcal{A}_0+\mathcal{R}+\Phi+\alpha_N I$ *(with* $\alpha_N > 0$ *given by Proposition* 3.1 *and* (h_{α_N})*), where* $D(\Lambda_N) = \{h \in H_B;\ \emptyset \neq \Lambda_N(h) \subset H_B\}$*. Then* $D(\Lambda_N) = D(\mathcal{A}) \cap D(\Phi)$ *and* Λ_N *is maximal monotone in* $H_B \times H_B$*.*

Moreover, there exists a constant $\bar{C}_N > 0$ *such that*

$$|\mathcal{A}h| \leq \bar{C}_N(1 + |h|^2 + |(\mathcal{A} + B_N + \mathcal{A}_0 + \mathcal{R})h + \Phi_\lambda(h)|^2)^{\frac{3}{2}}, \quad \forall h \in D(\mathcal{A}),\ \forall \lambda > 0, \tag{18}$$

$$|\mathcal{A}h| \leq \bar{C}_N(1 + |h|^2 + |(\mathcal{A} + B_N + \mathcal{A}_0 + \mathcal{R})h + \eta|^2)^{\frac{3}{2}}, \quad \forall h \in D(\mathcal{A}) \cap D(\Phi),\ \forall \eta \in \Phi(h). \tag{19}$$

Proof. For $\alpha_N > 0$ given as in the hypothesis, the operator Υ_N carries $D(\mathcal{A})$ into H_B and it is (maximal) monotone in $H_B \times H_B$. Then $D(\mathcal{A}) \cap D(\Phi) \subset D(\Lambda_N)$ and Λ_N is the sum of two monotone operators and by consequence is monotone. So, in order to obtain the maximal monotony of Λ_N, it remains to prove that $R(I + \Lambda_N) = H_B$.

Let $g \in H_B$ and $\lambda > 0$ be fixed. We approximate the equation

$$\zeta + \mathcal{A}\zeta + B_N\zeta + \mathcal{A}_0\zeta + \mathcal{R}\zeta + \Phi(\zeta) + \alpha_N\zeta \ni g, \text{ where } \zeta = \begin{pmatrix} w \\ \xi \end{pmatrix} \in H_B, \tag{20}$$

by the equation

$$\zeta_\lambda + \mathcal{A}\zeta_\lambda + B_N\zeta_\lambda + \mathcal{A}_0\zeta_\lambda + \mathcal{R}\zeta_\lambda + \Phi_\lambda(\zeta_\lambda) + \alpha_N\zeta_\lambda = g, \tag{21}$$

that is

$$\zeta_\lambda + \Upsilon_N\zeta_\lambda + \Phi_\lambda(\zeta_\lambda) = g, \tag{22}$$

where Φ_λ is the Yosida approximation of Φ. It is known that Φ_λ is demicontinuous monotone and thus its sum with the maximal monotone operator Υ_N is maximal monotone. This implies the existence of a solution $\zeta_\lambda = \begin{pmatrix} w_\lambda \\ \xi_\lambda \end{pmatrix} \in D(\mathcal{A}) \cap D(\Phi_\lambda)$ for equation (21). The uniqueness follows by monotony arguments.

Let $\mu_N = \alpha_N + 1$; then equation (21) reads

$$\mathcal{A}\zeta_\lambda + B_N\zeta_\lambda + \mathcal{A}_0\zeta_\lambda + \mathcal{R}\zeta_\lambda + \Phi_\lambda(\zeta_\lambda) + \mu_N\zeta_\lambda = g,\ \lambda > 0. \tag{23}$$

We first multiply equation (23) in H_B by ζ_λ, and using (3), we get that

$$\begin{aligned}&\nu\|w_\lambda\|^2+k\|\xi_\lambda\|^2+(\mathcal{A}_0\zeta_\lambda,\zeta_\lambda)-\gamma(\xi_\lambda e_d,w_\lambda)\\ &+(\Phi_\lambda(\zeta_\lambda),\zeta_\lambda)+\mu_N|\zeta_\lambda|^2=(g,\zeta_\lambda).\end{aligned}\tag{24}$$

But Φ_λ is monotone, with $0\in D(\Phi_\lambda)=H_B$, which yields

$$-(\Phi_\lambda(\zeta_\lambda),\zeta_\lambda)\le-(\Phi_\lambda(0),\zeta_\lambda)\le 1/2(|\Phi(0)|^2+|\zeta_\lambda|^2).$$

Also, $\gamma(\xi_\lambda e_d,w_\lambda)\le\gamma|\xi_\lambda||w_\lambda|\le\gamma/2|\zeta_\lambda|^2$ and, from (3), (5) and (7),

$$\begin{aligned}&-(\mathcal{A}_0\zeta_\lambda,\zeta_\lambda)=-b(w_\lambda,y_e,w_\lambda)-\bar b(w_\lambda,\theta_e,\xi_\lambda)\\ &\le\begin{cases}C_b\left(\|w_\lambda\||w_\lambda|\|y_e\|+|w_\lambda|^{\frac12}\|w_\lambda\|^{\frac12}|\xi_\lambda|^{\frac12}\|\xi_\lambda\|^{\frac12}\|\theta_e\|\right), & d=2\\ C_b(\|w_\lambda\|\|y_e\|_{3/2}|w_\lambda|+\|w_\lambda\|\|\theta_e\|_{3/2}|\xi_\lambda|), & d=3\end{cases}\\ &\le C_b|\zeta_\lambda|\|z_e\|_a\|\zeta_\lambda\|,\ \text{ where } a=\begin{cases}1, & d=2\\ 3/2, & d=3.\end{cases}\end{aligned}$$

Equation (24) implies

$$\begin{aligned}&m_{\nu k}\|\zeta_\lambda\|^2+\mu_N|\zeta_\lambda|^2\le\frac12|g|^2+\frac12|\Phi(0)|^2+|\zeta_\lambda|^2\\ &+\frac{\gamma}{2}|\zeta_\lambda|^2+C_b|\zeta_\lambda|\|z_e\|_a\|\zeta_\lambda\|\\ &\le\frac12|g|^2+\frac12|\Phi(0)|^2+\left(1+\frac{\gamma}{2}+\frac{C_b^2\|z_e\|_a^2}{2m_{\nu k}}\right)|\zeta_\lambda|^2+\frac{m_{\nu k}}{2}\|\zeta_\lambda\|^2.\end{aligned}$$

Hence, knowing that $\mu_N=\alpha_N+1$ and using hypothesis (h_{α_N}), we get

$$|\zeta_\lambda|^2,\|\zeta_\lambda\|^2\le C(1+|g|^2),\forall\lambda>0.\tag{25}$$

Here and thereafter the constant $C>0$ does not depend on λ or $|g|$.

Next, equation (23) is multiplied in H_B by $\mathcal{A}\zeta_\lambda$, which yields

$$\begin{aligned}&|\mathcal{A}\zeta_\lambda|^2+(B_N\zeta_\lambda,\mathcal{A}\zeta_\lambda)+(\mathcal{A}_0\zeta_\lambda,\mathcal{A}\zeta_\lambda)-(\gamma\xi_\lambda e_d,\nu Aw_\lambda)\\ &+(\Phi_\lambda(\zeta_\lambda),\mathcal{A}\zeta_\lambda)+\mu_N(\nu\|w_\lambda\|^2+k\|\xi_\lambda\|^2)=(g,\mathcal{A}\zeta_\lambda).\end{aligned}\tag{26}$$

But, if $d=2$, by (4) and Young's inequality, we obtain

$$\begin{aligned}|(B_N\zeta_\lambda,\mathcal{A}\zeta_\lambda)| &\le |\nu b(w_\lambda,w_\lambda,Aw_\lambda)|+|k\bar b(w_\lambda,\xi_\lambda,A_1\xi_\lambda)|\\ &\le C_b\left(\nu|w_\lambda|^{\frac12}\|w_\lambda\||Aw_\lambda|^{\frac32}+k|w_\lambda|^{\frac12}\|w_\lambda\|^{\frac12}\|\xi_\lambda\|^{\frac12}|A_1\xi_\lambda|^{\frac32}\right)\\ &\le C|\zeta_\lambda|^{\frac12}\|\zeta_\lambda\||\mathcal{A}\zeta_\lambda|^{\frac32}\le\frac{1}{12}|\mathcal{A}\zeta_\lambda|^2+C|\zeta_\lambda|^2\|\zeta_\lambda\|^4.\end{aligned}$$

If $d=3$, by (6) and Young's inequality, we get

$$\begin{aligned}|(B_N\zeta_\lambda,\mathcal{A}\zeta_\lambda)| &\leq |\nu b(w_\lambda,w_\lambda,Aw_\lambda)|+|k\bar{b}(w_\lambda,\xi_\lambda,A_1\xi_\lambda)|\\ &\leq C_b\left(\nu\|w_\lambda\|^{\frac{3}{2}}|Aw_\lambda|^{\frac{3}{2}}+k\|w_\lambda\|\|\xi_\lambda\|^{\frac{1}{2}}|A_1\xi_\lambda|^{\frac{3}{2}}\right)\\ &\leq C\|\zeta_\lambda\|^{\frac{3}{2}}|\mathcal{A}\zeta_\lambda|^{\frac{3}{2}}\leq\frac{1}{12}|\mathcal{A}\zeta_\lambda|^2+C\|\zeta_\lambda\|^6.\end{aligned}$$

Consequently, in both cases $d=2,3$, by (25), we have that

$$|(B_N\zeta_\lambda,\mathcal{A}\zeta_\lambda)|\leq\frac{1}{12}|\mathcal{A}\zeta_\lambda|^2+C(1+|g|^2)^3. \tag{27}$$

From (4), we also infer for $d=2$ that

$$\begin{aligned}|(\mathcal{A}_0\zeta_\lambda,\mathcal{A}\zeta_\lambda)| &\leq |\nu b(w_\lambda,y_e,Aw_\lambda)|+|\nu b(y_e,w_\lambda,Aw_\lambda)|\\ &\quad+|k\bar{b}(w_\lambda,\theta_e,A_1\xi_\lambda)|+|k\bar{b}(y_e,\xi_\lambda,A_1\xi_\lambda)|\\ &\leq C_b\left[\nu\|w_\lambda\|^{\frac{1}{2}}|Aw_\lambda|\|y_e\|^{\frac{1}{2}}\left(|w_\lambda|^{\frac{1}{2}}|Ay_e|^{\frac{1}{2}}+|y_e|^{\frac{1}{2}}|Aw_\lambda|^{\frac{1}{2}}\right)\right.\\ &\quad+k|A_1\xi_\lambda|\left(|w_\lambda|^{\frac{1}{2}}\|w_\lambda\|^{\frac{1}{2}}\|\theta_e\|^{\frac{1}{2}}|A_1\theta_e|^{\frac{1}{2}}\right.\\ &\quad\left.\left.+|y_e|^{\frac{1}{2}}\|y_e\|^{\frac{1}{2}}\|\xi_\lambda\|^{\frac{1}{2}}|A_1\xi_\lambda|^{\frac{1}{2}}\right)\right]\\ &\leq C\left(|\zeta_\lambda|^{\frac{1}{2}}\|\zeta_\lambda\|^{\frac{1}{2}}|\mathcal{A}\zeta_\lambda|+\|\zeta_\lambda\|^{\frac{1}{2}}|\mathcal{A}\zeta_\lambda|^{\frac{3}{2}}\right)\\ &\leq \frac{1}{12}|\mathcal{A}\zeta_\lambda|^2+C(|\zeta_\lambda|\|\zeta_\lambda\|+\|\zeta_\lambda\|^2).\end{aligned}$$

