ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" DIN IAŞI (S.N.)
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ON THE HOMOTOPY FUNCTOR OF AN I-CATEGORY

BY

I. POP

Abstract. For an I-category $(\mathbf{C}, \mathbf{cof}, \boldsymbol{I}, \emptyset)$ we consider the homotopy category $h\mathbf{C}$ and the homotopy functor $\hbar : \mathbf{C} \to h\mathbf{C}$. The purpose of this paper is to study the reflection by $\hbar$ of the I-category properties of category $\mathbf{C}$ on the category $h\mathbf{C}$. In order to obtain some interesting results, we introduce a particular class of I-categories, namely, those satisfying the so-called homotopy bisection hypothesis (HBH).

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Key words: I-category, cofibration category, homotopy category, weak push-out diagram, homotopy bisection hypothesis (HBH), almost I-category.

1. Introduction. In this paper we consider the homotopy category $h\mathbf{C}$ of an I-category $\mathbf{C}$ and study this category from the point of view of an I-category. For the necessity of this study we consider a particular class of I-categories satisfying the so-called homotopy bisection hypothesis (HBH).

At first, we recall the necessary notions from the books of Baues [1] and of Baues and Quintero [3].

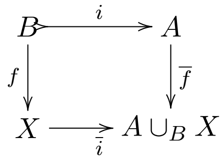
Definition 1.1. An *I-category* is a category $\mathbf{C}$ with the structure $(\mathbf{C}, \boldsymbol{cof}, \boldsymbol{I}, \emptyset)$. Here, $\boldsymbol{cof}$ is a class of morphisms in $\mathbf{C}$, called *cofibrations*. $\boldsymbol{I}$ is a *cylinder functor* and $\emptyset$ is an *initial object* of $\boldsymbol{C}$. The structure satisfies the following axioms (I1)-(I5).

(I1) Cylinder axiom. The cylinder functor $\boldsymbol{I}$ is a covariant functor $\boldsymbol{I} : \mathbf{C} \to \mathbf{C}$ together with natural transformations

$$i_0, i_1 : id_{\mathbf{C}} \to \boldsymbol{I} \text{ and } p : \boldsymbol{I} \to id_{\mathbf{C}}$$

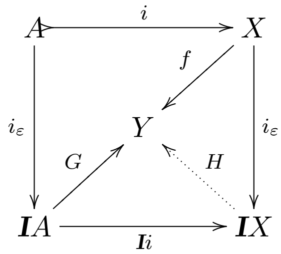
such that for all objects X, the composite $pi_\varepsilon : X \to \boldsymbol{I}X \to X$, with $\varepsilon = 0, 1$, is the identity of X. The transformations i_0, i_1, and p are called the structure maps of the cylinder. Moreover, $\boldsymbol{I}\emptyset = \emptyset$.

(I2) Push-out axiom. For a cofibration $i : B \rightarrowtail A$ and a morphism $f : B \to X$, the push-out



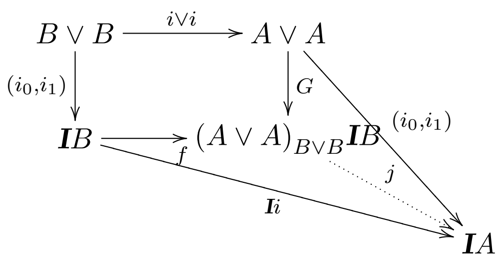
exists and the induced morphism $\bar{i}$ is a cofibration. Moreover, the functor $\boldsymbol{I}$ carries such a push-out diagram to a push-out diagram, i.e., $\boldsymbol{I}(A \cup_B X) = \boldsymbol{I}A \cup_{\boldsymbol{I}B} \boldsymbol{I}X$.

(I3)Cofibration axiom. Each isomorphism in $\mathbf{C}$ is a cofibration and for each object X, the morphism $\emptyset \to X$ is a cofibration. We thus have, by (I2), the *sum* $X \cup_\emptyset Y =: X \vee Y$. A composite of cofibrations is a cofibration. Moreover, a cofibration $i : A \rightarrowtail X$ has the following *homotopy extension property*(HEP) in $\mathbf{C}$. Let $\varepsilon \in \{0, 1\}$. For each commutative diagram in $\mathbf{C}$, with continuous arrows,



i.e., $G \circ i_\varepsilon = f \circ i$, there is an $H : \boldsymbol{I}X \to Y$ with $H(\boldsymbol{I}i) = G$ and $Hi_\varepsilon = f$.

(I4)Relative cylinder axiom. For a cofibration $i : B \rightarrowtail A$, the map j defined by the following push-out diagram is a cofibration



Equivalently, $(IB, B \vee B) \to (IA, A \vee A)$ is a cofibration in **Pair(C)**.

(I5)Interchange axiom. For the double cylinder $\boldsymbol{II}X$ of any object X, there is a morphism $T : \boldsymbol{II}X \to \boldsymbol{II}X$ in $\mathbf{C}$ with $T \circ i_\varepsilon(\boldsymbol{I}X) = \boldsymbol{I}(i_\varepsilon(X))$ and $T \circ \boldsymbol{I}(i_\varepsilon(X)) = i_\varepsilon(\boldsymbol{I}X)$ for $\varepsilon = 0$ and $\varepsilon = 1$. The morphism T is called an *interchange morphism.* The interchange morphism is called natural if for all $f : X \to Y$ in $\mathbf{C}$ we have $T(\boldsymbol{II}f) = (\boldsymbol{II}f)T$.

It is proved that for an object A in an I-category $\mathbf{C}$ the following morphisms are cofibrations: $i_\varepsilon : A \rightarrowtail \boldsymbol{I}A$, $i'_\varepsilon : A \rightarrowtail A \vee A$. If $i : B \rightarrowtail A$ is a cofibration in $\mathbf{C}$, then the morphisms $i \vee 1_A$, $i \vee i$, $\overline{i \vee i}$ and $\boldsymbol{I}i : \boldsymbol{I}B \rightarrowtail \boldsymbol{I}B$ are also cofibrations.

Important examples of I-categories are: the category **Top** of the topological spaces, the category **Topp** of compact maps, the category **End** of ended spaces, and the category $(\mathbf{Chain}^+_R)_c$ of free chain complexes of R-modules, which are bounded below, and chain maps ([1], p. 42).

