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CONCERNING TOPOLOGIES INDUCED BY PRINCIPAL GRILLS

BY

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Abstract. Given a grill $\mathcal{G}$ on a topological space (X, τ), we consider here a new topology $\tau_{\mathcal{G}}$ on X, induced by the operator $\Psi : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$, given by $\Psi(A) = A \cup \Phi(A)$ (for $A \in \mathcal{P}(X)$), where $\Phi(A) = \{x \in X : U \cap A \in \mathcal{G}$, for all open neighbourhoods U of $x\}$. In an earlier paper, we verified that Ψ is indeed a Kuratowski's closure operator and investigated therein some properties of this topology $\tau_{\mathcal{G}}$. In the present paper, we continue the study and it is shown that some interesting properties and behaviours of this topology $\tau_{\mathcal{G}}$ are encountered if $\mathcal{G}$ belongs to a particular class of grills, introduced here and termed as the principal grills. Certain separation axioms and some well known covering properties are investigated in this article, as to the simultaneous sharing of these properties by this topology $\tau_{\mathcal{G}}$ (with $\mathcal{G}$ a principal grill) and the preassigned topology on a set.

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1. Introduction. In an earlier paper [8] we introduced a typical topology $\tau_{\mathcal{G}}$ on the underlying set X of a topological space (X, τ), constructed in a rather natural way and associated with a given grill $\mathcal{G}$ (see [12]) for grill) on X. We also found therein some interesting features of this type of topologies. Our purpose in this paper is to continue the investigation, but this time with a particular type of grills, called principal grills. We shall see that the proposed topology $\tau_{\mathcal{G}}$ on a space X, where $\mathcal{G}$ is a principal grill on X, has some pleasing properties, e.g. certain covering properties are found

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to be simultaneously shared by this topology and the given topology on a set.

In what follows in this section we recall some definitions and results which will be used in course of the deliberations in the next two sections. We begin with the definition of a grill, as first proposed by Choquet [2].

Definition 1.1. *A nonempty collection $\mathcal{G}$ of nonempty sets in a topological space X is called a grill on X if (i) $A \in \mathcal{G}$ and $A \subseteq B \subseteq X \Rightarrow B \in \mathcal{G}$, and (ii) $A \cup B \in \mathcal{G}$ $(A, B \subseteq X) \Rightarrow A \in \mathcal{G}$ or $B \in \mathcal{G}$.*

Throughout the paper by a space X we shall denote a topological space (X, τ). For a subset A of X, the closure and interior of A will be denoted by cl$A$ and int$A$ respectively. The notations $\mathcal{P}(X)$ and $\tau(x)$ shall stand for the power set of X and the set of all open neighbourhoods of x $(\in X)$ respectively. As already said, a new kind of topology $\tau_{\mathcal{G}}$, induced by the topology τ and a grill $\mathcal{G}$ on a set X, was initiated in [8]. We give below a brief description of such a topology.

Definition 1.2. [8] *Let $\mathcal{G}$ be a grill on a topological space (X, τ). Consider the mapping $\Phi : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ given by $\Phi(A) = \{x \in X : U \cap A \in \mathcal{G}$, for all $U \in \tau(x)\}$ for $A \subseteq X$. Then the operator $\Psi : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$, defined by $\Psi(A) = A \cup \Phi(A)$ (for $A \in \mathcal{P}(X)$) is a Kuratowski's closure operator and hence gives rise to a topology on X, to be denoted by $\tau_{\mathcal{G}}$.*

We shall need the following results frequently in the sequel.

Theorem 1.3. [8] *Let (X, τ) be a topological space.*
(a) If $\mathcal{G}$ is a grill on X, then $\tau \subseteq \tau_{\mathcal{G}}$.
(b) If $\mathcal{G}_1, \mathcal{G}_2$ be two grills on X such that $\mathcal{G}_1 \subseteq \mathcal{G}_2$, then $\tau_{\mathcal{G}_2} \subseteq \tau_{\mathcal{G}_1}$.