For $d=3$, by (6), we have that

$$\begin{aligned}|(\mathcal{A}_0\zeta_\lambda,\mathcal{A}\zeta_\lambda)| &\leq C_b\left[\nu\|w_\lambda\|^{\frac{1}{2}}|Aw_\lambda|\|y_e\|^{\frac{1}{2}}\left(\|w_\lambda\|^{\frac{1}{2}}|Ay_e|^{\frac{1}{2}}+\|y_e\|^{\frac{1}{2}}|Aw_\lambda|^{\frac{1}{2}}\right)\right.\\ &\quad\left.+k|A_1\xi_\lambda|\left(\|w_\lambda\|\|\theta_e\|^{\frac{1}{2}}|A_1\theta_e|^{\frac{1}{2}}+\|y_e\|\|\xi_\lambda\|^{\frac{1}{2}}|A_1\xi_\lambda|^{\frac{1}{2}}\right)\right]\\ &\leq C\left(\|\zeta_\lambda\||\mathcal{A}\zeta_\lambda|+\|\zeta_\lambda\|^{\frac{1}{2}}|\mathcal{A}\zeta_\lambda|^{\frac{3}{2}}\right)\leq\frac{1}{12}|\mathcal{A}\zeta_\lambda|^2+C\|\zeta_\lambda\|^2\end{aligned}$$

Hence, in both cases $d=2,3$, by (25), we obtain that

$$|(\mathcal{A}_0\zeta_\lambda,\mathcal{A}\zeta_\lambda)|\leq\frac{1}{12}|\mathcal{A}\zeta_\lambda|^2+C(1+|g|^2). \tag{28}$$

Using $(\gamma\xi_\lambda e_d, \nu A w_\lambda) \le 3\gamma^2/2|\zeta_\lambda|^2 + 1/6|\mathcal{A}\zeta_\lambda|^2$, (27), (28) and hypothesis (12), equation (26) implies

$$\begin{aligned}&|\mathcal{A}\zeta_\lambda|^2 - C(1+|g|^2)^3 - C(1+|g|^2) - \frac{3\gamma^2}{2}|\zeta_\lambda|^2 - \frac{1}{3}|\mathcal{A}\zeta_\lambda|^2 \\ &-\alpha_0(1+\|\zeta_\lambda\|^2) - \alpha|\Phi_\lambda(\zeta_\lambda)|^2 + \mu_N m_{\nu k}\|\zeta_\lambda\|^2 \le \frac{3}{2}|g|^2 + \frac{1}{6}|\mathcal{A}\zeta_\lambda|^2.\end{aligned}$$

If $\mu_N m_{\nu k} \ge \alpha_0$, we ignore the term $(\mu_N m_{\nu k} - \alpha_0)\|\zeta_\lambda\|^2 \ge 0$; if $\mu_N m_{\nu k} < \alpha_0$, we pass this term in the right hand side of the inequality. Anyway, by (25), the above relation yields

$$\frac{1}{2}|\mathcal{A}\zeta_\lambda|^2 \le \alpha|\Phi_\lambda(\zeta_\lambda)|^2 + C(1+|g|^2)^3, \ \forall\lambda > 0. \tag{29}$$

Finally, we multiply equation (23) in H_B by $\Phi_\lambda(\zeta_\lambda)$ and obtain

$$\begin{aligned}&(\mathcal{A}\zeta_\lambda, \Phi_\lambda(\zeta_\lambda)) + (B_N\zeta_\lambda, \Phi_\lambda(\zeta_\lambda)) + (\mathcal{A}_0\zeta_\lambda, \Phi_\lambda(\zeta_\lambda)) + (\mathcal{R}\zeta_\lambda, \Phi_\lambda(\zeta_\lambda)) \\ &+|\Phi_\lambda(\zeta_\lambda)|^2 + \mu_N(\zeta_\lambda, \Phi_\lambda(\zeta_\lambda)) = (g, \Phi_\lambda(\zeta_\lambda)).\end{aligned} \tag{30}$$

As shown before,

$$\mu_N(\zeta_\lambda, \Phi_\lambda(\zeta_\lambda)) \ge -\frac{\mu_N}{2}(|\Phi(0)|^2 + |\zeta_\lambda|^2).$$

If we denote the components of $\Phi_\lambda(\zeta_\lambda)) \in H_B = H \times L^2(\Omega)$ by $[\Phi_\lambda(\zeta_\lambda)]_1 \in H$, $[\Phi_\lambda(\zeta_\lambda)]_2 \in L^2(\Omega)$, we get

$$-(\mathcal{R}\zeta_\lambda, \Phi_\lambda(\zeta_\lambda)) = (\gamma\xi_\lambda e_d, [\Phi_\lambda(\zeta_\lambda)]_1) \le \frac{2\gamma^2}{1-\alpha}|\zeta_\lambda|^2 + \frac{1-\alpha}{8}|\Phi_\lambda(\zeta_\lambda)|^2.$$

Then,

$$|(B_N\zeta_\lambda, \Phi_\lambda(\zeta_\lambda))| \le |b(w_\lambda, w_\lambda, [\Phi_\lambda(\zeta_\lambda)]_1)| + |\bar{b}(w_\lambda, \xi_\lambda, [\Phi_\lambda(\zeta_\lambda)]_2)|.$$

In the case $d = 2$, using (4), the right hand side terms are bounded by

$$C\left(|w_\lambda|^{\frac{1}{2}}\|w_\lambda\||Aw_\lambda|^{\frac{1}{2}}|[\Phi_\lambda(\zeta_\lambda)]_1| + |w_\lambda|^{\frac{1}{2}}\|w_\lambda\|^{\frac{1}{2}}\|\xi_\lambda\|^{\frac{1}{2}}|A_1\xi_\lambda|^{\frac{1}{2}}|[\Phi_\lambda(\zeta_\lambda)]_2|\right)$$

$$\le C|\zeta_\lambda|^{\frac{1}{2}}\|\zeta_\lambda\||\mathcal{A}\zeta_\lambda|^{\frac{1}{2}}|\Phi_\lambda(\zeta_\lambda)|.$$

In the dimension $d = 3$, using (6), the same expression is bounded by

$$C\left(\|w_\lambda\|^{\frac{3}{2}}|Aw_\lambda|^{\frac{1}{2}}|[\Phi_\lambda(\zeta_\lambda)]_1| + \|w_\lambda\|\|\xi_\lambda\|^{\frac{1}{2}}|A_1\xi_\lambda|^{\frac{1}{2}}|[\Phi_\lambda(\zeta_\lambda)]_2|\right)$$
$$\leq C\|\zeta_\lambda\|^{\frac{3}{2}}|\mathcal{A}\zeta_\lambda|^{\frac{1}{2}}|\Phi_\lambda(\zeta_\lambda)|.$$

Hence, in both cases $d = 2, 3$, by (25), we have that

$$|(B_N\zeta_\lambda, \Phi_\lambda(\zeta_\lambda))| \leq \frac{1-\alpha}{8}|\Phi_\lambda(\zeta_\lambda)|^2 + C(1+|g|^2)^{\frac{3}{2}}|\mathcal{A}\zeta_\lambda|. \tag{31}$$

Proceeding in the same way, we infer that for $d = 2$

$$|(\mathcal{A}_0\zeta_\lambda, \Phi_\lambda(\zeta_\lambda))| \leq |b(w_\lambda, y_e, [\Phi_\lambda(\zeta_\lambda)]_1)| + |b(y_e, w_\lambda, [\Phi_\lambda(\zeta_\lambda))]_1)|$$
$$+|\bar{b}(w_\lambda, \theta_e, [\Phi_\lambda(\zeta_\lambda)]_2)| + |\bar{b}(y_e, \xi_\lambda, [\Phi_\lambda(\zeta_\lambda)]_2)|$$
$$\leq C_b\Big[\|w_\lambda\|^{\frac{1}{2}}|[\Phi_\lambda(\zeta_\lambda)]_1|\|y_e\|^{\frac{1}{2}}\left(|w_\lambda|^{\frac{1}{2}}|Ay_e|^{\frac{1}{2}} + |y_e|^{\frac{1}{2}}|Aw_\lambda|^{\frac{1}{2}}\right)$$
$$+|[\Phi_\lambda(\zeta_\lambda)]_2|\Big(|w_\lambda|^{\frac{1}{2}}\|w_\lambda\|^{\frac{1}{2}}\|\theta_e\|^{\frac{1}{2}}|A_1\theta_e|^{\frac{1}{2}}$$
$$+|y_e|^{\frac{1}{2}}\|y_e\|^{\frac{1}{2}}\|\xi_\lambda\|^{\frac{1}{2}}|A_1\xi_\lambda|^{\frac{1}{2}}\Big)\Big] \leq C|\Phi_\lambda(\zeta_\lambda)|\|\zeta_\lambda\|^{\frac{1}{2}}\left(|\zeta_\lambda|^{\frac{1}{2}} + |\mathcal{A}\zeta_\lambda|^{\frac{1}{2}}\right)$$
$$\leq \frac{1-\alpha}{8}|\Phi_\lambda(\zeta_\lambda)|^2 + C\|\zeta_\lambda\|(|\zeta_\lambda| + |\mathcal{A}\zeta_\lambda|).$$

For $d = 3$ we have

$$|(\mathcal{A}_0\zeta_\lambda, \Phi_\lambda(\zeta_\lambda))| \leq C_b\Big[\|w_\lambda\|^{\frac{1}{2}}|[\Phi_\lambda(\zeta_\lambda)]_1|\|y_e\|^{\frac{1}{2}}\Big(\|w_\lambda\|^{\frac{1}{2}}|Ay_e|^{\frac{1}{2}}$$
$$+\|y_e\|^{\frac{1}{2}}|Aw_\lambda|^{\frac{1}{2}}\Big) + |[\Phi_\lambda(\zeta_\lambda)]_2|\Big(\|w_\lambda\|\|\theta_e\|^{\frac{1}{2}}|A_1\theta_e|^{\frac{1}{2}}$$
$$+\|y_e\|\|\xi_\lambda\|^{\frac{1}{2}}|A_1\xi_\lambda|^{\frac{1}{2}}\Big)\Big] \leq C|\Phi_\lambda(\zeta_\lambda)|\|\zeta_\lambda\|^{\frac{1}{2}}\left(\|\zeta_\lambda\|^{\frac{1}{2}} + |\mathcal{A}\zeta_\lambda|^{\frac{1}{2}}\right)$$
$$\leq \frac{1-\alpha}{8}|\Phi_\lambda(\zeta_\lambda)|^2 + C(\|\zeta_\lambda\|^2 + \|\zeta_\lambda\||\mathcal{A}\zeta_\lambda|).$$

Hence, in both cases $d = 2, 3$, by (25), we obtain that

$$|(\mathcal{A}_0\zeta_\lambda, \Phi_\lambda(\zeta_\lambda)| \leq \frac{1-\alpha}{8}|\Phi_\lambda(\zeta_\lambda)|^2 + C(1+|g|^2) + C(1+|g|^2)^{1/2}|\mathcal{A}\zeta_\lambda|. \tag{32}$$

Together with (12), (31), (32), equation (30) implies

$$-\alpha_0(1+\|\zeta_\lambda\|^2) - \alpha|\Phi_\lambda(\zeta_\lambda)|^2 - \frac{1-\alpha}{4}|\Phi_\lambda(\zeta_\lambda)|^2 - C(1+|g|^2)^{\frac{3}{2}}|\mathcal{A}\zeta_\lambda|$$
$$-C(1+|g|^2) - C(1+|g|^2)^{1/2}|\mathcal{A}\zeta_\lambda| + |\Phi_\lambda(\zeta_\lambda)|^2 - \frac{\mu_N}{2}(|\Phi(0)|^2 + |\zeta_\lambda|^2)$$
$$\leq \frac{2}{1-\alpha}|g|^2 + \frac{2\gamma^2}{1-\alpha}|\zeta_\lambda|^2 + \frac{1-\alpha}{4}|\Phi_\lambda(\zeta_\lambda)|^2.$$