If $\mathbf{C}$ is an I-category, then the category **Pair(C)** of pairs in $\mathbf{C}$ can be also organized as an I-category. The objects (X, X_0) of the category **Pair(C)** are morphisms $i_X : X_0 \to X$ in $\mathbf{C}$ and the morphisms $(f, f_0) : (A, A_0) \to (X, X_0)$ are the commutative diagrams

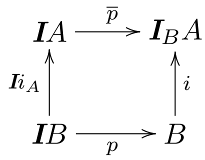
$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \uparrow{\scriptstyle i_A} & & \uparrow{\scriptstyle i_X} \\ A_0 & \xrightarrow[f_0]{} & X_0 \end{array}$$

This morphism is a cofibration in **Pair(C)** if f_0 and $(f, i_X) : A \cup_{A_0} X_0 \to X$ are cofibrations in $\mathbf{C}$. The cylinder functor on **Pair(C)** is defined by $\boldsymbol{I}(X, X_0) = (\boldsymbol{I}X, \boldsymbol{I}X_0)$ with $i_{\boldsymbol{I}X} = \boldsymbol{I}(i_X)$. If $\emptyset$ is the initial object in $\mathbf{C}$, then $(\emptyset, \emptyset)$ is the initial object in **Pair(C)**.

If $\mathbf{C}$ is an I-category, consider the category $\mathbf{C}^B$ of the objects and morphisms under B. A morphism

$$\begin{array}{ccc} & B & \\ \swarrow & & \searrow \\ X & \xrightarrow[f]{} & Y \end{array}$$

is a cofibration in $\mathbf{C}^B$ if f is a cofibration in $\mathbf{C}$. An object $B \to X$ in $\mathbf{C}^B$ is called *cofibrant* if the morphism $B \to X$ is a cofibration in $\mathbf{C}$. The full subcategory of the category $\mathbf{C}^B$ of cofibrant objects is denoted by $(\mathbf{C}^B)_c$. For a cofibrant object $i_A : B \rightarrowtail A$ the *relative cylinder* $\boldsymbol{I}_B A$ is defined by the following push-out diagram



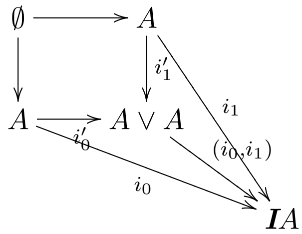
where p is the structure map of the cylinder $\boldsymbol{I}B$. Then the category $(\mathbf{C}^B)_c$ with the cofibrations and cylinder functor $\boldsymbol{I}_B$ as above is an I-category. An important example is obtained for $\mathbf{C}$=$\mathbf{Top}$ and $B = *$. Then $\mathbf{C}^B = \mathbf{Top}^*$ the category of pointed space and $(\mathbf{Top}^*)_c$ is the category of *well pointed spaces.*

Definition 1.2. Let $\mathbf{C}$ be an I-category. It is said that morphisms $f_0, f_1 : X \to Y$ are *homotopic* in $\mathbf{C}$, and it is written $H : f_0 \simeq f_1$, if there is a morphism $H : \boldsymbol{I}X \to Y$ with $Hi_0 = f_0$ and $Hi_1 = f_1$. We will denote also $Hi_\varepsilon := H_\varepsilon$. A morphism $f : X \to Y$ is a *homotopy equivalence* in $\mathbf{C}$ if there exists a morphism $g : Y \to X$ with $gf \simeq 1_X$ and $fg \simeq 1_Y$. In this case X and Y have the same *homotopy type* in $\mathbf{C}$, and we write $X \simeq Y$.

Homotopy in $\mathbf{C}$ is a natural equivalence relation. It is useful, for a subsequent comparison, to recall the proof in detail.

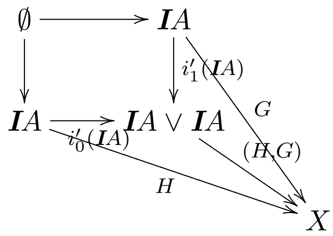
Lemma 1.3. *Homotopy in* $\mathbf{C}$ *is a natural equivalence relation.*

Proof. For some morphisms $f, f_0, f_1 \in \mathbf{C}(A, X)$, let $H : f \simeq f_0$ and $G : f \simeq f_1$. We will prove that $f_0 \simeq f_1$ and $f_0 \simeq f$. First, to fix the notations, we consider the following commutative diagrams



and this implies

$$fp(i_0, i_1)i'_0 = f, fp(i_0, i_1)i'_1 = f; \tag{1.1}$$



and this implies

(1.2) $$(H,G)i'_0(\boldsymbol{I}A)i_0 = Hi_0 = f, (H,G)i'_1(\boldsymbol{I}A)i_0 = Gi_0 = f$$

By these formulas we obtain

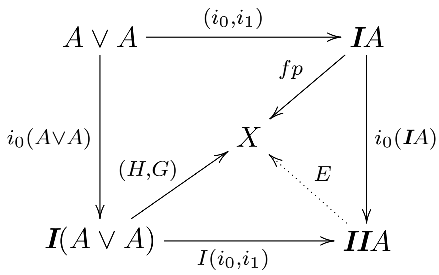
(1.3) $$(H,G)i_0(A \vee A)i'_0 = (H,G)i'_0(\boldsymbol{I}A)i_0 = f = fp(i_0,i_1)i'_0,$$

(1.4) $$(H,G)i_0(A \vee A)i'_1 = (H,G)i'_1(\boldsymbol{I}A)i_0 = f = fp(i_0,i_1)i'_1$$

and this implies

(1.5) $$(H,G)i_0(A \vee A) = fp(i_0,i_1)$$

Now, using the fact that $(i_0,i_1) : A \vee A \to A$ is a cofibration, there is a morphism $E : \boldsymbol{II}A \to X$ such that the following diagram is commutative



We consider the morphism

(1.6) $$Ei_1(\boldsymbol{I}A) : \boldsymbol{I}A \to X.$$

We have $E\boldsymbol{I}(i_0,i_1) = (H,G)$, $(H,G)i'_0(\boldsymbol{I}A) = H$, $(H,G)i'_1(\boldsymbol{I}A) = G$. Then we can write : $Ei_1(\boldsymbol{I}A)i_0 = E\boldsymbol{I}(i_0,i_1)i'_0(\boldsymbol{I}A)i_1 = (H,G)i'_0(\boldsymbol{I}A)i_1 = Hi_1 = f_0$ and $Ei_1(\boldsymbol{I}A)i_1 = E\boldsymbol{I}(i_0,i_1)i'_1(\boldsymbol{I}A)i_1 = (H,G)i'_1(\boldsymbol{I}A)i_1 = Gi_1 = f_1$. Therefore, $Ei_1(\boldsymbol{I}A) : f_0 \simeq f_1$, and this morphism is denoted by $-H + G$. If we consider $f_1 = f$ and $G = fp$, the corresponding homotopy is denoted $-H : f_0 \simeq f$. □

Example 1.4. 1) $1_{\boldsymbol{I}A} : \boldsymbol{I}A \to \boldsymbol{I}A$ is a homotopy between $i_0 : A \to IA$ and $i_1 : A \to \boldsymbol{I}A$. In concordance with the above notation, we will denote by $-1_{\boldsymbol{I}A}$ the induced homotopy of i_1 with i_0.