2. Principal grills and the induced topologies. Let us begin with the definition of the special type of grills that we shall be concerned with.

Definition 2.1. *Let X be a nonempty set and $(\emptyset \neq)$ $A \subseteq X$. Then $[A] = \{B \subseteq X : A \cap B \neq \emptyset\}$ is a grill (easily verifiable) on X. We call this grill the principal grill on X generated by A.*

From now on we shall use $[A]$ to stand for the principal grill on a space X generated by $A \neq \emptyset$. We shall most of the time use the principal grill $[X \setminus A]$ on X, and in such a case it will naturally be assumed that $A \subsetneq X$.

Remark 2.2. If $A = X$, then it is easy to see that $[A] = \mathcal{P}(X)\setminus\{\emptyset\}$ and $\tau_{[A]} = \tau$, and this is surely not an interesting case. Thus while considering the principal grill $[X \setminus A]$ on X, we generally assume, to avoid triviality, that $A \neq \emptyset$.

We see that if $A \subseteq B \subseteq X$, then $[A] \subseteq [B]$, and hence by Theorem 1.3(b) we at once have:

Corollary 2.3. *Let A and B be two nonempty subsets of a space X with $A \subseteq B$. Then $\tau_{[B]} \subseteq \tau_{[A]}$.*

That the converse of the above corollary (hence that of Theorem 1.3(b)) is false, follows from the example below.

Example 2.4. Consider the topological space (X, τ), where $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$. Taking $A = \{a, c\}$, we find that $[A] = \{\{a\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$, $\Psi(\{a\}) = \{a\}$, $\Psi(\{b\}) = \{b\}$, $\Psi(\{c\}) = \Psi(\{b, c\}) = \{b, c\}$, $\Psi(\{a, b\}) = \{a, b\}$, $\Psi(\{a, c\}) = X$, so that $\tau_{[A]} = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$. Also, for $B = \{b, c\}$ we have on calculation, $[B] = \{\{b\}, \{c\}, \{b, c\}, \{a, c\}, \{a, b\}, X\}$, and $\Psi(\{a\}) = \{a\}$, $\Psi(\{b\}) = \Psi(\{c\}) = \Psi(\{b, c\}) = \{b, c\}$ and $\Psi(\{a, b\}) = \Psi(\{a, c\}) = X$, so that $\tau_{[B]} = \{\emptyset, X, \{a\}, \{b, c\}\}$. Thus here $\tau_{[B]} \subseteq \tau_{[A]}$ but $A \not\subseteq B$.

However, we have

Theorem 2.5. *Let A be a nonempty subset of a topological space (X, τ) such that $\tau = \tau_{[A]}$. Then for any $B \supseteq A$, $\tau_{[B]} = \tau_{[A]}$.*

Proof. By using Theorem 1.3(a) and Corollary 2.3 we have, $\tau \subseteq \tau_{[B]} \subseteq \tau_{[A]} \subseteq \tau$. □

In [8] we found the following convenient expression for the topology $\tau_{\mathcal{G}}$ (where $\mathcal{G}$ is any grill on the space (X, τ)) under a specific condition.

Theorem 2.6.[8] *Let $\mathcal{G}$ be a grill on a space X with the condition that $(A \cap \Phi(A) = \emptyset, A \subseteq X \Rightarrow A \notin \mathcal{G})$. Then $\tau_{\mathcal{G}} = \{V \setminus B : V \in \tau \text{ and } B \notin \mathcal{G}\}$.*

We now see that in case of a principal grill, the condition of the above theorem is automatically satisfied.

Theorem 2.7. *For any proper subset A of a space (X, τ), $\tau_{[X\setminus A]} = \{V \setminus B : V \in \tau \text{ and } B \subseteq A\}$.*

Proof. It suffices to show, in view of Theorem 2.6, that for any $B \subseteq X$, $B \cap \Phi(B) = \emptyset \Rightarrow B \subseteq A$.