By (25), it follows

$$|\Phi_\lambda(\zeta_\lambda)|^2 \le C(1+|g|^2)^{\frac{3}{2}}|\mathcal{A}\zeta_\lambda| + C(1+|g|^2),\ \forall\lambda > 0. \tag{33}$$

Substituting (33) into (29), we obtain

$$\frac{1}{2}|\mathcal{A}\zeta_\lambda|^2 \le C(1+|g|^2)^{\frac{3}{2}}|\mathcal{A}\zeta_\lambda| + C(1+|g|^2)^3,$$

which implies

$$|\mathcal{A}\zeta_\lambda| \le C(1+|g|^2)^{\frac{3}{2}} \tag{34}$$

and

$$|\Phi_\lambda(\zeta_\lambda)|^2 \le C(1+|g|^2)^3. \tag{35}$$

From the boundedness in H_B of the sequences $(\zeta_\lambda)_{\lambda>0}$, $(\Phi_\lambda(\zeta_\lambda))_{\lambda>0}$, $(g_\lambda)_{\lambda>0}$, where $g_\lambda = g - \zeta_\lambda - \Phi_\lambda(\zeta_\lambda) = \Upsilon_N\zeta_\lambda$, it follows that, on a sequence $\lambda_j \searrow 0$, we have the weak convergences in H_B,

$$\zeta_{\lambda_j} \rightharpoonup \zeta,\ \Phi_{\lambda_j}(\zeta_{\lambda_j}) \rightharpoonup g_1,\ g_{\lambda_j} = \Upsilon_N\zeta_{\lambda_j} \rightharpoonup g_2.$$

Because, by (25), (ζ_{λ_j}) is bounded in V_B, we get that $\zeta_{\lambda_j} \to \zeta$ in H_B.

Passing to the weak limit in the equality $g - \zeta_{\lambda_j} - \Phi_{\lambda_j}(\zeta_{\lambda_j}) = g_{\lambda_j}$, we obtain $g = \zeta + g_1 + g_2$. If we would prove that $g_2 = \Upsilon_N\zeta,\ g_1 \in \Phi(\zeta)$, it will follow that $g \in \zeta + \Upsilon_N\zeta + \Phi(\zeta)$, as claimed.

We multiply by $\zeta_\lambda - \zeta_\mu$ the difference of equation (22) written for $\lambda > 0$ and the same equation written for $\mu > 0$. We find

$$(\Phi_\lambda(\zeta_\lambda) - \Phi_\mu(\zeta_\mu), \zeta_\lambda - \zeta_\mu) + ((\Upsilon_N + I)\zeta_\lambda - (\Upsilon_N + I)\zeta_\mu, \zeta_\lambda - \zeta_\mu) = 0. \tag{36}$$

Since $\Upsilon_N + I$ is the sum of two monotone operators and by consequence monotone, we get $(\Phi_\lambda(\zeta_\lambda) - \Phi_\mu(\zeta_\mu), \zeta_\lambda - \zeta_\mu) \le 0,\ \forall\lambda,\mu > 0$. From the maximal monotonicity of Φ it follows that $(\zeta, g_1) \in \Phi$ and $\lim_{\lambda,\mu\searrow 0}(\Phi_\lambda(\zeta_\lambda) - \Phi_\mu(\zeta_\mu), \zeta_\lambda - \zeta_\mu) = 0$ (see [1], Prop. 1.3 iv), p. 49).

Relation (36) implies that

$$\lim_{\lambda,\mu\searrow 0}((\Upsilon_N + I)\zeta_\lambda - (\Upsilon_N + I)\zeta_\mu, \zeta_\lambda - \zeta_\mu) = 0.$$

Using also $\zeta_{\lambda_j} \to \zeta,\ \Upsilon_N\zeta_{\lambda_j} \rightharpoonup g_2$ and the fact that $\Upsilon_N + I$ is maximal monotone (Υ_N maximal monotone), it follows (see [1], Lemma 1.3, p. 49) that $(\zeta, \zeta + g_2) \in \Upsilon_N + I$ and thus $\Upsilon_N\zeta = g_2$.

From $(\zeta, g_1) \in \Phi$ and $\Upsilon_N \zeta = g_2$ we also get $\zeta \in D(\Upsilon_N) \cap D(\Phi) = D(\mathcal{A}) \cap D(\Phi)$. Consequently, $D(\Lambda_N) = D(\mathcal{A}) \cap D(\Phi)$.

Let us prove now (18), (19).

For the first one, we consider $\lambda > 0$ fixed, $h \in D(\mathcal{A})$ and let $g_\lambda = \mathcal{A}h + B_N h + \mathcal{A}_0 h + \mathcal{R}h + \Phi_\lambda(h) + \mu_N h$. In the same way as we obtained (34), we get $|\mathcal{A}h| \leq C(1 + |g_\lambda|^2)^{\frac{3}{2}}$. Hence

$$\begin{aligned} |\mathcal{A}h|^{\frac{2}{3}} &\leq C(1 + |g_\lambda|^2) = C(1 + |(\mathcal{A} + B_N + \mathcal{A}_0 + \mathcal{R})h + \Phi_\lambda(h) + \mu_N h|^2) \\ &\leq C(1 + 2\mu_N^2 |h|^2 + 2|(\mathcal{A} + B_N + \mathcal{A}_0 + \mathcal{R})h + \Phi_\lambda(h)|^2). \end{aligned}$$

Thus (18) holds true.

In order to prove the second relation, we take $h \in D(\mathcal{A}) \cap D(\Phi)$ and $\eta \in \Phi(h)$. Let $g = (\mathcal{A} + B_N + \mathcal{A}_0 + \mathcal{R})h + \eta + \mu_N h$. For this g we may construct as in the first part of the proof a sequence $(h_\lambda)_{\lambda > 0} \subset H_B$ such that

$$(\mathcal{A} + B_N + \mathcal{A}_0 + \mathcal{R})h_\lambda + \Phi_\lambda(h_\lambda) + \mu_N h_\lambda = g, \ \forall \lambda > 0.$$

Moreover, $h_\lambda \to h$, $\mathcal{A}h_\lambda \rightharpoonup \mathcal{A}h$ in H_B because Λ_N is maximal monotone.

Passing to the limit with $\lambda \searrow 0$ in (34) written for $(h_\lambda)_{\lambda > 0}$, we obtain $|\mathcal{A}h| \leq C(1 + |g|^2)^{\frac{3}{2}}$, hence

$$\begin{aligned} |\mathcal{A}h|^{\frac{2}{3}} &\leq C(1 + |g|^2) = C(1 + |(\mathcal{A} + B_N + \mathcal{A}_0 + \mathcal{R})h + \eta + \mu_N h|^2) \\ &\leq C(1 + 2\mu_N^2 |h|^2 + 2|(\mathcal{A} + B_N + \mathcal{A}_0 + \mathcal{R})h + \eta|^2), \end{aligned}$$

which proves relation (19). This concludes the proof of Proposition 3.2. □

Proposition 3.3. *Let us assume that $\Phi \subset H_B \times H_B$ satisfies the hypotheses in Proposition* 3.2. *Let $F \in W^{1,1}(0, T; H_B)$, $z_0 \in D(\mathcal{A}) \cap D(\Phi)$. Then there exists a unique strong solution*

$$z_N \in W^{1,\infty}(0, T; H_B) \cap L^\infty(0, T; D(\mathcal{A})) \cap C([0, T]; V_B)$$

of the problem (17). *Moreover, z_N is right differentiable, $\frac{d^+}{dt} z_N$ is right continuous and*

$$\frac{d^+}{dt} z_N(t) + ((\mathcal{A} + B_N + \mathcal{A}_0 + \mathcal{R})z_N(t) + \Phi(z_N(t)) - F(t))^0 = 0,$$

(37) $\quad \forall t \in [0, T)$.

Proof. From Proposition 3.2 and [1], Th. 1.4, Th. 1.6, p. 214-216, it follows that problem (17) has a unique solution $z_N \in W^{1,\infty}(0,T;H_B)$ satisfying relation (37). In order to prove that $z_N \in L^\infty(0,T;D(\mathcal{A})) \cap C([0,T];V_B)$, let $\eta_N(t) \in \Phi(z_N(t))$ be such that

$$\frac{dz_N}{dt}(t) + (\mathcal{A} + B_N + \mathcal{A}_0 + \mathcal{R})z_N(t) + \eta_N(t) = F(t).$$

We know that $F - \frac{dz_N}{dt} \in L^\infty(0,T;H_B)$, so $(\mathcal{A} + B_N + \mathcal{A}_0 + \mathcal{R})z_N + \eta_N \in L^\infty(0,T;H_B)$. Applying (19) for $z_N(t)$ and $\eta_N(t) \in \Phi(z_N(t))$, we get $\mathcal{A}z_N \in L^\infty(0,T;H_B)$, which implies $z_N \in L^\infty(0,T;D(\mathcal{A}))$. Together with $\frac{dz_N}{dt} \in L^\infty(0,T;H_B)$ we infer that $z_N \in C([0,T];V_B)$. □

We get a similar result if we use the Yosida approximation Φ_λ instead of Φ:

Proposition 3.4. *Let $N \in \mathbb{N}^*$ be fixed. Assume that $\Phi \subset H_B \times H_B$ is a maximal monotone operator satisfying hypotheses (h_2), (h_3). Let $z_0 \in D(\mathcal{A}) \cap D(\Phi)$ and $F \in W^{1,1}(0,T;H_B)$. Then for all $\lambda > 0$ there exists a unique strong solution*

$$z_N^\lambda \in W^{1,\infty}(0,T;H_B) \cap L^\infty(0,T;D(\mathcal{A})) \cap C([0,T];V_B)$$

for the problem

$$\text{(38)} \qquad \begin{cases} \dfrac{dz_N^\lambda}{dt}(t) + \mathcal{A}z_N^\lambda(t) + B_N z_N^\lambda(t) + \mathcal{A}_0 z_N^\lambda(t) + \mathcal{R}z_N^\lambda(t) \\ \qquad\qquad + \Phi_\lambda(z_N^\lambda(t)) = F(t), \quad a.e.\ t \in (0,T) \\ z_N^\lambda(0) = z_0. \end{cases}$$

Moreover, z_N^λ is right differentiable, $\frac{d^+}{dt}z_N^\lambda(t)$ is right continuous and

$$\text{(39)} \qquad \frac{d^+}{dt}z_N^\lambda(t) + \mathcal{A}z_N^\lambda(t) + B_N z_N^\lambda(t) + \mathcal{A}_0 z_N^\lambda(t) + \mathcal{R}z_N^\lambda(t)$$
$$+\Phi_\lambda(z_N^\lambda(t)) = F(t),\ \forall t \in [0,T).$$

Proof. The operator $\Upsilon_N = \mathcal{A} + B_N + \mathcal{A}_0 + \mathcal{R} + \alpha_N I$ is maximal monotone (for α_N given by Proposition 3.1) and Φ_λ is demicontinuous monotone, so ([1, 4]) $\Upsilon_N + \Phi_\lambda$ is maximal monotone in $H_B \times H_B$. Then, (38) has a unique solution $z_N^\lambda \in W^{1,\infty}(0,T;H_B)$ verifying relation (39). Consequently $(\mathcal{A} + B_N + \mathcal{A}_0 + \mathcal{R})z_N^\lambda + \Phi_\lambda(z_N^\lambda) = F - \frac{dz_N^\lambda}{dt} \in L^\infty(0,T;H_B)$.