2) By Interchange Axiom (I5), $T : \boldsymbol{II}A \to \boldsymbol{II}A$ is a homotopy of $\boldsymbol{I}i_0 : \boldsymbol{I}A \to \boldsymbol{II}A$ and $\boldsymbol{I}i_1 : \boldsymbol{I}A \to \boldsymbol{II}A$, and we will denote by $-T$ the induced homotopy equivalence between $\boldsymbol{I}i_1$ and $\boldsymbol{I}i_0$.

3) If $f, g \in \mathbf{C}(A, B)$, with $H : f \simeq g$, and $h \in \mathbf{C}(A', A), k \in \mathbf{C}(B, B')$, then $H \circ \boldsymbol{I}h : f \circ h \simeq g \circ h$ and $k \circ H : k \circ f \simeq k \circ g$.

Let

$$h\mathbf{C}(X, Y) = \mathbf{C}(X, Y)/ \simeq \tag{1.7}$$

be the set of homotopy classes $[f] : X \to Y$. Then the homotopy category $h\mathbf{C}$ is also defined. The objects of $h\mathbf{C}$ are all objects of $\mathbf{C}$ and the morphisms in $h\mathbf{C}$ are the homotopy classes of morphisms, with the composition law $[g][f] = [gf]$. We will denote by

$$\hbar : \mathbf{C} \to h\mathbf{C} \tag{1.8}$$

the homotopy covariant functor defined by $\hbar(X) = X$ and $\hbar(f) = [f]$.

Clearly, a morphism f in $\mathbf{C}$ is a homotopy equivalence in $\mathbf{C}$ if and only if $[f]$ is an isomorphism in $h\mathbf{C}$. Such homotopy equivalences in $\mathbf{C}$ are examples of "weak equivalences", as in the following fundamental notion of the homotopy theory.

Definition 1.5. A *cofibration category* is a category $\mathbf{C}$ with an additional structure **(C,cof,we)** subject to axioms C1-C4. Here, **cof** and **we** are classes of morphisms in $\mathbf{C}$, called *cofibrations* and *weak equivalences*, respectively. It is written $A \rightarrowtail B$ for a cofibration and $A \xrightarrow{\sim} B$ for a weak equivalence. If $\mathbf{C}$ has an intial object $\emptyset$, then an object A in $\mathbf{C}$ is *cofibrant* if $\emptyset \rightarrowtail A$ is a cofibration.

(C1) Composition axion. The isomorphisms in $\mathbf{C}$ are weak equivalences and cofibrations. For two morphisms $f : A \to B$, $g : B \to C$, if any two of f, g, and gf are weak equivalences, then so is the third. The composition of cofibrations is a cofibration.

(C2) Push-out axiom. For a cofibration $i : B \rightarrowtail A$ and a morphism

$f : B \to Y$ the push-out diagram

$$\begin{array}{ccc} B & \xrightarrow{\ i\ } & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \overline{f}} \\ Y & \xrightarrow[\ \overline{i}\ ]{} & A \cup_B Y \end{array}$$

exists in C and $\overline{i}$ is a cofibration. Moreover:

(a) if f is a weak equivalence, so is $\overline{f}$;

(b) if i is a weak equivalence, so is $\overline{i}$.

(C3) Factorization axiom. For a morphism $f : B \to Y$ in C there exists a commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{\ f\ } & Y \\ & {\scriptstyle i}\searrow \quad \nearrow{\scriptstyle g} & \\ & A & \end{array}$$

where i is a cofibration and g is a weak equivalence.

(C4) Retraction axiom. Each cofibration $i : B \overset{\sim}{\rightarrowtail} A$ that is a weak equivalence admits a retraction $r : A \to B$, i.e. $ri = 1_B$.

Theorem 1.6. *Let* $(\boldsymbol{C}, \boldsymbol{cof}, \boldsymbol{I}, \emptyset)$ *be an I-category. Then* $(\boldsymbol{C},\boldsymbol{cof},\boldsymbol{we})$ *is also a cofibration category, where weak equivalences are homotopy equivalence in* $\boldsymbol{C}$.

For the factorization axiom (C3) in I-categories we can use the following mapping cylinder Z_f of a morphism $f : B \to Y$. One defines Z_f by the push-out diagram

$$\begin{array}{ccc} B & \xrightarrow{\ f\ } & Y \\ {\scriptstyle i_0}\downarrow & & \downarrow{\scriptstyle \overline{i}_0} \\ \boldsymbol{I}B & \xrightarrow[\ \overline{f}\ ]{} & Z_f \end{array}$$

For this the following diagram is commutative

$$\begin{array}{ccc} \boldsymbol{I}B & \xrightarrow{\ \overline{f}\ } & Z_f \\ {\scriptstyle p}\downarrow & \nearrow{\scriptstyle j} & \downarrow{\scriptstyle q} \\ B & \xrightarrow[\ f\ ]{} & Y \end{array}$$

with $j = \overline{f} i_1 : B \to Z_f$ a cofibration, and $q : Z_f \to Y$ a homotopy equivalence and $qj = f$.

Definition 1.7. Let $B \rightarrowtail A$ be a cofibration in the I-category $\mathbf{C}$. Let $f_0, f_1 : A \to X$ be morphisms in $\mathbf{C}$ that coincide on B, i.e., $f_0 \mid B = f_1 \mid B$. Then it is said that f_0 and f_1 are *homotopic relative* B (or under B), and it is written $f_0 \simeq f_1$ rel B, if there is a commutative diagram

$$\begin{array}{ccc} A \cup_B A & \xrightarrow{(i_0,i_1)} & \boldsymbol{I}_B A \\ & {\scriptstyle (f_0,f_1)} \searrow \quad \swarrow {\scriptstyle H} & \\ & X & \end{array}$$

Here, $\boldsymbol{I}_B A$ is the relative cylinder and the morphism $(i_0, i_1) : A \cup_B A \rightarrowtail \boldsymbol{I}_B A$ is corresponding to $(i_0, i_1) : A \vee A \rightarrowtail \boldsymbol{I}A$ in the category $(\mathbf{C}^B)_c$. H is called a homotopy from f_0 to f_1 rel B and this is denoted by $H : f_0 \simeq f_1$ rel B. If $B = \emptyset$ is the initial object, then a homotopy rel $\emptyset$ is the same as a homotopy $f_0 \simeq f_1$ since $\boldsymbol{I}_\emptyset A = \boldsymbol{I}A$.