If possible, let $B \nsubseteq A$. Then there exists $x \in B$ such that $x \notin A$. Since $B \cap \Phi(A) = \emptyset$, $x \notin \Phi(B)$. Hence there exists some $U_x \in \tau(x)$ such that $U_x \cap B \notin [X \setminus A]$ i.e., $U_x \cap B \subseteq A$. This is a contradiction, as $x \in U_x \cap B$ but $x \notin A$. □

Theorem 2.8. *Let A be a proper subset of a space (X, τ) such that (A, τ_A) is a discrete space. Then $\tau = \tau_{[X\setminus A]}$.*

Proof. Let $G \in \tau_{[X\setminus A]}$. Then by the above theorem, $G = V \setminus B$, where $V \in \tau$ and $B \subseteq A$. Since A is closed in (X, τ), B is closed in (X, τ). So $V \setminus B (= G)$ is open in (X, τ). Now, in view of Theorem 1.3(a) the result follows. □

Coming to subspace topologies, we observe that for any two nonempty subsets A and B of a space X, $(\tau)_A \subseteq (\tau_{[B]})_A$ (as $\tau \subseteq \tau_{[B]}$), where $(\tau)_A$ and $(\tau_{[B]})_A$ denote the respective subspace topologies on A. But the following example shows that this inclusion cannot, in general, be reversed.

Example 2.9. We consider the subsets $A = \{b, c\}$ and $B = \{a, c\}$ of the topological space (X, τ), where $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$. Then $\tau_{[B]} = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$ (see Example 2.4) and hence $(\tau_{[B]})_A = \{\emptyset, \{c\}, A\}$, whereas $(\tau)_A = \{\emptyset, A\}$.

Nevertheless, if $B = A$ then we have the desired equality.

Theorem 2.10. *For any nonempty set A in a space (X, τ), $(\tau)_A = (\tau_{[A]})_A$.*

Proof. It is only to be proved that $(\tau_{[A]})_A \subseteq \tau_A$. We have, $U \in (\tau_{[A]})_A \Rightarrow U = V \cap A$, where $V \in \tau_{[A]} \Rightarrow V = G \setminus B$ where $G \in \tau$ and $B \subseteq X \setminus A$ (by Theorem 2.7), and $U = (G \setminus B) \cap A = G \cap (X \setminus B) \cap A = G \cap A \Rightarrow U \in (\tau)_A$.

We now wish to look for suitable conditions under which the closure of a subset with respect to the given topology and that induced by a principal grill, may coincide. This will ultimately help us to ascertain the simultaneous sharing of certain covering properties by these two topologies. In this connection, we first observe, in view of Theorem 1.3(b), that

Observation 2.11. For subsets A, B of a space (X, τ) with $A \neq X$, we have

(a) $d_{\tau_{[X \setminus A]}}(B) \subseteq d_\tau(B)$ and

(b) $b_{\tau_{[X \setminus A]}}(B) \subseteq b_\tau(B)$, where $d_\tau(B)$ and $b_\tau(B)$ respectively denote the derived set and the boundary set of B in (X, τ) (similar meaning for other notations).

In what follows we shall sometimes impose an additional condition on a subset A of X, to be referred to as 'Condition $(*)$' which is given as follows. "A is such a subset of a space (X, τ) that $A \cap U = \emptyset$, for all $U \in \tau \setminus \{X\}$".

Theorem 2.12. *Let A be a proper subset of a space (X, τ) satisfying Condition $(*)$. Then τ and $\tau_{[X \setminus A]}$ have identical dense subsets.*

Proof. As $\tau \subseteq \tau_{[X \setminus A]}$, each $\tau_{[X \setminus A]}$-dense set is clearly τ-dense. For the converse, let D be a τ-dense subset of X. To show D to be $\tau_{[X \setminus A]}$-dense, let $G(\neq \emptyset) \in \tau_{[X \setminus A]}$. Then by Theorem 2.7, $G = U \setminus B$, where $U \in \tau$ and $B \subseteq A$.