Applying (18) for $z_N^\lambda(t) \in D(\mathcal{A})$, we get $\mathcal{A}z_N^\lambda \in L^\infty(0,T;H_B)$, which implies $z_N^\lambda \in L^\infty(0,T;D(\mathcal{A}))$. Together with $\frac{dz_N^\lambda}{dt} \in L^\infty(0,T;H_B)$, we obtain $z_N^\lambda \in C([0,T];V_B)$. □

3.2. Estimates for the solution of problem (38)**.** By Proposition 3.3, problem (17) has a unique strong solution

$$z_N \in W^{1,\infty}(0,T;H_B) \cap L^\infty(0,T;D(\mathcal{A})) \cap C([0,T];V_B).$$

However, in order to get better estimates we shall further approximate problem (17) by problem (38), which also has a unique strong solution in the spaces above (by Proposition 3.4).

First, we multiply (38) by $z_N^\lambda(t) = \begin{pmatrix} y_N^\lambda(t) \\ \theta_N^\lambda(t) \end{pmatrix}$ in H_B and integrate on $(0,t)$. Using (3), we get

$$\int_0^t \frac{d}{ds}\left(\frac{1}{2}|z_N^\lambda(s)|^2\right) ds + \int_0^t (\mathcal{A}z_N^\lambda(s), z_N^\lambda(s))ds + \int_0^t (\mathcal{A}_0 z_N^\lambda(s), z_N^\lambda(s))ds$$
$$+ \int_0^t (\mathcal{R}z_N^\lambda(s), z_N^\lambda(s))ds + \int_0^t (\Phi_\lambda(z_N^\lambda(s)), z_N^\lambda(s))ds = \int_0^t (F(s), z_N^\lambda(s))ds.$$

But

$$(\mathcal{A}z_N^\lambda(s), z_N^\lambda(s)) = \nu\|y_N^\lambda(s)\|^2 + k\|\theta_N^\lambda(s)\|^2 \geq m_{\nu k}\|z_N^\lambda(s)\|^2;$$
$$-(\mathcal{R}z_N^\lambda(s), z_N^\lambda(s)) \leq \gamma|\theta_N^\lambda(s)||y_N^\lambda(s)| \leq (\gamma/2)|z_N^\lambda(s)|^2$$

and, proceeding as in the proof of Proposition 3.2,

$$\begin{aligned} -(\mathcal{A}_0 z_N^\lambda(s), z_N^\lambda(s)) &\leq |b(y_N^\lambda(s), y_e, y_N^\lambda(s))| + |\bar{b}(y_N^\lambda(s), \theta_e, \theta_N^\lambda(s))| \\ &\leq C_b|z_N^\lambda(s)|\|z_e\|_a\|z_N^\lambda(s)\|, \text{ where } a = \begin{cases} 1, & d=2 \\ 3/2, & d=3. \end{cases} \end{aligned}$$

Also, from the monotony of Φ_λ, $(\Phi_\lambda(z_N^\lambda(s)), z_N^\lambda(s)) \geq (\Phi_\lambda(0), z_N^\lambda(s))$. So,

$$\frac{1}{2}|z_N^\lambda(t)|^2 + m_{\nu k}\int_0^t \|z_N^\lambda(s)\|^2 ds \leq \frac{1}{2}|z_0|^2 + \frac{\gamma}{2}\int_0^t |z_N^\lambda(s)|^2 ds$$
$$+ \int_0^t C_b|z_N^\lambda(s)|\|z_e\|_a\|z_N^\lambda(s)\|ds - \int_0^t (\Phi_\lambda(0), z_N^\lambda(s))ds$$

$$+\int_0^t (F(s), z_N^\lambda(s))ds \le \frac{1}{2}|z_0|^2 + \frac{m_{\nu k}}{2}\|z_N^\lambda(s)\|^2 ds$$
$$+\left(\frac{\gamma}{2} + \frac{C_b^2\|z_e\|_a^2}{2m_{\nu k}} + 1\right)\int_0^t |z_N^\lambda(s)|^2 ds + \frac{1}{2}\int_0^t (|\Phi(0)|^2 + |F(s)|^2)ds.$$

In particular, it follows that

$$|z_N^\lambda(t)|^2 \le |z_0|^2 + \int_0^T (|\Phi(0)|^2 + |F(s)|^2)ds + C\int_0^t |z_N^\lambda(s)|^2 ds$$

and by Gronwall's inequality

$$|z_N^\lambda(t)|^2 \le \left(|z_0|^2 + \int_0^T (|\Phi(0)|^2 + |F(s)|^2)ds\right) e^{Ct}.$$

Finally, we infer

$$|z_N^\lambda(t)|^2 + m_{\nu k}\int_0^t \|z_N^\lambda(s)\|^2 ds \le e^{Ct}\left[|z_0|^2 + \int_0^T (|\Phi(0)|^2 + |F(s)|^2)ds\right],$$

and thus

$$|z_N^\lambda(t)|^2 + m_{\nu k}\int_0^t \|z_N^\lambda(s)\|^2 ds \le C_1(\|F\|^2_{L^2(0,T;H_B)}, |z_0|^2), \tag{40}$$

where C_1 is a positive bounded function depending on $\|F\|^2_{L^2(0,T;H_B)}$,$|z_0|^2$, but independent of N, λ.

Next we multiply (38) in H_B with $\mathcal{A}z_N^\lambda(t)$ and integrate on $(0,t)$,

$$\int_0^t \frac{d}{ds}\left(\frac{\nu}{2}\|y_N^\lambda(s)\|^2 + \frac{k}{2}\|\theta_N^\lambda(s)\|^2\right) ds + \int_0^t |\mathcal{A}z_N^\lambda(s)|^2 ds$$
$$+\int_0^t ((B_N + \mathcal{A}_0)z_N^\lambda(s), \mathcal{A}z_N^\lambda(s))ds + \int_0^t (\mathcal{R}z_N^\lambda(s), \mathcal{A}z_N^\lambda(s))ds$$
$$+\int_0^t (\Phi_\lambda(z_N^\lambda(s)), \mathcal{A}z_N^\lambda(s))ds = \int_0^t (F(s), \mathcal{A}z_N^\lambda(s))ds.$$

Recalling (9) and (12), this yields

$$\frac{m_{\nu k}}{2}\|z_N^\lambda(t)\|^2 + \int_0^t |\mathcal{A}z_N^\lambda(s)|^2 ds - \alpha_0\int_0^t (1 + \|z_N^\lambda(s)\|^2)ds$$
$$-\alpha\int_0^t |\Phi_\lambda(z_N^\lambda(s))|^2 ds \le \frac{m_{\nu k}}{2}\|z_0\|^2 + \gamma\int_0^t \left(\nu A y_N^\lambda(s), \theta_N^\lambda(s)e_d\right) ds$$
$$-\int_0^t ((B_N + \mathcal{A}_0)z_N^\lambda(s), \mathcal{A}z_N^\lambda(s))ds + \int_0^t (F(s), \mathcal{A}z_N^\lambda(s))ds.$$

But

$$\gamma \int_0^t \left(\nu A y_N^\lambda(s), \theta_N^\lambda(s) e_d\right) ds \leq \frac{1}{6}\int_0^t |\mathcal{A} z_N^\lambda(s)|^2 ds + \frac{3\gamma^2}{2}\int_0^t |z_N^\lambda(s)|^2 ds.$$

Moreover, using the same estimates leading to (27), we obtain in the case $d = 2$ that

$$\left|\int_0^t (B_N z_N^\lambda(s), \mathcal{A} z_N^\lambda(s)) ds\right| \leq \frac{1}{12}\int_0^t |\mathcal{A} z_N^\lambda(s)|^2 ds + C\int_0^t |z_N^\lambda(s)|^2 \|z_N^\lambda(s)\|^4 ds$$

$$\leq \frac{1}{12}\int_0^t |\mathcal{A} z_N^\lambda(s)|^2 ds + C\int_0^t \|z_N^\lambda(s)\|^4 ds \ (|z_N^\lambda(s)| \text{ is bounded from (40)}).$$

In the case $d = 3$, we have

$$\left|\int_0^t (B_N z_N^\lambda(s), \mathcal{A} z_N^\lambda(s)) ds\right| \leq \frac{1}{12}\int_0^t |\mathcal{A} z_N^\lambda(s)|^2 ds + C\int_0^t \|z_N^\lambda(s)\|^6 ds.$$

In both cases the constant C is independent of N, λ.

Using the same estimates leading to inequality (28), we obtain in the case $d = 2$ that

$$\left|\int_0^t (\mathcal{A}_0 z_N^\lambda(s), \mathcal{A} z_N^\lambda(s)) ds\right| \leq \frac{1}{12}\int_0^t |\mathcal{A} z_N^\lambda(s)|^2 ds + C\left(\int_0^t |z_N^\lambda(s)| \|z_N^\lambda(s)\| ds\right.$$

$$\left. + \int_0^t \|z_N^\lambda(s)\|^2 ds\right) \leq \frac{1}{12}\int_0^t |\mathcal{A} z_N^\lambda(s)|^2 ds + C \text{ (from (40))}.$$

In the case $d = 3$, we have

$$\begin{aligned}\left|\int_0^t (\mathcal{A}_0 z_N^\lambda(s), \mathcal{A} z_N^\lambda(s)) ds\right| &\leq \frac{1}{12}\int_0^t |\mathcal{A} z_N^\lambda(s)|^2 ds + C\int_0^t \|z_N^\lambda(s)\|^2 ds \\ &\leq \frac{1}{12}\int_0^t |\mathcal{A} z_N^\lambda(s)|^2 ds + C \text{ (from (40))}.\end{aligned}$$

We get

$$\frac{m_{\nu k}}{2}\|z_N^\lambda(t)\|^2 + \int_0^t |\mathcal{A} z_N^\lambda(s)|^2 ds \leq \frac{m_{\nu k}}{2}\|z_0\|^2 + \frac{1}{2}\int_0^t |\mathcal{A} z_N^\lambda(s)|^2 ds$$

$$+\frac{3\gamma^2}{2}\int_0^t |z_N^\lambda(s)|^2 ds + \frac{3}{2}\int_0^t |F(s)|^2 ds + C\int_0^t \|z_N^\lambda(s)\|^{2d} ds$$

$$+\alpha_0\left(t+\int_0^t \|z_N^\lambda(s)\|^2 ds\right)+\alpha\int_0^t |\Phi_\lambda(z_N^\lambda(s))|^2 ds + C.$$

Using also (40), it follows that

$$\frac{m_\nu k}{2}\|z_N^\lambda(t)\|^2+\frac{1}{2}\int_0^t |\mathcal{A}z_N^\lambda(s)|^2 ds \le \alpha\int_0^t |\Phi_\lambda(z_N^\lambda(s))|^2 ds \tag{41}$$

$$+C_2\int_0^t \|z_N^\lambda(s)\|^{2d} ds + \tilde{C}_2,$$

where $\tilde{C}_2, C_2$ are positive bounded functions of $\|F\|^2_{L^2(0,T;H_B)}, |z_0|^2, \|z_0\|^2$, but independent of N, λ.