It is proved that homotopy rel B is an equivalence relation.

2. Homotopy cylinder functor. Let $(\mathbf{C}, \mathbf{cof}, \boldsymbol{I}, \emptyset)$ be an I-category and $h\mathbf{C}$ the corresponding homotopy category.

At first it is obvious that we can take as initial object of $h\mathbf{C}$ the initial object $\emptyset$ of $\mathbf{C}$ since, if for an object X of $\mathbf{C}$ (and of $h\mathbf{C}$), we denote by $\emptyset_X$ the unique morphism $\emptyset \to X$ in $\mathbf{C}$, then $[\emptyset_X]$ is the unique morphism in $h\mathbf{C}$ from $\emptyset$ to X.

Now we will define the *homotopy cylinder functor* $h\mathbf{I} : h\mathbf{C} \to h\mathbf{C}$. We take

$$h\mathbf{I}(X) = \boldsymbol{I}X \tag{2.1}$$

for every object X. Then, if $[f] : X \to Y$ is a morphism in $h\mathbf{C}$, we can take

$$h\mathbf{I}([f]) = [\boldsymbol{I}f]. \tag{2.2}$$

It is sufficient to verify that this definition depends only on the homotopy class $[f]$. Let $H : f \simeq g : X \to Y$, i.e., $H : \boldsymbol{I}X \to Y$, with $Hi_0(X) = f$ and $Hi_1(X) = g$. We need to verify that $\boldsymbol{I}f \simeq \boldsymbol{I}g$. We define a homotopy $hH : \boldsymbol{I}\boldsymbol{I}X \to \boldsymbol{I}Y$ by the following composition : $hH = (\boldsymbol{I}H)T$, where T is the morphism from axiom (I5). Then we have $(hH)i_0(\boldsymbol{I}X) = ((\boldsymbol{I}H)T)(i_0(IX)) = (\boldsymbol{I}H)(Ti_X(\boldsymbol{I}X)) = (\boldsymbol{I}H)(\boldsymbol{I}i_0(X)) = \boldsymbol{I}(Hi_0(X)) = \boldsymbol{I}f$. Similarly, the relation $(hH)i_1(\boldsymbol{I}X) = \boldsymbol{I}g$ is verified. Therefore, we obtain $hH : \boldsymbol{I}f \simeq \boldsymbol{I}g$, which implies $[\boldsymbol{I}f] = [\boldsymbol{I}g]$ and by this we conclude that $h\boldsymbol{I}([f])$ is well defined by the above formula.

Now we verify that $h\boldsymbol{I}$ is a covariant functor. First, it is clear that for an object X, $[1_X]$ is the identity morphism in $h\mathbf{C}$ and for this we have $h\boldsymbol{I}([1_X])) = [\boldsymbol{I}1_X] = [1_{\boldsymbol{I}X}]$. Then, if $[f] : X \to Y$ and $[g] : Y \to Z$ are morphisms in $h\mathbf{C}$, we have $h\boldsymbol{I}([g][f]) = h\boldsymbol{I}([gf]) = [\boldsymbol{I}(gf)] = [\boldsymbol{I}g[\mathbf{I}f] = [\boldsymbol{I}g][If] = h\boldsymbol{I}([g])h\boldsymbol{I}([f])$. Therefore, $h\boldsymbol{I}$ is indeed a covariant functor. We will refer to it as the *homotopy cylinder functor* of the category $\mathbf{C}$. For this obviously we have $h\boldsymbol{I}(\emptyset) = \boldsymbol{I}(\emptyset) = \emptyset$.

Now we will see which are the structure transformations in $h\mathbf{C}$ for the homotopy cylinder functor $h\boldsymbol{I}$. We need to define some natural transformations $hi_\varepsilon : id_{h\mathbf{C}} \to h\boldsymbol{I}, \varepsilon \in \{0,1\}$, and $hp : h\boldsymbol{I} \to id_{h\mathbf{C}}$. This is simple since we can take $hi_\varepsilon = [i_\varepsilon]$ and $hp = [p]$. Analogously, we can define for any object X the interchange morphism $hT : h\boldsymbol{I}h\boldsymbol{I}X \to h\boldsymbol{I}h\boldsymbol{I}X$ by $hT = [T]$. Now it is immediate that the axioms (I1) and (I5) are verified and we can state the following theorem.

Theorem 2.1. *The homotopy category $h\boldsymbol{C}$ of an I-category $\boldsymbol{C}$ is a category with a cylinder functor which verifies the axiom $(I1)$ and $(I5)$ of an I-category.*

Now it is interesting to see what means the homotopy relation in the category $h\mathbf{C}$ with respect to the homotopy cylinder functor $h\boldsymbol{I}$. Let $[f], [g] : X \to Y$ be morphisms in $h\boldsymbol{I}$. We have $[f] \simeq [g]$ if there exists a morphism $[H] : \boldsymbol{I}X \to Y$ in $h\mathbf{C}$, such that $[H](hi_0) = [f]$ and $[H](hi_1) = [g]$, which means $H_0 \simeq f$ and $H_1 \simeq g$, in $\mathbf{C}$, where $H_\varepsilon = Hi_\varepsilon$. Therefore, there exist some morphisms $K, L : \boldsymbol{I}X \to Y$ in $\mathbf{C}$, such that $Ki_0 = Hi_0$, $Ki_1 = f$, $Li_0 = Hi_1$ and $Li_1 = g$. It follows that we have $-K + H + L : f \simeq g$. Conversely, if $f \simeq g$, then $[f] = [g]$ and trivially $[f] \simeq [g]$. Thus we can state the following corollaries.