Case 1. If $U = X$, then $G \cap D = (X \setminus B) \cap D = D \setminus B$. We claim that $D \setminus B \neq \emptyset$. For, if not, then $D \subseteq B \subseteq A$ so that $V \cap D = \emptyset$, for all $V \in \tau \setminus \{X\}$, a contradiction (as D is dense in (X, τ).

Case 2. If $U \neq X$, then $G \cap D = (U \setminus B) \cap D = U \cap D \cap (X \setminus B) = (U \cap D) \setminus (U \cap D \cap B) = U \cap D$ (as $U \cap B \subseteq U \cap A = \emptyset$, for all $U \in \tau \setminus \{X\}$) $\neq \emptyset$. □

Theorem 2.13. *Let A be a proper subset of a space (X, τ) satisfying Condition $(*)$. Then τ-cl$G = \tau_{[X \setminus A]}$-cl$G$, for all $G \in \tau_{[X \setminus A]}$.*

Proof. Since $\tau \subseteq \tau_{[X \setminus A]}$, we have $\tau_{[X \setminus A]}$-cl$G \subseteq \tau$-cl$G$ for any $G \subseteq X$. Suppose $x \in \tau$-cl$G$ and $U \in \tau_{[X \setminus A]}$ be such that $x \in U$. It is only to be shown that $G \cap U \neq \emptyset$. Now, $U, G \in \tau_{[X \setminus A]} \Rightarrow U = V \setminus B$ and $G = V' \setminus B'$, where $V, V' \in \tau$ and $B \subseteq A$, $B' \subseteq A$.

Case 1. If $V' = X$, then $G \cap U = (X \setminus B') \cap U = U \setminus B'$, and we claim that $U \setminus B' \neq \emptyset$. For otherwise, $U \subseteq B' \subseteq A \Rightarrow U \cap A \neq \emptyset$, a contradiction to the hypothesis. Thus $G \cap U \neq \emptyset$.

Case 2. If $V' = X$, then $G \cap B \subseteq V' \cap B \subseteq V' \cap A = \emptyset$ (by hypothesis), i.e., $G \cap B = \emptyset$. Now, $G \cap U = G \cap (V \setminus B) = (G \cap V) \setminus (G \cap B) = G \cap V$ (as $G \cap B = \emptyset$) $\neq \emptyset$ (as $x \in \tau$-cl$G$).

In the next theorem we show that for subsets B of $(X \setminus A)$, the assumed conditions of Theorem 2.13 can be disposed of. We note incidentally that each subset B of A is $\tau_{[X \setminus A]}$-closed; in fact, $B \subseteq A \Rightarrow B \notin [X \setminus A] \Rightarrow \Phi(B) \subseteq B \Rightarrow B$ is $\tau_{[X \setminus A]}$-closed. □

Theorem 2.14. *Let A be a proper subset of a space (X, τ). Then for any subset B of $X \setminus A$, τ-cl$B$ = $\tau_{[X \setminus A]}$-cl$B$.*

Proof. Since $\tau \subseteq \tau_{[X\setminus A]}$, $\tau_{[X\setminus A]}$-cl$B \subseteq \tau$-cl$B$. Next let $x \in \tau$-cl$B$ but $x \notin \tau_{[X\setminus A]}$-cl$B$. Then $G \cap B = \emptyset$, for some $G \in \tau_{[X\setminus A]}$ with $x \in G$. Now, $G = V \setminus D$, where $V \in \tau$ and $D \subseteq A$. Clearly, $x \in V$ and so $V \cap B \neq \emptyset$ (as $x \in \tau$-cl$B$). Again, since $D \subseteq A$ and $B \subseteq X \setminus A$, we have $B \cap D = \emptyset$ and hence $(V \setminus D) \cap B \neq \emptyset$, i.e., $G \cap B \neq \emptyset$, a contradiction. □