Finally, we multiply (38) in H_B with $\Phi_\lambda(z_N^\lambda(t))$ and integrate on $(0,t)$. We recall that $\Phi = \partial\varphi$ and that the Yosida approximation Φ_λ is the Fréchet differential $\nabla\varphi_\lambda$ of φ_λ, where

$$\varphi_\lambda(u)=\inf\left\{\frac{|u-v|^2}{2\lambda}+\varphi(v);\ v\in H_B\right\},\ \forall u\in H_B$$

is the regularization of φ. So,

$$\int_0^t\left(\frac{d}{ds}z_N^\lambda(s),(\nabla\varphi_\lambda)(z_N^\lambda(s))\right)ds+\int_0^t(\mathcal{A}z_N^\lambda(s),\Phi_\lambda(z_N^\lambda(s)))ds$$
$$+\int_0^t(B_N z_N^\lambda(s),\Phi_\lambda(z_N^\lambda(s)))ds+\int_0^t(\mathcal{A}_0 z_N^\lambda(s),\Phi_\lambda(z_N^\lambda(s)))ds$$
$$+\int_0^t(\mathcal{R}z_N^\lambda(s),\Phi_\lambda(z_N^\lambda(s)))ds+\int_0^t|\Phi_\lambda(z_N^\lambda(s))|^2 ds=\int_0^t(F(s),\Phi_\lambda(z_N^\lambda(s)))ds.$$

Using $\left(\frac{d}{ds}z_N^\lambda(s),(\nabla\varphi_\lambda)(z_N^\lambda(s))\right)=\frac{d}{ds}\left[\varphi_\lambda(z_N^\lambda(s))\right]$, (9) and (12), we get

$$\begin{aligned}(1-\alpha)\int_0^t|\Phi_\lambda(z_N^\lambda(s))|^2 ds &\le \varphi_\lambda(z_0)-\varphi_\lambda(z_N^\lambda(t))\\ &+\alpha_0\left(\int_0^t\|z_N^\lambda(s)\|^2 ds+t\right)+\gamma\int_0^t\left(\theta_N^\lambda(s)e_d,\Phi_\lambda(z_N^\lambda(s))\right)ds\\ &+\int_0^t(F(s),\Phi_\lambda(z_N^\lambda(s)))ds-\int_0^t(B_N z_N^\lambda(s),\Phi_\lambda(z_N^\lambda(s)))ds\\ &-\int_0^t(\mathcal{A}_0 z_N^\lambda(s),\Phi_\lambda(z_N^\lambda(s)))ds.\end{aligned}\tag{42}$$

Since every proper lower semicontinuous convex function is bounded from below by an affine function, it follows that there are $h \in H_B$ and $p \in \mathbb{R}$ such that

$$\varphi(u) \geq (u, h) + p, \ \forall u \in H_B.$$

Also, $J_\lambda = (\lambda\Phi + I)^{-1}$ is bounded on bounded subsets of H_B and $\varphi(J_\lambda(u)) \leq \varphi_\lambda(u) \leq \varphi(u), \ \forall \lambda > 0, \ \forall u \in H_B$. Using again (40), we infer that

$$-\varphi_\lambda(z_N^\lambda(t)) \leq -\varphi(J_\lambda(z_N^\lambda(t))) \leq -(J_\lambda(z_N^\lambda(t)), h) - p \leq |J_\lambda(z_N^\lambda(t))||h| + |p| \leq c,$$

c constant not depending of N, λ, t.

On the other hand, $\varphi_\lambda(z_0) \leq \varphi(z_0)$.

Using the same estimates leading to (31), we get that

$$\begin{aligned}
&\left|\int_0^t (B_N z_N^\lambda(s), \Phi_\lambda(z_N^\lambda(s)))ds\right| \\
&\leq \begin{cases} \displaystyle\int_0^t C|z_N^\lambda(s)|^{\frac{1}{2}} \|z_N^\lambda(s)\| |\mathcal{A} z_N^\lambda(s)|^{\frac{1}{2}} |\Phi_\lambda(z_N^\lambda(s))|ds, & d = 2 \\ \displaystyle\int_0^t C\|z_N^\lambda(s)\|^{\frac{3}{2}} |\mathcal{A} z_N^\lambda(s)|^{\frac{1}{2}} |\Phi_\lambda(z_N^\lambda(s))|ds, & d = 3 \end{cases} \\
&\leq \varepsilon \int_0^t |\mathcal{A} z_N^\lambda(s)|^2 ds + \frac{1-\alpha}{8} \int_0^t |\Phi_\lambda(z_N^\lambda(s))|^2 ds \\
&+ \frac{C}{\varepsilon(1-\alpha)^2} \int_0^t \|z_N^\lambda(s)\|^{2d} ds
\end{aligned}$$

($|z_N^\lambda(s)|^2$ being bounded by C_1 from (40)). The constants $C, \varepsilon > 0$ do not depend on N, λ. While C (related to (4), (6) respectively) is fixed, ε is at our choice and will be precised later.

Using the same estimates leading to (32), we get that

$$\begin{aligned}
&\left|\int_0^t (\mathcal{A}_0 z_N^\lambda(s), \Phi_\lambda(z_N^\lambda(s)))ds\right| \leq \frac{1-\alpha}{8} \int_0^t |\Phi_\lambda(z_N^\lambda(s))|^2 ds \\
&+ C \int_0^t \|z_N^\lambda(s)\| |\mathcal{A} z_N^\lambda(s)| ds + \begin{cases} C \displaystyle\int_0^t |z_N^\lambda(s)| \|z_N^\lambda(s)\| ds, & d = 2 \\ C \displaystyle\int_0^t \|z_N^\lambda(s)\|^2 ds, & d = 3 \end{cases} \\
&\leq \frac{1-\alpha}{8} \int_0^t |\Phi_\lambda(z_N^\lambda(s))|^2 ds + \varepsilon \int_0^t |\mathcal{A} z_N^\lambda(s)|^2 ds + C_\varepsilon
\end{aligned}$$

(from (40)). Here C_ε is a constant depending on ε.

Then (42) reads

$$(1-\alpha)\int_0^t |\Phi_\lambda(z_N^\lambda(s))|^2 ds \le \varphi(z_0) + c + \alpha_0\left(T + \frac{C_1}{m_{\nu k}}\right) + \frac{2}{1-\alpha}\int_0^t |F(s)|^2 ds + \frac{2\gamma^2}{1-\alpha}\int_0^t |z_N^\lambda(s)|^2 ds + \frac{1-\alpha}{2}\int_0^t |\Phi_\lambda(z_N^\lambda(s))|^2 ds + 2\varepsilon\int_0^t |\mathcal{A}z_N^\lambda(s)|^2 ds + C_\varepsilon + \frac{C}{\varepsilon(1-\alpha)^2}\int_0^t \|z_N^\lambda(s)\|^{2d} ds, \text{ that is}$$

$$\int_0^t |\Phi_\lambda(z_N^\lambda(s))|^2 ds \le \frac{4\varepsilon}{1-\alpha}\int_0^t |\mathcal{A}z_N^\lambda(s)|^2 ds + \frac{2C}{\varepsilon(1-\alpha)^3}\int_0^t \|z_N^\lambda(s)\|^{2d} ds + C_3\left(\|F\|^2_{L^2(0,T;H_B)}, |z_0|^2, \varphi(z_0), \varepsilon\right). \tag{43}$$

Now we substitute relation (43) into (41).

$$\frac{m_{\nu k}}{2}\|z_N^\lambda(t)\|^2 + \frac{1}{2}\int_0^t |\mathcal{A}z_N^\lambda(s)|^2 ds \le \frac{4\alpha\varepsilon}{1-\alpha}\int_0^t |\mathcal{A}z_N^\lambda(s)|^2 ds + C_4\int_0^t \|z_N^\lambda(s)\|^{2d} ds + C_5,$$

where $C_4 = C_2 + \frac{2\alpha C}{\varepsilon(1-\alpha)^3}$, $C_5 = \tilde{C}_2 + \alpha C_3$. We want that $\epsilon = 1/2 - 4\alpha\varepsilon/(1-\alpha) > 0$, so, we take $\varepsilon \in (0, (1-\alpha)/(8\alpha))$. This yields

$$\frac{m_{\nu k}}{2}\|z_N^\lambda(t)\|^2 + \epsilon\int_0^t |\mathcal{A}z_N^\lambda(s)|^2 ds \le C_4\int_0^t \|z_N^\lambda(s)\|^{2d} ds + C_5. \tag{44}$$

Case $d = 2$: global boundedness results. From (44) we have, in particular, that $\|z_N^\lambda(t)\|^2 \le \tilde{C}_4\int_0^t \|z_N^\lambda(s)\|^4 ds + \tilde{C}_5$. By Gronwall's lemma, we infer that $\|z_N^\lambda(t)\|^2 \le \tilde{C}_5 e^{\tilde{C}_4\int_0^t \|z_N^\lambda(s)\|^2 ds}$ and using that $\int_0^t \|z_N^\lambda(s)\|^2 ds$ is bounded by $C_1/m_{\nu k}$ from (40), we get

$$\|z_N^\lambda(t)\|^2 \le \tilde{C}_5 \exp[(\tilde{C}_4 C_1)/m_{\nu k}] = C_6, \quad \text{a.e. } t \in [0,T]. \tag{45}$$

Substituting (45) into (44), we obtain

$$\frac{m_{\nu k}}{2}\|z_N^\lambda(t)\|^2 + \epsilon\int_0^t |\mathcal{A}z_N^\lambda(s)|^2 ds \le C_7(\|F\|^2_{L^2(0,T;H_B)}, \|z_0\|^2, \varphi(z_0)), \quad \text{a.e. } t \in [0,T]. \tag{46}$$

Relation (46) implies $\int_0^t |\mathcal{A}z_N^\lambda(s)|^2 ds \le C_7/\epsilon$ and, together with (45), transforms (43) into

$$\text{(47)}\quad \int_0^t |\Phi_\lambda(z_N^\lambda(s))|^2 ds \le C_8(\|F\|^2_{L^2(0,T;H_B)}, \|z_0\|^2, \varphi(z_0)), \quad \text{a.e. } t \in [0,T].$$

Case $d = 3$: local boundedness results. From (44), we have, in particular, that

$$\|z_N^\lambda(t)\|^2 \le \tilde{C}_4 \int_0^t \|z_N^\lambda(s)\|^6 ds + \tilde{C}_5.$$

Using a comparison result, we infer that $\|z_N^\lambda(t)\|^2 \le W(t)$, where

$$\begin{cases} W'(t) = \tilde{C}_4 W^3(t) \\ W(0) = \tilde{C}_5 \end{cases}, \text{ that is } W(t) = \frac{\tilde{C}_5}{\sqrt{1 - 2\tilde{C}_4\tilde{C}_5^2 t}}.$$

The solution W exists on a maximal interval $[0, \overline{T^*}), \overline{T^*} = 1/(2\tilde{C}_4\tilde{C}_5^2)$. Let $T^* = \min\{T, \overline{T^*}\}$.

We get $\|z_N^\lambda(t)\|^2 \le \frac{\tilde{C}_5}{(1-2\tilde{C}_4\tilde{C}_5^2 t)^{\frac{1}{2}}}$, $t \in [0, T^*)$. Substituting into (44), we obtain

$$\frac{m_\nu k}{2}\|z_N^\lambda(t)\|^2 + \epsilon \int_0^t |\mathcal{A}z_N^\lambda(s)|^2 ds \le C_5 + C_4 \int_0^t \frac{\tilde{C}_5^3}{(1 - 2\tilde{C}_4\tilde{C}_5^2 s)^{\frac{3}{2}}} ds, \ t \in [0, T^*).$$

The term $\int_0^t \frac{\tilde{C}_5^3}{(1-2\tilde{C}_4\tilde{C}_5^2 s)^{\frac{3}{2}}} ds$ is of the order of $\int_0^t \frac{1}{(\overline{T^*}-s)^{\frac{3}{2}}} ds = \frac{2}{\sqrt{\overline{T^*}-t}} - \frac{2}{\sqrt{\overline{T^*}}}$, which blows up in $t = \overline{T^*}$. Consequently it is bounded on intervals of the type $[0, T^* - \delta]$, $\delta \in (0, T^*)$.