Corollary 2.2. *If $f, g : X \to Y$ are morphisms in $\boldsymbol{C}$, then $[f] \simeq [g]$ in $h\boldsymbol{C}$ if and only if $f \simeq g$ in $\boldsymbol{C}$.*

Corollary 2.3. *Two objects X, Y are homotopy equivalent in $h\boldsymbol{C}$ if and only if they are isomorphic in $h\boldsymbol{C}$.*

Corollary 2.4. *If for the category $h\boldsymbol{C}$ the Factorization Axiom (C3) is satisfied, then every morphism of this category is a cofibration.*

3. Homotopy bisection hypothesis

Definition 3.1. Suppose that $(\mathbf{C}, \mathbf{cof}, \boldsymbol{I}, \emptyset)$ is an I-category. We say that this category satisfies the *Homotopy Bisection Hypothesis* (HBH) if there exist two natural transformations

$$b_0, b_1 : \boldsymbol{I} \to \boldsymbol{I} \tag{3.1}$$

satisfying the following conditions:

(a) $b_\varepsilon \circ i_\varepsilon = i_\varepsilon$ and $p \circ b_\varepsilon = p$, $\varepsilon = 0, 1$;

(b) For any object A the following diagram is commutative and, moreover, it is a push-out diagram

$$\begin{array}{ccc} A & \xrightarrow{i_1} & IA \\ {\scriptstyle i_0}\downarrow & & \downarrow{\scriptstyle b_0} \\ IA & \xrightarrow[b_1]{} & IA \end{array}$$

Remark 3.2. The motivation for this name (HBH) is the following: If $H : \boldsymbol{I}A \to X$ is an arbitrary homotopy, then putting $F := H \circ b_0$ and $G := H \circ b_1$, for these morphisms we have $F \circ i_1 = H \circ b_0 \circ i_1 = H \circ b_1 \circ i_0 = G \circ i_0$, so that there exists $(F, G) : \boldsymbol{I}A \to X$, and $H = (F, G)$, with $H \circ i_0 = H \circ b_0 \circ i_0 = F \circ i_0$ and $H \circ i_1 = H \circ b_1 \circ i_1 = G \circ i_1$.

Remark 3.3. In the general case of an arbitrary I-category $\mathbf{C}$, if $f, g, h : A \to X$ are morphisms and $F : f \simeq g$, $G : g \simeq h$, then the homotopy $F + G : f \simeq h$ is obtained not very simply as we have seen in the proof of Lemma 1.3. But if we suppose that (HBH) is satisfied, then from the conditions $F \circ i_1 = g = G \circ i_0$, it follows a morphism $(F, G) : \boldsymbol{I}A \to X$ satisfying the conditions $(F, G) \circ b_0 = F$ and $(F, G) \circ b_1 = G$. For this morphism we have $(F, G) \circ i_0 = (F, G) \circ b_0 \circ i_0 = F \circ i_0 = f$ and $(F, G) \circ i_1 = (F, G) \circ b_1 \circ i_1 = G \circ i_1 = h$, so that $(F, G) : f \simeq h$.

Example 3.4. The I-category **Top** satisfies the HBH. Denote the elements of the $(A \times I) \cup_A (A \times I)$ by $[a, t]_0$ and $[a, t]_1$ with the identification $[a, 0]_0 = [a, 1]_1$. Then this space is homeomorphic with the cylinder $A \times I$

by the map $\chi : (A \times I) \cup_A (A \times I) \to A \times I$, defined by $\chi([a,t]_0) = (a, \frac{1+t}{2})$ and $\chi([a,t]_1) = (a, \frac{t}{2})$, with the inverse $\chi^{-1} : A \times I \to (A \times I) \cup_A (A \times I)$

$$\chi^{-1}(a,t) = \begin{cases} [a,2t]_1, & \text{if } 0 \leq t \leq \frac{1}{2}, \\ [a,2t-1]_0, & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

The push-out morphisms $(a,t) \to [a,t]_0$, opposite to i_1, and $(a,t) \to [a,t]_1$, opposite to i_0, become $b_1(a,t) = (a, \frac{1+t}{2})$ and respectively $b_0(a,t) = (a, \frac{t}{2})$. These maps b_0, b_1 are natural transformations and satisfy: $(b_0 \circ i_0)(a) = b_0(a,0) = (a,0) = i_0(a)$; $(b_1 \circ i_1)(a) = b_1(a,1) = (a,1) = i_1(a)$, and $(p \circ b_0)(a,t) = p(a, \frac{t}{2}) = a = p(a, \frac{1+t}{2}) = (p \circ b_1)(a,t)$.

Example 3.5. The I-category **Topp** satisfies the HBH. The verification is as in Example 3.4. The single fact that must be specified is that the maps b_0, b_1 above defined are compact and if $f, g : A \times I \to Y$ are two compact maps, with $f(a,0) = g(a,1)$, then the map $(f,g) : A \times I \to X$ defined by

$$(f,g)(a,t) = \begin{cases} g(a,2t), & \text{if } 0 \leq t \leq \frac{1}{2}, \\ f(a,2t-1), & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

is a compact map also.

Example 3.6. The I-category **End** satisfies the HBH. If A is an ended space $(\widehat{X}, X, E)$, let $\boldsymbol{I}A = (I_E\widehat{X}, X \times I, E)$, where $I_E\widehat{X}$ is given by the following push-out diagram in **Top**:

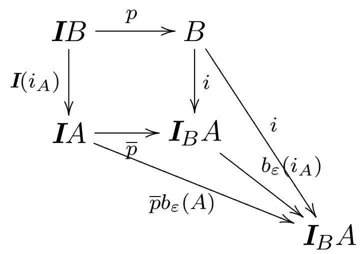
$$\begin{array}{ccc} E \times I & \xrightarrow{p} & E \\ {\scriptstyle e\times 1_I}\downarrow & & \downarrow{\scriptstyle h} \\ \widehat{X} \times I & \xrightarrow[k]{} & I_E\widehat{X} \end{array}$$

where $e : E \to \widehat{X}$ is the inclusion. Then the maps $i_0, i_1 : (\widehat{X}, X, E) \to (I_E\widehat{X}, IX, E)$ are defined by $i_\varepsilon(\widehat{x}) = [\widehat{x}, \varepsilon]$, where $[\widehat{x}, t]$ denotes the elements of $I_E\widehat{X}$, and we have $[e,t] = [e]$, $\forall e \in E, t \in I$. The maps $\bar{i}_0, \bar{i}_1 : (I_E\widehat{X}, IX, E) \to (I_E\widehat{X}, IX, E)$ are defined by $b_0([\widehat{x}, t] = [\widehat{x}, \frac{t}{2}]$, $b_1([\widehat{x}, t]) = [\widehat{x}, \frac{1+t}{2}]$ and $b_\varepsilon([e]) = [e]$, for $\widehat{x} \in \widehat{X}, e \in E, t \in I$.