We recall that a set A in a space X is called regular open if $A =$ intcl$A$, and the complements of regular open sets are called regular closed sets [5]. The set of regular open (regular closed) subsets of a space (X, τ) will be denoted by $RO(X, \tau)$ (resp. $RC(X, \tau)$). It is well known [7] that

Result 2.15. If τ and $\tau^{'}$ be topologies on a set X with $\tau \subseteq \tau^{'}$. Then $RO(X, \tau) = RO(X, \tau^{'})$ iff τ-cl$G = \tau'$-cl$G$, for all $G \in \tau'$.

It then follows by virtue of Theorem 2.13 and Result 2.15 that

Theorem 2.16. *If A is any proper subset of a space (X, τ), satisfying Condition $(*)$, then*

(a) $RO(X, \tau) = RO(X, \tau_{[X\setminus A]})$ and

(b) $RC(X, \tau) = RC(X, \tau_{[X\setminus A]})$.

We now show that the Condition $(*)$ in the above theorem cannot be dropped.

Example 2.17. We consider the subset $A = \{b, c\}$ of the topological space (X, τ), where $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Then $[X \setminus A] = [\{a\}] = \{\{a\}, \{a, b\}, \{a, c\}, X\}$. It is then a routine work to check that $\tau_{[X\setminus A]} = \{X, \emptyset, \{a\}, \{b\}, \{a, c\}, \{a, b\}\}$, $RO(X, \tau) = \{X, \emptyset, \{a\}, \{b\}\}$ and $RO(X, \tau_{[X\setminus A]}) = \{\{b\}, \{a, c\}, X, \emptyset\}$. Here A does not satisfy the Condition $(*)$ and we have $RO(X, \tau) \neq RO(X, \tau_{[X\setminus A]})$.

Mashhour et al. [6] defined a set A in a space X to be preopen (preclosed) if $A \subseteq$intcl$A$ (resp.clint$A \subseteq A$). For any set $A (\subseteq X)$, the intersection of all preclosed sets containing A is called the preclosure of A [6], denoted by pcl$A$. It is known that A is preclosed iff $(X \setminus A)$ is preopen iff $A =$ pcl$A$. We shall denote by $PO(X, \tau)$ the set of all preopen sets in a space (X, τ). It is proved in [3] that a set A in a space (X, τ) is preopen iff $A = W \cap D$, for some regular open set W and a dense subset D of X. It then follows in virtue of Theorems 2.12 and 2.16 that

Theorem 2.18. *Let A be a proper subset of a space (X, τ) satisfying Condition $(*)$. Then $PO(X, \tau) = PO(X, \tau_{[X \setminus A]})$.*

The definitions of some well known covering properties are now recalled.

Definition 2.19. *A topological space (X, τ) is called*

(*a*) *nearly compact* [10] (*S-closed* [11], *strongly compact* [4]) *if every cover of X by regular open (resp. regular closed, preopen) sets has a finite subcover;*

(*b*) *p-closed* [1] *if for every cover $\{V_\alpha : \alpha \in \Lambda\}$ of X by preopen sets V_α of X, there exists a finite subset Λ_0 of Λ such that $X = \bigcup_{\alpha \in \Lambda_0} pclU_\alpha$.*

We are now equipped enough to obtain the desired results.

Theorem 2.20. *Let A be any proper subset of a topological space (X, τ), satisfying Condition $(*)$. Then*

(*a*) *(X, τ) is nearly compact iff $(X, \tau_{[X \setminus A]})$ is nearly compact,*

(*b*) *(X, τ) is S-closed iff $(X, \tau_{[X \setminus A]})$ is S-closed,*

(*c*) *(X, τ) is strongly compact iff $(X, \tau_{[X \setminus A]})$ is strongly compact,*

(*d*) *(X, τ) is p-closed iff $(X, \tau_{[X \setminus A]})$ is p-closed.*

Proof. (a) and (b) follow from Theorem 2.16, while (c) and (d) follow from Theorem 2.18. □

3. Separation axioms and principal grill topology. In this section we shall take up some well known separation [5] axioms and would like to ascertain whether these are possessed by the topology $(X, \tau_{[X \setminus A]})$ whenever the original space (X, τ) possesses them and vice versa.