Finally, for $d = 3$, we have

$$\text{(48)}\quad \begin{aligned} &\frac{m_\nu k}{2}\|z_N^\lambda(t)\|^2 + \epsilon \int_0^t |\mathcal{A}z_N^\lambda(s)|^2 ds \\ &\le C_7(\|F\|^2_{L^2(0,T;H_B)}, \|z_0\|^2, \varphi(z_0), \delta), \quad \text{a.e. } t \in [0, T^* - \delta]. \end{aligned}$$

Using the local boundedness of $\|z_N^\lambda(t)\|^2$ and $\int_0^t |\mathcal{A}z_N^\lambda(s)|^2 ds$, the estimate (43) reads

$$\text{(49)}\quad \int_0^t |\Phi_\lambda(z_N^\lambda(s))|^2 ds \le C_8(\|F\|^2_{L^2(0,T;H_B)}, \|z_0\|^2, \varphi(z_0), \delta), \ \text{a.e. } t \in [0, T^* - \delta].$$

If $\overline{T}^* > T$, the estimates (48), (49) hold true a.e $t \in [0, T]$.

Estimates (40) and (46),(47) in the case $d = 2$, respectively (48), (49) in the case $d = 3$, will allow us to pass to the limit for $\lambda \searrow 0$ (maintaining N fixed). The positive bounded functions C_1, C_7, C_8 are independent of λ, N.

3.3. Passing to the limit for $\boldsymbol{\lambda \searrow 0}$. We recall that Proposition 3.4 implies

$$z_N^\lambda \in W^{1,\infty}(0, T; H_B) \cap L^\infty(0, T; D(\mathcal{A})) \cap C([0, T]; V_B).$$

Let $T_0 = T$ for $d = 2$ and $T_0 < T^*$ for $d = 3$ (we may take $T_0 = T$ if $\overline{T}^* > T$). We have

$$(z_N^\lambda)_\lambda \text{ is bounded in } C([0, T_0]; V_B) \cap L^2(0, T_0; D(\mathcal{A})), \tag{50}$$

$$(\mathcal{A}z_N^\lambda)_\lambda,\ (\Phi_\lambda(z_N^\lambda))_\lambda \text{ are bounded in } L^2(0, T_0; H_B). \tag{51}$$

Because $\mathcal{R} \in L(H_B, H_B)$, it follows that

$$(\mathcal{R}z_N^\lambda)_\lambda \text{ is bounded in } L^\infty(0, T_0; H_B). \tag{52}$$

From (4), (6), (40), (46) and (48) we infer for $d = 2$ that

$$|B_N z_N^\lambda(s)| \le C|z_N^\lambda(s)|^{\frac{1}{2}} \|z_N^\lambda(s)\| |\mathcal{A}z_N^\lambda(s)|^{\frac{1}{2}} \le C|\mathcal{A}z_N^\lambda(s)|^{\frac{1}{2}},$$

$$|\mathcal{A}_0 z_N^\lambda(s)| \le C\|z_N^\lambda(s)\|^{\frac{1}{2}} \left(|z_N^\lambda(s)|^{\frac{1}{2}} + |\mathcal{A}z_N^\lambda(s)|^{\frac{1}{2}}\right) \le C(1 + |\mathcal{A}z_N^\lambda(s)|^{\frac{1}{2}})$$

and for $d = 3$ that

$$|B_N z_N^\lambda(s)| \le C\|z_N^\lambda(s)\|^{\frac{3}{2}} |\mathcal{A}z_N^\lambda(s)|^{\frac{1}{2}} \le C|\mathcal{A}z_N^\lambda(s)|^{\frac{1}{2}},$$

$$|\mathcal{A}_0 z_N^\lambda(s)| \le C\left(\|z_N^\lambda(s)\| + \|z_N^\lambda(s)\|^{\frac{1}{2}} |\mathcal{A}z_N^\lambda(s)|^{\frac{1}{2}}\right) \le C(1 + |\mathcal{A}z_N^\lambda(s)|^{\frac{1}{2}})$$

The constants C do not depend on N, λ. Together with (51), we get

$$(B_N z_N^\lambda)_\lambda,\ (\mathcal{A}_0 z_N^\lambda)_\lambda \text{ are bounded in } L^2(0, T_0; H_B). \tag{53}$$

From (51), (52), (53) and equation (38), we have also

$$\left(\frac{dz_N^\lambda}{dt}\right)_\lambda \text{ is bounded in } L^2(0, T_0; H_B). \tag{54}$$

From (50) and (54) we infer that

(55) $\{z_N^\lambda; \lambda > 0\}$ is relatively compact in $C([0,T_0]; H_B)$.

These yield that, on a subsequence again denoted by $(z_N^\lambda)_\lambda$, we have for $\lambda \searrow 0$

(56) $$z_N^\lambda \to z_N \text{ in } C([0,T_0]; H_B)$$

and the following weak convergences in $L^2(0,T_0;H_B)$:

$$\frac{dz_N^\lambda}{dt} \rightharpoonup \frac{dz_N}{dt}, \quad \mathcal{A}z_N^\lambda \rightharpoonup \mathcal{A}z_N, \quad \mathcal{R}z_N^\lambda \rightharpoonup \mathcal{R}z_N, \quad \mathcal{A}_0 z_N^\lambda \rightharpoonup \mathcal{A}_0 z_N,$$

$$B_N z_N^\lambda \rightharpoonup \beta_N, \text{ and } \Phi_\lambda(z_N^\lambda) \rightharpoonup \eta_N.$$

Moreover, by Aubin's compactness theorem,

(57) $$z_N^\lambda \to z_N \text{ in } L^2(0,T_0;V_B).$$

But $\Phi_\lambda(z_N^\lambda) = \Phi(I+\lambda\Phi)^{-1}(z_N^\lambda)$ and $(I+\lambda\Phi)^{-1}(z_N^\lambda) \to z_N$ in $L^2(0,T_0;H_B)$. Φ being maximal monotone, it follows that $\eta_N(t) \in \Phi(z_N(t))$ a.e. $t \in [0,T_0]$.

We prove now that $\beta_N(t) = B_N z_N(t)$ a.e. $t \in [0,T_0]$. In the dimension $d = 2$, from (4), we obtain that, for any $\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \in V_B$,

$$\begin{aligned}
&|(B_N z_N^\lambda(t) - B_N z_N(t), \psi)| \le |b(y_N^\lambda(t) - y_N(t), y_N^\lambda(t), \psi_1)| \\
&\quad + |b(y_N(t), y_N^\lambda(t) - y_N(t), \psi_1)| + |\bar{b}(y_N^\lambda(t) - y_N(t), \theta_N^\lambda(t), \psi_2)| \\
&\quad + |\bar{b}(y_N(t), \theta_N^\lambda(t) - \theta_N(t), \psi_2)| \\
&\quad \le C_b \Big[|\psi_1| \|y_N^\lambda(t) - y_N(t)\|^{\frac12} \Big(|y_N^\lambda(t) - y_N(t)|^{\frac12} \cdot \|y_N^\lambda(t)\|^{\frac12} |Ay_N^\lambda(t)|^{\frac12} \\
&\quad + |y_N(t)|^{\frac12} \|y_N(t)\|^{\frac12} |A(y_N^\lambda(t) - y_N(t))|^{\frac12} \Big) \\
&\quad + |\psi_2| \Big(|y_N^\lambda(t) - y_N(t)|^{\frac12} \|y_N^\lambda(t) - y_N(t)\|^{\frac12} \|\theta_N^\lambda(t)\|^{\frac12} |A_1 \theta_N^\lambda(t)|^{\frac12} \\
&\quad + |y_N(t)|^{\frac12} \cdot \|y_N(t)\|^{\frac12} \|\theta_N^\lambda(t) - \theta_N(t)\|^{\frac12} |A_1(\theta_N^\lambda(t) - \theta_N(t))|^{\frac12} \Big) \Big] \\
&\quad \le C|\psi| \|z_N^\lambda(t) - z_N(t)\|^{\frac12} \cdot \Big(|z_N^\lambda(t) - z_N(t)|^{\frac12} \|z_N^\lambda(t)\|^{\frac12} |\mathcal{A}z_N^\lambda(t)|^{\frac12} \\
&\quad + |z_N(t)|^{\frac12} \|z_N(t)\|^{\frac12} |\mathcal{A}(z_N^\lambda(t) - z_N(t))|^{\frac12} \Big).
\end{aligned}$$

Using (56), (40) and (46), we get that

$$|B_N z_N^\lambda(t) - B_N z_N(t)|$$
$$\leq C\|z_N^\lambda(t) - z_N(t)\|^{\frac{1}{2}} \left(|\mathcal{A}z_N^\lambda(t)|^{\frac{1}{2}} + |\mathcal{A}(z_N^\lambda(t) - z_N(t))|^{\frac{1}{2}}\right),\ t \in (0,T).$$

In the case $d = 3$, by (6), we obtain that

$$|(B_N z_N^\lambda(t) - B_N z_N(t), \psi)| \leq C_b \Big[|\psi_1|\|y_N^\lambda(t) - y_N(t)\|^{\frac{1}{2}}$$
$$\cdot\left(\|y_N^\lambda(t) - y_N(t)\|^{\frac{1}{2}}\|y_N^\lambda(t)\|^{\frac{1}{2}}|Ay_N^\lambda(t)|^{\frac{1}{2}} + \|y_N(t)\||A(y_N^\lambda(t) - y_N(t))|^{\frac{1}{2}}\right)$$
$$+|\psi_2|\left(\|y_N^\lambda(t) - y_N(t)\|\|\theta_N^\lambda(t)\|^{\frac{1}{2}}|A_1\theta_N^\lambda(t)|^{\frac{1}{2}} + \|y_N(t)\|\|\theta_N^\lambda(t) - \theta_N(t)\|^{\frac{1}{2}}\right.$$
$$\left.\cdot|A_1(\theta_N^\lambda(t) - \theta_N(t))|^{\frac{1}{2}}\right)\Big] \leq C\|z_N^\lambda(t) - z_N(t)\|^{\frac{1}{2}}\left(\|z_N^\lambda(t) - z_N(t)\|^{\frac{1}{2}}\|z_N(t)\|^{\frac{1}{2}}\right.$$
$$\left.\cdot|\mathcal{A}(z_N^\lambda(t))|^{\frac{1}{2}} + \|z_N^\lambda(t)\||\mathcal{A}(z_N^\lambda(t) - z_N(t))|^{\frac{1}{2}}\right)|\psi|,\ \forall\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \in V_B;$$

hence, using also (48), we infer that

$$|B_N z_N^\lambda(t) - B_N z_N(t)|$$
$$\leq C\|z_N^\lambda(t) - z_N(t)\|^{\frac{1}{2}} \left(|\mathcal{A}z_N^\lambda(t)|^{\frac{1}{2}} + |\mathcal{A}(z_N^\lambda(t) - z_N(t))|^{\frac{1}{2}}\right),\ t \in (0,T_0).$$

So, in both cases $d = 2, 3$, we obtain

$$\int_0^{T_0} |B_N z_N^\lambda(t) - B_N z_N(t)|^2 dt \leq C\left(\int_0^{T_0} \|z_N^\lambda(t) - z_N(t)\|^2 dt\right)^{\frac{1}{2}}$$
$$\cdot\left[\int_0^{T_0}\left(|\mathcal{A}z_N^\lambda(t)|^2 + |\mathcal{A}(z_N^\lambda(t) - z_N(t))|^2\right)dt\right]^{\frac{1}{2}}.$$

Using (46), (48) and (57), we conclude

$$B_N z_N^\lambda \to B_N z_N \text{ in } L^2(0, T_0; H_B).$$

Letting λ tend to zero in (38), we obtain that z_N satisfies problem (17).