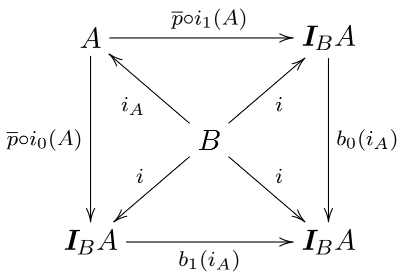
Example 3.7. If **C** is an I-category satisfying HBH, then the category of pairs **Par**(**C**) has the same property also. For an object (X, X_0) of **Pair**(**C**) given by a morphism $i_X : X_0 \to X$, define $b_\varepsilon(X, X_0) = (b_\varepsilon(X), b_\varepsilon(X_0))$:

$(\boldsymbol{I}X, \boldsymbol{I}X_0) \to (\boldsymbol{I}X, \boldsymbol{I}X_0)$. Since $i_\varepsilon(X, X_0) : (X, X_0) \to (IX, IX_0)$ is the pair $(i_\varepsilon(X), i_\varepsilon(X_0))$, the condition (a) from Definition 3.1 is verified. Then, since a push-out in **Pair**(**C**) is simply a pair of push-outs, the condition (b) is verified also.

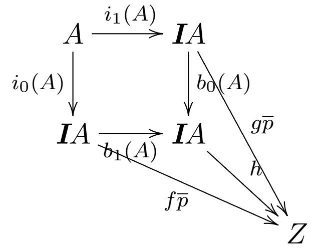
Example 3.8. If **C** is an I-category satisfying HBH, and if B is an object in **C**, then the category $(\mathbf{C}^B)_c$, of cofibrant objects and morphisms under B , also has the HBH. For a cofibrant object $i_A : B \rightarrowtail A$, consider the following pair of morphisms with respect to the push-out diagram which defines the relative cylinder $\boldsymbol{I}_B A$: $i : B \to \boldsymbol{I}_B A$ and $\overline{p} \circ b_\varepsilon(A) : \boldsymbol{I}A \to \boldsymbol{I}_B A$. For these morphisms we have $i \circ p = \overline{p} \circ b_\varepsilon \circ \boldsymbol{I}(i_A)$. Then there exists a morphism $b_\varepsilon(i_A) := (i, \overline{p} \circ b_\varepsilon(A))$, satisfying $b_\varepsilon(i_A) \circ i = i$ and $b_\varepsilon(i_A) \circ \overline{p} = \overline{p} \circ b_\varepsilon(A)$.



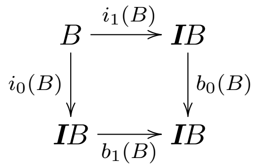
Then, since $i_\varepsilon(i_A) = \overline{p} \circ i_\varepsilon(A)$, we have $b_\varepsilon(i_A) \circ i_\varepsilon(i_A) = b_\varepsilon(i_A) \circ \overline{p} \circ i_\varepsilon(A) = \overline{p} \circ \overline{i}_\varepsilon(A) \circ i_\varepsilon(A) = \overline{p} \circ i_\varepsilon(A) = i_\varepsilon(i_A)$. Then we have $p(i_A) = (p, i_A)$, and we need to prove that $p(i_A) \circ b_\varepsilon(i_A) = p(i_A)$. But we have $p(i_A) \circ \overline{i}_\varepsilon(i_A) \circ \overline{p} = p(i_A) \circ \overline{p} \circ \overline{i}_\varepsilon(A) = p \circ \overline{i}_\varepsilon(A) = p$ and $b_\varepsilon(i_A) \circ i = p(i_A) \circ i = i$ and by the uniqueness in the push-out diagram which defines $p(i_A)$, we deduce $p(i_A) \circ b_\varepsilon(i_A) = p(i_A)$. Thus the conditions (a) are satisfied. For the condition (b), at first we observe that $b_0(i_A) \circ \overline{p} \circ i_1(A) = p \circ b_0(A) \circ i_1(A) = p \circ b_1(A) \circ i_0(A) = \overline{p} \circ b_0(A) \circ i_0(A) = b_1(i_A) \circ \overline{p} \circ i_0(A)$.



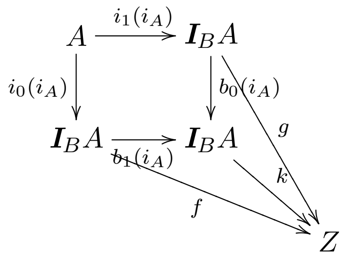
Now we suppose that we have an object $i_Z : B \rightarrowtail Z$ in $(\mathbf{C}^B)_c$ and two morphisms $f : I_B A \to Z$, with $f \circ i = i_Z$, and $g : I_B A \to Z$, with $g \circ i = i_Z$, satisfying $f \circ (\overline{p} \circ i_0(A)) = g \circ (\overline{p} \circ i_1(A))$. If we write the last relation as $(f\overline{p}) \circ i_0(A) = (gp) \circ i_1(A)$, then by the HBH for the category $\mathbf{C}$, there exists (unique) $h := (f\overline{p}, g\overline{p}) : \boldsymbol{I}A \to Z$, such that $h \circ b_1(A) = f \circ p$ and $h \circ b_0(A) = g \circ \overline{p}$.



Now we consider the push-out diagram



and the morphism $h \circ \boldsymbol{I}(i_A), i_Z \circ p(B) : \boldsymbol{I}B \to Z$. For these morphisms we have $h \circ \boldsymbol{I}(i_a) \circ b_1(B) = h \circ b_1(A) \circ \boldsymbol{I}(i_A) = f \circ p \circ \boldsymbol{I}(i_A) = f \circ i \circ p(B) = i_Z \circ p(B) = i_Z \circ p(B) \circ b_1(B)$. This implies $h \circ \boldsymbol{I}(i_A) = i_Z \circ p(B)$. Thus we can consider with respect to the push out diagram of the relative cylinder $\boldsymbol{I}_B A$ the morphism $k := (h, i_Z) : \boldsymbol{I}_B A \to Z$, with the properties $k \circ i = i_Z$ and $k \circ \overline{p} = h$.The first relation shows that k is a morphism in the category $(\mathbf{C}^B)_c$. The second is used to verify the relations $k \circ b_1(i_A) = f$ and $k \circ b_0(i_A) = g$. Again we use the push-out diagram of $\boldsymbol{I}_B A$. We have $k \circ b_1(A) \circ \overline{p} = k \circ \overline{p} \circ b_1(A) = h \circ b_1(A) = f \circ \overline{p}$ and $k \circ b_1(i_A) \circ i = k \circ i = i_Z = f \circ i$, and by this we can conclude that $k \circ b_1(i_A) = f$. Similarly, it result that $k \circ b_0(i_A) = g$.