To start with we observe that T_0, T_1 or T_2 separation axioms are expansive properties, and hence

Theorem 3.1. *For any proper subset A of a topological space (X, τ), $(X, \tau_{[X \setminus A]})$ is T_0 or T_1 or T_2 according as (X, τ) has the respective property.*

That the converses of the above results are false, is shown by the following example.

Example 3.2. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$. Then (X, τ) is a topological space which is not even T_0. Let $A = \{b, c\}$. Then $[X \setminus A] = \{\{a\}, \{a, b\}, \{a, c\}, X\}$. One can check that $\Psi(B) = B$ for any $B \subseteq X$, and hence $\tau_{[X \setminus A]}$ is the discrete topology on X so that $(X, \tau_{[X \setminus A]})$ is a T_5-space.

Definition 3.3. [9] *A space X is said to be almost regular if for any regular closed set A and any point $x \in X \setminus A$, there exist disjoint open sets strongly separating A and x.*

Theorem 3.4. *Let A be a proper subset of an almost regular space (X, τ) such that A satisfies Condition $(*)$. Then $(X, \tau_{[X \setminus A]})$ is almost regular.*

Proof. The proof follows from Theorem 2.16(b) and the fact that $\tau \subseteq \tau_{[X \setminus A]}$. □

For the cases of regularity and normality, we need the following result.

Lemma 3.5. *Let $A(\neq \emptyset, X) \subseteq X$ be open in (X, τ) and $B \subseteq A$. Then $B \in \tau_{[X \setminus A]}$.*

Proof. We need to show that $(X \setminus B)$ is closed in $(X, \tau_{[X \setminus A]})$ i.e., to show that $\Phi(X \setminus B) \subseteq (X \setminus B)$. Now, $x \in \Phi(X \setminus B) \Rightarrow U_x \cap (X \setminus B) \in [X \setminus A]$, i.e., $U_x \cap (X \setminus B) \cap (X \setminus A) \neq \emptyset$, for all $U_x \in \tau(x) \Rightarrow U_x \cap (X \setminus A) \neq \emptyset$ (as $X \setminus B \supseteq X \setminus A$), for all $U_x \in \tau(x) \Rightarrow x \in \tau\text{-cl}(X \setminus A) = X \setminus A$ (as $A \in \tau$) $\Rightarrow x \in X \setminus B$. □

Theorem 3.6. *Let (X, τ) be a regular space and A be a proper open subset of X. Then $(X, \tau_{[X \setminus A]})$ is regular.*

Proof. Since A is a proper open subset of X, we have by Lemma 3.5 that every subset of A is $\tau_{[X \setminus A]}$ open. Let F be any nonempty closed subset of $(X, \tau_{[X \setminus A]})$ and $x \in X \setminus F$. Then $X \setminus F = U \setminus B$, for some $U \in \tau$ and $B \subseteq A$. Thus $F = (X \setminus U) \cup B$ and so, $x \notin (X \setminus U)$ and $x \notin B$. Since (X, τ) is regular and $x \notin (X \setminus U)$, there exist disjoint τ-open sets G and H such that $x \in G$ and $(X \setminus U) \subseteq H$.

If $x \notin A$, then $(G \setminus A)$ and $(H \cup B)$ are disjoint $\tau_{[X \setminus A]}$-open sets such that $x \in (G \setminus A)$ and $F \subseteq (H \cup B)$.