3.4. The uniqueness of the solution for the problem (17)**.** We intend to prove that the solution obtained by passing to the limit with $\lambda \searrow 0$ is unique. So, assume that $z_N^l = \begin{pmatrix} y_N^l \\ \theta_N^l \end{pmatrix} \in C([0,T_0];H_B) \cap L^2(0,T_0;D(\mathcal{A})) \cap L^\infty(0,T_0;V_B)$, $l = 1,2$ are two solutions for (17). Then $(z_N^1 - z_N^2)(0) = 0$ and

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|z_N^1(t) - z_N^2(t)|^2 + \nu\|y_N^1(t) - y_N^2(t)\|^2 + k\|\theta_N^1(t) - \theta_N^2(t)\|^2 \\
&\quad -\gamma((\theta_N^1(t) - \theta_N^2(t))e_d, y_N^1(t) - y_N^2(t)) \\
&\quad +(B_N z_N^1(t) - B_N z_N^2(t), z_N^1(t) - z_N^2(t)) \\
&\quad +(\mathcal{A}_0 z_N^1(t) - \mathcal{A}_0 z_N^2(t), z_N^1(t) - z_N^2(t)) \\
&\quad +(\eta_N^1(t) - \eta_N^2(t), z_N^1(t) - z_N^2(t)) = 0, \\
&\quad \eta_N^l(t) \in \Phi(z_N^l(t)),\ l = 1,2.
\end{aligned}$$

We use the monotony of Φ and the estimates

$$\begin{aligned}
&\gamma((\theta_N^1(t) - \theta_N^2(t))e_d, y_N^1(t) - y_N^2(t)) \le \frac{\gamma}{2}|z_N^1(t) - z_N^2(t)|^2; \\
&|(B_N z_N^1(t) - B_N z_N^2(t), z_N^1(t) - z_N^2(t))| \\
&\le |b(y_N^1(t) - y_N^2(t), y_N^1(t), y_N^1(t) - y_N^2(t))| \\
&+|\bar{b}(y_N^1(t) - y_N^2(t), \theta_N^1(t), \theta_N^1(t) - \theta_N^2(t))| \le (\text{by } (6))\ C_b\|y_N^1(t) - y_N^2(t)\| \\
&\cdot \left(\|y_N^1(t)\|^{\frac{1}{2}} |Ay_N^1(t)|^{\frac{1}{2}} |y_N^1(t) - y_N^2(t)| + \|\theta_N^1(t)\|^{\frac{1}{2}} |A_1\theta_N^1(t)|^{\frac{1}{2}} |\theta_N^1(t) - \theta_N^2(t)| \right) \\
&\le C\|z_N^1(t) - z_N^2(t)\| \|z_N^1(t)\|^{\frac{1}{2}} |\mathcal{A}z_N^1(t)|^{\frac{1}{2}} |z_N^1(t) - z_N^2(t)| \\
&\le \frac{m_\nu k}{4}\|z_N^1(t) - z_N^2(t)\|^2 + C|\mathcal{A}z_N^1(t)||z_N^1(t) - z_N^2(t)|^2 \\
&(\text{the solution } z_N^1 \in L^\infty(0,T_0;V_B)); \\
&|(\mathcal{A}_0 z_N^1(t) - \mathcal{A}_0 z_N^2(t), z_N^1(t) - z_N^2(t))| \\
&\le |b(y_N^1(t) - y_N^2(t), y_e, y_N^1(t) - y_N^2(t))| \\
&+|\bar{b}(y_N^1(t) - y_N^2(t), \theta_e, \theta_N^1(t) - \theta_N^2(t))| \le (\text{by } (6))\ C_b\|y_N^1(t) - y_N^2(t)\| \\
&\cdot \left(\|y_e\|^{\frac{1}{2}} |Ay_e|^{\frac{1}{2}} |y_N^1(t) - y_N^2(t)| + \|\theta_e\|^{\frac{1}{2}} |A_1\theta_e|^{\frac{1}{2}} |\theta_N^1(t) - \theta_N^2(t)| \right) \\
&\le C\|z_N^1(t) - z_N^2(t)\| |z_N^1(t) - z_N^2(t)| \le \frac{m_\nu k}{4}\|z_N^1(t) - z_N^2(t)\|^2 \\
&+C|z_N^1(t) - z_N^2(t)|^2
\end{aligned}$$

and we get

$$\frac{1}{2}\frac{d}{dt}|z_N^1(t)-z_N^2(t)|^2+m_{\nu k}\|z_N^1(t)-z_N^2(t)\|^2\leq\frac{m_{\nu k}}{2}\|z_N^1(t)-z_N^2(t)\|^2$$
$$+C|z_N^1(t)-z_N^2(t)|^2(1+|\mathcal{A}z_N^1(t)|),$$

which implies

$$\begin{cases}\dfrac{d}{dt}|z_N^1(t)-z_N^2(t)|^2\leq C(1+|\mathcal{A}z_N^1(t)|)|z_N^1(t)-z_N^2(t)|^2\\(z_N^1-z_N^2)(0)=0.\end{cases}$$

Then $|z_N^1(t)-z_N^2(t)|^2\leq C\int_0^t(1+|\mathcal{A}z_N^1(s)|)|z_N^1(s)-z_N^2(s)|^2ds$ and by Gronwall's inequality,

$$|z_N^1(t)-z_N^2(t)|^2\leq 0\cdot exp\left(C\int_0^t(1+|\mathcal{A}z_N^1(s)|)ds\right)=0,\ t\in[0,T_0]$$

($[0,T_0]$ bounded and $\mathcal{A}z_N^1\in L^2(0,T_0;H_B)\subset L^1(0,T_0;H_B)$).

We infer that $z_N^1(t)=z_N^2(t)$, $t\in[0,T_0]$.

3.5. Proof of Theorem 2.1(the final part). We know that problem (17)

- has a unique solution in $C([0,T_0];H_B)\cap L^2(0,T_0;D(\mathcal{A}))\cap L^\infty(0,T_0;V_B)$, obtained by letting $\lambda\searrow 0$ in problem (38);
- has a solution in $C([0,T];V_B)\cap L^\infty(0,T;D(\mathcal{A}))\cap W^{1,\infty}(0,T;H_B)$ (unique), given by Proposition 3.3.

Thus the two solutions must coincide and the resulting function has the regularity properties given by Proposition 3.3. Moreover, the solution of problem (17) satisfies a.e. on $[0,T_0]$ the estimates

$$|z_N(t)|^2+m_{\nu k}\int_0^t\|z_N(s)\|^2ds\leq C_1, \tag{58}$$

$$\frac{m_{\nu k}}{2}\|z_N(t)\|^2+\epsilon\int_0^t|\mathcal{A}z_N(s)|^2ds\leq C_7, \tag{59}$$

$$\int_0^t |\eta_N(s)|^2 ds \le C_8, \tag{60}$$

where

$$\eta_N(t) = F(t) - \left(\frac{dz_N}{dt}(t) + (\mathcal{A} + \mathcal{R} + B_N + \mathcal{A}_0)z_N(t)\right) \in \Phi(z_N(t)),$$

$$\begin{aligned} \int_0^{T_0} &\left(\left|\frac{dz_N}{dt}(t)\right|^2 + |B_N z_N(t)|^2 + |\mathcal{A}_0 z_N(t)|^2\right) dt \\ &\le C_9(\|F\|^2_{L^2(0,T;H_B)}, \|z_0\|^2, \varphi(z_0)). \end{aligned} \tag{61}$$

The positive bounded functions $C_1,\ C_7,\ C_8,\ C_9$ do not depend on N.

From (59), we infer that

$$\|z_N(t)\|^2 \le \frac{2C_7}{m_{\nu k}},\ t \in [0, T_0].$$

It yields that, for N large enough, $\|z_N(t)\| \le N,\ t \in [0, T_0]$ and, by consequence, $B_N z_N = B z_N$ in $[0, T_0]$. So, $z_N = z$ is a solution (defined on $[0, T_0]$) of the initial problem (16), preserving on $[0, T_0]$ all regularity properties of z_N. The uniqueness comes from the uniqueness of the solution of problem (17). □

Remark 3.1. If Φ is single valued, it is no longer necessary to use approximate problem (38) because hypothesis (12) implies

$$(\mathcal{A}h, \Phi(h)) \ge -\alpha_0(1 + \|h\|^2) - \alpha|\Phi(h)|^2,\ \forall h \in D(\mathcal{A}) \cap D(\Phi). \tag{62}$$

4. Proof of Theorem 2.2. The idea of the proof is to approximate the initial data with sequences of functions satisfying the hypotheses of Theorem 2.1 and then to pass to the limit.

Let $(z_0^j)_{j\in\mathbb{N}} \subset D(\mathcal{A}) \cap D(\Phi)$ and $(F_j)_{j\in\mathbb{N}} \subset W^{1,1}(0, T; H_B)$ such that

$$z_0^j \to z_0 \text{ in } V_B,\quad F_j \to F \text{ in } L^2(0, T; H_B).$$

According to Theorem 2.1, the problem

$$\begin{cases} \dfrac{dz_j(t)}{dt} + (\mathcal{A} + B + \mathcal{A}_0 + \mathcal{R})z_j(t) + \Phi(z_j(t)) \ni f_j(t), \text{ a.e. } t \in (0, T) \\ z_j(0) = z_0^j \end{cases} \tag{63}$$

has an unique solution

$$z_j \in W^{1,\infty}(0,T_0;H_B) \cap L^\infty(0,T_0;D(\mathcal{A})) \cap C([0,T_0];V_B),$$

where $T_0 = T$ if $d = 2$ and $T_0 \le T$ in $d = 3$. Moreover, z_j satisfy the estimates,

$$|z_j(t)|^2 + m_{\nu k}\int_0^t \|z_j(s)\|^2 ds \le C,\ t \in [0,T_0],$$

$$\frac{m_{\nu k}}{2}\|z_j(t)\|^2 + \epsilon\int_0^t |\mathcal{A}z_j(s)|^2 ds \le C,\ t \in [0,T_0],$$

$$\int_0^t |\eta_j(s)|^2 ds \le C,\ t \in [0,T_0],$$

where $\eta_j(t) = F_j(t) - (\frac{dz_j}{dt}(t) + (\mathcal{A} + \mathcal{R} + B + \mathcal{A}_0)z_j(t)) \in \Phi(z_j(t))$,

$$\int_0^{T_0}\left(\left|\frac{dz_j}{dt}(t)\right|^2 + |Bz_j(t)|^2 + |\mathcal{A}_0 z_j(t)|^2\right)dt \le C.$$

The constants are independent of j.