Now for the uniqueness of k, suppose that we have also $k' : \boldsymbol{I}_B A \to Z$, with $k' \circ i = i_Z$ and $k' \circ b_1(i_A) = f$, $k' \circ b_0(i_A) = g$. It is sufficient to verify that these relations imply $k' \circ \overline{p} = h$ and $k' \circ i = i_Z$. The second relation is given. Then we have $k' \circ \overline{p} \circ b_1(A) = k' \circ b_1(i_A) \circ \overline{p} = f \circ \overline{p} = h \circ b_1(A)$ and $k' \circ \overline{p} \circ b_0(A) = k' \circ b_0(i_A) \circ \overline{p} = g \circ \overline{p} = h \circ b_0(A)$, which implies $k' \circ \overline{p} = h$. Thus we can obtain from the definition of k that we have $k' = k$. This finishes this sufficiently laborious example, but we hope that it is instructive.

Example 3.9. * Let $(\mathbf{Chain}_R^+)_c$ be the category of the free chain complexes of R-modules , which are bounded below, and of the chain maps. We recall in short the structure of this I-category. The trivial chain complex $C = 0$ is the initial object. For a chain complex $C = (C_n, d_n)$, the cylinder is $((\boldsymbol{I}C)_n, (\boldsymbol{I}d)_n)$ where $(\boldsymbol{I}C)_n = \{(c'_n, c''_n, c_{n-1}) | c'_n, c''_n \in C_n, c_{n-1} \in C_{n-1}\}$, and $(\boldsymbol{I}d)_n(c'_n, c''_n, c_{n-1}) = (dc'_n - c_{n-1}, dc''_n + c_{n-1}, -dc_{n-1})$. For a chain map $f = (f_n) : C = (C_n, d_n) \to C' = (C'_n, d'_n)$, $(\boldsymbol{I}f) : (\boldsymbol{I}C) \to (\boldsymbol{I}C')$ is defined by $(\boldsymbol{I}f)_n(c'_n, c''_n, c_{n-1}) = (f_n(c'_n), f_n(c''_n), f_{n-1}(c_{n-1}))$. The maps $i_0, i_1 : (C_n, d_n) \to ((IC)_n, (Id)_n)$ are defined by $(i_0)_n(c_n) = (c_n, 0, 0)$ and $(i_1)_n(c_n) = (0, c_n, 0)$ for $c_n \in C_n$. The map $p : \boldsymbol{I}C \to C$ is given by $p_n(c'_n, c''_n, c_{n-1}) = c'_n + c''_n$. A cofibration is an injective chain map $i = (i_n) : C = (C_n, d_n) \to C' = (C'_n, d'_n)$ for which the cokernel $C'/i(C)$ is a free chain complex. The push-out of chain complexes is obtained as a push-out of modules. Then if we consider the push-out diagram over the pair morphisms i_0, i_1, then the fourth vertex of the diagram is not the cylinder $((IC)_n, (Id)_n)$, so that the category $(\mathbf{Chain}_R^+)_c$ does not satisfy the HBH.

*This example was offered by Professor Antonio Quintero to whom the author is very indebted.

Remark 3.10. Actually, if an I-category C satisfies the HBH, then for every object A the morphisms $b_\varepsilon : IA \to IA$, $\varepsilon = 0, 1$, are cofibrations.

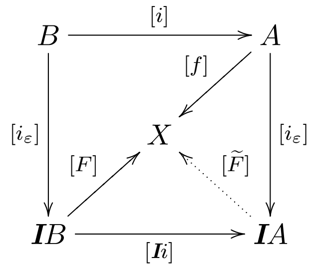
Remark 3.11. The proof of the symmetry of the homotopy relation can be simplified if we suppose that the I-category verifies the following *homotopy symmetry hypothesis*(HSH): there exists a natural transformation $\sigma : \boldsymbol{I} \to \boldsymbol{I}$, with $\sigma \circ i_0 = i_1$, $\sigma \circ i_1 = i_0$, and $p \circ \sigma = p$. Then, if $F : f \simeq g$, the morphism $\widetilde{F} = F \circ \sigma$ satisfies $\widetilde{F} \circ i_0 = F \circ \sigma \circ i_0 = F \circ i_1 = g$ and $\widetilde{F} \circ i_0 = F \circ \sigma \circ i_1 = F \circ i_0 = f$. The I-categories **Top**, **Topp** and **End** have the HSH. If $\mathbf{C}$ is an abstract I-category with the HSH, then $\mathbf{Pair}(\mathbf{C})$ and $(\mathbf{C}^B)_c$ also have this property. In the case of the last category $\sigma_c^B : \boldsymbol{I}_B A \to \boldsymbol{I}_B A$ is defined by $\sigma_c^B = (\overline{p}\sigma, i)$.

We remark that this hypothesis will not be used on. This is given only for the 'symmetry' with the HBH.

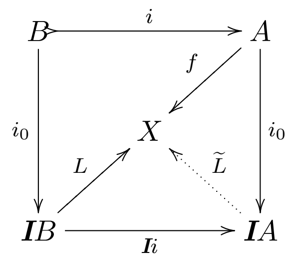
4. The reflection of cofibrations

Proposition 4.1. *Let $(\boldsymbol{C}, \boldsymbol{cof}, \boldsymbol{I}, \emptyset)$ be an I-category which satisfies the HBH. Then the homotopy functor $\hbar : \boldsymbol{C} \to h\boldsymbol{C}$ preserves the HEP, i.e., if $i : B \to A$ is a morphism in $\boldsymbol{C}$ with the HEP, then the morphism $[i] : B \to A$ has the HEP in $h\boldsymbol{C}$.*

Proof. Let X be an arbitrary object and $[f] : A \to X$, $[F] : \boldsymbol{I}B \to X$ morphisms in $h\mathbf{C}$ such that $[F][i_\varepsilon] = [f][i]$, $\varepsilon \in \{0, 1\}$.



We need to prove that there is a morphism $[\widetilde{F}] : \boldsymbol{I}A \to X$ with the properties $[\widetilde{F}][i_\varepsilon] = [f]$ and $[\widetilde{F}][\boldsymbol{I}i] = [F]$. At first, we consider the case $\varepsilon = 0$. We have $F \circ i_0 \simeq f \circ i$ and let $H : \boldsymbol{I}B \to X$ be a homotopy morphism such that $H \circ i_0 = f \circ i$ and $H \circ i_1 = F \circ i_0$. Then we can consider the homotopy morphism $L : \boldsymbol{I}B \to X$, defined by $L = H + F$. For this morphism we have $L \circ i_0 = H \circ i_0 = f \circ i$. Then, by the HEP for i, there exists a homotopy extension $\widetilde{L} : \boldsymbol{I}A \to X$ such that $\widetilde{L} \circ i_0 = f$ and $\widetilde{L} \circ \boldsymbol{I}i = L$.