If $x \in A$, then $x \in (G \setminus B)$, $F \subseteq (H \cup B)$, where $(G \setminus B)$ and $(H \cup B)$ are disjoint $\tau_{[X \setminus A]}$-open sets. Hence $(X, \tau_{[X \setminus A]})$ is regular. □

Theorem 3.7. *Let A be a proper open subset of a normal space (X, τ). Then $(X, \tau_{[X \setminus A]})$ is normal.*

Proof. Let F_1 and F_2 be two disjoint closed subsets of $(X, \tau_{[X \setminus A]})$. Then $X \setminus F_1 = U_1 \setminus B_1$ and $X \setminus F_2 = U_2 \setminus B_2$, where $U_1, U_2 \in \tau$ and $B_1, B_2 \subseteq A$.

Thus $F_1 = (X\setminus U_1)\cup B_1$ and $F_2 = (X\setminus U_2)\cup B_2$. As $(X\setminus U_1)$ and $(X\setminus U_2)$ are two disjoint closed subsets of the normal space (X, τ), there exist disjoint τ-open sets G and H such that $(X \setminus U_1) \subseteq G$ and $(X \setminus U_2) \subseteq H$. Then $(X \setminus U_1) \subseteq G \setminus B_2$ (as $(X \setminus U_1) \cap B_2 = \emptyset$) and $(X \setminus U_2) \subseteq (H \setminus B_1)$ (as $(X\setminus U_2)\cap B_1 = \emptyset$). Also, $G, H \in \tau$ and $B_1, B_2 \subseteq A \Rightarrow G\setminus B_2, H\setminus B_1 \in \tau_{[X\setminus A]}$. Then by Lemma 3.5, $(G\setminus B_2)\cup B_1, (H\setminus B_1)\cup B_2 \in \tau_{[X\setminus A]}$ and clearly they are disjoint (as $B_1 \cap B_2 = \emptyset$). Also, $F_1 \subseteq (G\setminus B_2)\cup B_1$, and $F_2 \subseteq (H\setminus B_1)\cup B_2$. Hence $(X, \tau_{[X\setminus A]})$ is normal. □

Remark 3.8. The converses of Theorems 3.6 and 3.7 are false, and moreover, the openness of A in these theorems cannot be suppressed. These assertions are apparent from the examples that follow.

Examples 3.9 Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Then (X, τ) is not a regular space. Put $A = \{a, b\}$. Then $[X \setminus A] = \{\{c\}, \{a, c\}, \{b, c\}, X\}$. It is easy to check that $\tau_{[X\setminus A]}$ is the discrete topology on X and $(X, \tau_{[X\setminus A]})$ is regular.

Example 3.10. Consider the topological space (X, τ), where $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$. We take $A = \{b\}$, which is not open in (X, τ). Then $\tau_{[X\setminus A]} = \{\emptyset, \{a\}, \{c\}, \{b, c\}, \{a, c\}, X\}$ (see Example 2.4). Clearly (X, τ) is regular but $(X, \tau_{[X\setminus A]})$ is not.

Example 3.11. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$ and $A = \{a, b\}$. Then a routine check gives that (X, τ) is normal, $[X \setminus A] = \{\{c\}, \{a, c\}, \{b, c\}, X\}$ and $\tau_{[X\setminus A]}$ is the discrete topology on X, and hence $(X, \tau_{[X\setminus A]})$ is a normal space.

Example 3.12. Let us take $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$. Then (X, τ) is a normal space. If we put $A = \{b\}$, then A is not open in (X, τ) and $[X\setminus A] = \{\{a\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}, X\}$. A straight away verification gives $\Psi(\{a\}) = \Psi(\{a, b\}) = \Psi(\{a, c\}) = X, \Psi(\{b\}) = \{b\}, \Psi(\{c\}) = \{c\}$ and $\Psi(\{b, c\}) = \{b, c\}$. Thus $\tau_{[X\setminus A]} = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$ and $(X, \tau_{[X\setminus A]})$ is not normal.

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