Consequently,

$$(z_j)_j \text{ is bounded in } C([0,T_0];V_B) \cap L^2(0,T_0;D(\mathcal{A})), \tag{64}$$

which implies by $\mathcal{R} \in L(H_B,H_B)$ that

$$(\mathcal{R}z_j)_j \text{ is bounded in } L^\infty(0,T_0;H_B),$$

$$(\mathcal{A}z_j)_j,\ (Bz_j)_j,\ (\mathcal{A}_0 z_j)_j,\ (\eta_j)_j \text{ are bounded in } L^2(0,T_0;H_B),$$

$$\left(\frac{dz_j}{dt}\right)_j \text{ is bounded in } L^2(0,T_0;H_B). \tag{65}$$

(64) and (65) imply that

$$\{z_j; j \in \mathbb{N}\} \text{ is relatively compact in } C([0,T_0];H_B).$$

Then, on a subsequence again denoted by $(z_j)_j$, we have for $j \to \infty$,

$$z_j \to z \text{ in } C([0,T_0];H_B)$$

and the following weak convergences in $L^2(0,T_0;H_B)$,

$$\frac{dz_j}{dt} \rightharpoonup \frac{dz}{dt}, \quad \mathcal{A}z_j \rightharpoonup \mathcal{A}z, \quad \mathcal{R}z_j \rightharpoonup \mathcal{R}z, \quad \mathcal{A}_0 z_j \rightharpoonup \mathcal{A}_0 z,$$

$$Bz_j \rightharpoonup \beta \text{ and } \eta_j \rightharpoonup \eta.$$

Moreover, by Aubin's compactness theorem,

$$z_j \to z \text{ in } L^2(0,T_0;V_B). \tag{66}$$

Using (66) and the maximal monotony of Φ, we get $\eta(t) \in \Phi(z(t))$ a.e. t. Proceeding in the same way as we did in Theorem 2.1 to prove that $\beta_N(t) = B_N z_N(t)$, we deduce also that $\beta(t) = Bz(t)$ a.e. $t \in [0,T_0]$. Passing to the limit with $j \to \infty$, we prove the existence of the strong solution. In order to prove the uniqueness of the solution, we proceed as in the proof of Theorem 2.1, § 3.4. □

5. Feedback stabilization on closed convex sets. Let K be a set verifying the conditions

$$K \subset H_B \text{ is closed and convex, } 0 \in K \tag{h_K}$$

and

$$(I + \lambda\mathcal{A})^{-1}(K) \subset K, \ \forall \lambda > 0. \tag{67}$$

Let us consider the indicator function of the set K,

$$I_K : H_B \to H_B, \ I_K(h) = \begin{cases} 0, & \text{for } h \in K \\ +\infty, & \text{for } h \notin K \end{cases}$$

and its subdifferential,

$$\partial I_K(h) = \begin{cases} \emptyset, & h \notin K \\ \{0\}, & h \in \text{int } K \\ N_K(h) = \{z \in H_B; (z, h - h') \geq 0, \ \forall h' \in K\}, & h \in \partial K \end{cases}$$

($N_K(h)$ is the normal cone at K in h). The Yosida approximation of ∂I_K is

$$(\partial I_K)_\lambda(h) = \frac{1}{\lambda}(h - P_K h), \ \forall \lambda > 0, \ \forall h \in H_B,$$

where $P_K : H_B \to K$ is the orthogonal projection operator on K.

Consider the controlled system

$$\begin{cases} \dfrac{dz}{dt}(t) + \mathcal{A}z(t) + Bz(t) + \mathcal{R}z(t) = F_e + U(t), \quad t > 0 \\ z(0) = z_0, \end{cases} \tag{68}$$

where $z(t) = \begin{pmatrix} y(t) \\ \theta(t) \end{pmatrix}$, $z_0 = \begin{pmatrix} y_0 \\ \theta_0 \end{pmatrix} \in V_B$, $F_e \in H_B$. Let $z_e = \begin{pmatrix} y_e \\ \theta_e \end{pmatrix} \in D(\mathcal{A})$ be a steady state solution for (68), i.e. z_e satisfies

$$\mathcal{A}z_e + Bz_e + \mathcal{R}z_e = F_e. \tag{69}$$

We look for a feedback controller U such that $z(t) - z_e \in K$, $t \geq 0$ and $\lim_{t\to\infty} |z(t) - z_e| = 0$ exponentially.

We set $Z = z - z_e = \begin{pmatrix} y - y_e \\ \theta - \theta_e \end{pmatrix}$. Then (68) implies

$$\begin{cases} \dfrac{dZ}{dt}(t) + \mathcal{A}Z(t) + BZ(t) + \mathcal{A}_0 Z(t) + \mathcal{R}Z(t) = U(t), \quad t > 0 \\ Z(0) = z_0 - z_e, \end{cases}$$

where the operator $\mathcal{A}_0 \in L(V_B, V_B')$ is defined by (13) for z_e given by (69).

The following global stability result holds true in the bidimensional case,

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded set, with a smooth boundary. Let $F_e \in H_B$ and $z_e \in D(\mathcal{A})$ verifying* (69). *Assume that K satisfies the hypothesis (h_K) and that $z_0 - z_e \in \overline{D(\mathcal{A}) \cap K}^{V_B}$. Then, for $\sigma > 0$ large enough, problem*

$$\begin{cases} \dfrac{dZ}{dt}(t) + (\mathcal{A} + B + \mathcal{A}_0 + \mathcal{R})Z(t) + \partial I_K(Z(t)) + \sigma Z(t) \ni 0, \ t > 0 \\ Z(0) = Z_0 = z_0 - z_e, \end{cases} \tag{70}$$

has a unique strong solution $Z \in L^2(0,T; D(\mathcal{A})) \cap L^\infty(0,T; V_B) \cap C([0,T]; H_B)$, with $\frac{dZ}{dt} \in L^2(0,T; H_B)$, for all $T > 0$. In addition, $Z(t) \in K$, $t > 0$ and there exists $\delta = \delta(\sigma) > 0$ such that $\lim_{t\to\infty} |Z(t)| e^{\delta t} = 0$.

Remark 5.1. The above theorem says that there exists a feedback controller $U \in L^2(\mathbb{R}_+; H_B)$, $U = U(Z) = -\sigma Z - \eta$, $\eta \in \partial I_K(Z)$ such that

$$\frac{dZ}{dt}(t) + \mathcal{A}Z(t) + BZ(t) + \mathcal{A}_0 Z(t) + \mathcal{R}Z(t) = U(t), \ t > 0.$$

Actually, from the quasi-m-accretivity of the operator $\mathcal{A} + B_N + \mathcal{A}_0 + \mathcal{R} + \partial I_K + \sigma I$ we may deduce (using [1], Th. 1.6, p. 216) that the solution Z of problem (70) is right differentiable, d^+Z/dt is right continuous and the feedback controller $U(t) = -((\mathcal{A}+B+\mathcal{A}_0+\mathcal{R})Z(t)+\partial I_K(Z(t))+\sigma Z(t))^0 + \mathcal{A}Z(t) + BZ(t) + \mathcal{A}_0 Z(t) + \mathcal{R}Z(t),\ \forall t \in (0,T)$.

Proof of Theorem 5.1. In order to prove the existence and invariance part, we will apply Theorem 2.2. Let $\Phi \subset H_B \times H_B$, $\Phi = \partial I_K$. We know that $0 \in K = D(\partial I_K) = D(\Phi)$. Moreover, relation (67) yields

$$(\Phi_\lambda(W), \mathcal{A}W) \geq 0,\ \forall W \in D(\mathcal{A}),\ \forall \lambda > 0$$

(see [4], Prop. 4.5, i) $\Leftrightarrow$ ii) p. 131). As a result, Φ satisfies hypotheses $(h_1) - (h_3)$, Section 1. Moreover, I_K is bounded on the subsets of $K \cap D(\mathcal{A})$ which are bounded in V_B, because $I_K = 0$ on K and $V_B \subset H_B$ with continuous injection.

Let us fix $\sigma > \gamma/2 + C_b^2\|z_e\|^2/(2m_{\nu k})$. In the proof of Theorem 2.2, we may replace the operator $\mathcal{R}$ by $\mathcal{R}+\sigma I$ (where $I : H_B \to H_B$ is the identity), because the only property of $\mathcal{R}$ that we actually use is its boundedness. We also observe that hypothesis (h_{α_N}) will be no longer needed.

By consequence, for all $T > 0$, (70) has a unique solution

$$Z \in L^2(0,T;D(\mathcal{A}))\cap L^\infty(0,T;V_B)\cap C([0,T];H_B),\ \text{with}\ \frac{dZ}{dt} \in L^2(0,T;H_B),$$

satisfying the invariance property $Z(t) \in K,\ t > 0$.

For the stabilization part, we apply the idea in [2]. Let us multiply scalarly equation (70) by $Z(t)$ and use (3), (5) and the fact that ∂I_K is a monotone operator, with $0 \in \partial I_K(0)$ $(0 \in K)$. Then

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|Z(t)|^2 + \nu\|y(t) - y_e\|^2 + k\|\theta(t) - \theta_e\|^2 + \sigma|Z(t)|^2 \\
&\quad\leq -b(y(t) - y_e, y_e, y(t) - y_e) - \bar{b}(y(t) - y_e, \theta_e, \theta(t) - \theta_e) \\
&\quad+\gamma((\theta(t) - \theta_e)e_d, y(t) - y_e) \leq C_b\Bigg(|y(t) - y_e| \cdot \|y(t)-y_e\|\|y_e\| \\
&\quad+|y(t) - y_e|^{\frac{1}{2}}\|y(t)-y_e\|^{\frac{1}{2}}|\theta(t)-\theta_e|^{\frac{1}{2}}\|\theta(t) - \theta_e\|^{\frac{1}{2}}\|\theta_e\|\Bigg) \\
&\quad+\frac{\gamma}{2}|Z(t)|^2 \leq C_b|Z(t)|\|Z(t)\|\|z_e\| + \frac{\gamma}{2}|Z(t)|^2,
\end{aligned}$$

which yields

$$\frac{1}{2}\frac{d}{dt}|Z(t)|^2+m_{\nu k}\|Z(t)\|^2+\sigma|Z(t)|^2 \le \frac{m_{\nu k}}{2}\|Z(t)\|^2+\left(\frac{C_b^2\|z_e\|^2}{2m_{\nu k}}+\frac{\gamma}{2}\right)|Z(t)|^2.$$

We deduce

$$\frac{d}{dt}|Z(t)|^2 \le -2\left(\sigma-\frac{C_b^2\|z_e\|^2}{2m_{\nu k}}-\frac{\gamma}{2}\right)|Z(t)|^2, \text{ a.e. } t\ge 0.$$

Hence $|Z(t)|^2 \le e^{-2\delta_1 t}|z_0-z_e|^2$, a.e. $t\ge 0$, where, from the choice of σ, $\delta_1=\sigma-\frac{2C_b^2\|z_e\|^2}{m_{\nu k}}-\frac{\gamma}{2}$ is strictly positive. Consequently, there exists $\delta\in(0,\delta_1)$ such that $\lim_{t\to\infty}|Z(t)|e^{\delta t}=0$. □

5.1. Particular cases

5.1.1. Stability on finite dimensional sets. Let $e_1,e_2,\dots,e_r\in H_B$, with $r\in\mathbb{N}^*$, be eigenfunctions for the operator $\mathcal{A}$ and let $K=\text{linspan}\ \{e_i;\ i=\overline{1,r}\}\subset H_B$. Then the property (h_K) is satisfied. Consequently, the stabilization Theorem 5.1 holds.

5.1.2. Stability on bounded sets in V_B. Let the set $K\subset H_B$ fulfill hypothesis (h_K) and suppose in addition that K is bounded in V_B. Applying repeatedly Theorem 2.2, it results that problem (70) admits even in the tridimensional case a *global* strong solution, and we may write directly the stability argument.

Remark 5.2. In order to give a stability result we need to prove the *global* existence for the solution of the involved problem. But in the case $d=3$ the strong solution is in general only *local*. That is why we have to start from an existence result for weak solutions, which will be considered in a further paper.

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Received: 14.X.2008

Faculty of Mathematics,
University "Al.I. Cuza",
11, Bd. Carol I, 700506, Iaşi,
ROMANIA
and
Institute of Mathematics "Octav Mayer"
Romanian Academy, Iaşi Branch,
ROMANIA
ilefter@uaic.ro