Now we can define $\widetilde{F} : \boldsymbol{I}A \to X$ by

$$\widetilde{F} = \widetilde{L} \circ b_1 \tag{4.1}$$

Then we have

$$\widetilde{F} \circ \boldsymbol{I}i = \widetilde{L} \circ b_1 \circ \boldsymbol{I}i = \widetilde{L} \circ \boldsymbol{I}i \circ b_1 = L \circ b_1 = F, \tag{4.2}$$

such that we have $[\widetilde{F}][\boldsymbol{I}i] = [F]$.

Now we consider the homotopy $K : \boldsymbol{I}A \to X$ defined by

$$K = \widetilde{L} \circ b_0. \tag{4.3}$$

For this we have:

$$K \circ i_0 = \widetilde{L} \circ b_0 \circ i_0 = \widetilde{L} \circ i_0 = f \tag{4.4}$$

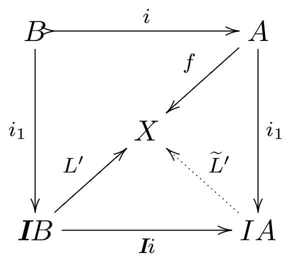
and

$$K \circ i_1 = \widetilde{L} \circ b_0 \circ i_1 = \widetilde{L} \circ b_1 \circ i_0 = \widetilde{F} \circ i_0 \tag{4.5}$$

Therefore, we conclude

$$K : f \simeq \widetilde{F} \circ i_0 \Rightarrow [f] = [\widetilde{F}][i_0]. \tag{4.6}$$

Now we consider the case $\varepsilon = 1$. We have $F \circ i_1 \simeq f \circ i$ and let $H' : \boldsymbol{I}B \to X$ be a homotopy morphism such that $H' \circ i_1 = f \circ i$ and $H' \circ i_0 = F \circ i_1$. Then can consider the homotopy morphism $L' : \boldsymbol{I}B \to X$, defined by $L' = F + H'$. For this morphism we have $L' \circ i_1 = H' \circ i_1 = f \circ i$. Then by the HEP for i there exists a homotopy extension $\widetilde{L}' : \boldsymbol{I}A \to X$ such that $\widetilde{L}' \circ i_1 = f$ and $\widetilde{L}' \circ \boldsymbol{I}i = L'$.



Now we can define $\widetilde{F}' : \boldsymbol{I}A \to X$ by

$$\widetilde{F}' = \widetilde{L}' \circ b_0 \tag{4.7}$$

Then we have

$$\widetilde{F}' \circ \boldsymbol{I}i = \widetilde{L}' \circ \bar{i}_0 \circ \boldsymbol{I}i = \widetilde{L}' \circ \boldsymbol{I}i \circ b_0 = L' \circ b_0 = F, \tag{4.8}$$

such that we have $[\widetilde{F}'][\boldsymbol{I}i] = [F]$.

Now we consider the homotopy $K' : IA \to X$ defined by

$$K' = \widetilde{L}' \circ b_1. \tag{4.9}$$

For this we have:

$$K' \circ i_1 = \widetilde{L}' \circ b_1 \circ i_1 = \widetilde{L}' \circ i_1 = f \tag{4.10}$$

and

$$K' \circ i_0 = \widetilde{L} \circ b_1 \circ i_0 = \widetilde{L}' \circ b_0 \circ i_1 = \widetilde{F}' \circ i_1 \tag{4.11}$$

Therefore, we conclude

$$K' : \widetilde{F}' \circ i_1 \simeq f \Rightarrow [\widetilde{F}'][i_1] = [f]. \tag{4.12}$$

□

Theorem 4.2. *Let $(\boldsymbol{C}, \boldsymbol{cof}, \boldsymbol{I}, \emptyset)$ be an I-category satisfying the HBH. Then in the homotopy category $h\boldsymbol{C}$ any morphism has the HEP.*

Proof. By Theorem 1.6 ([3],Theorem 4.5, p. 31), and by the Factorization Axiom (C3), any morphism $[f] : B \to Y$ in $h\mathbf{C}$ can be factorized as $[f] = [g] \circ [i]$, with $i : B \to A$ a cofibration in $\mathbf{C}$ and $g : A \to Y$ a homotopy equivalence in $\mathbf{C}$. By section 2, $\boldsymbol{I}g$ is a homotopy equivalence, so that $[g]$ is an isomorphism in $h\mathbf{C}$. Then, by the Cofibration Axiom (I3) and by Proposition 4.1, the morphism $[i]$ has the HEP in $h\mathbf{C}$. Now it can immediately be seen that $[f]$ has the HEP in $h\mathbf{C}$. □

Now we will give an example of an I-category for which the image of a cofibration by the functor $\hbar$ does not have the HEP.

Example 4.3. In the category $(\mathbf{Chain}_R^+)_c$, with R a nontrivial ring with unit 1, consider a chain complex C with $C_n = 0$ for all $n \neq 0$, and $C_0 = R$. For this we have

$$(\boldsymbol{I}C)_0 = R \oplus R, (\boldsymbol{I}C)_1 = R,\ (\boldsymbol{I}C_n) = 0 \text{ for } n \neq 0, 1 \text{ and } (\boldsymbol{I}d)_1(x) = (-x, x),$$
$$\text{for } x \in R, \text{ and } (\boldsymbol{I}d)_n = 0 \text{ for } n \neq 1.$$

For this we have

$$i_0(x) = (x, 0),\ \ i_1(x) = (0, x).$$

Then

$$(\boldsymbol{II}C)_0 = (\boldsymbol{I}C)_0 + (\boldsymbol{I}C)_0 = R \oplus R \oplus R \oplus R,$$

$$(\boldsymbol{IIC})_1 = (\boldsymbol{IC})_1 \oplus (\boldsymbol{IC})_1 \oplus (\boldsymbol{IC})_0 = R \oplus R \oplus R \oplus R,$$

$$(\boldsymbol{IIC})_2 = (\boldsymbol{IC})_2 \oplus (\boldsymbol{IC})_2 \oplus (\boldsymbol{IC})_1 = R;$$

$$(\boldsymbol{IC})_n = 0 \ \text{ for } \ n \neq 0, 1, 2,$$

and

$$(\boldsymbol{II}d)_2(x) = (-x, x, x, -x),$$

$$(\boldsymbol{II}d)_1(x_1, x_2, x_3, x_4) = (-x_1 - x_3, x_1 - x_4, -x_2 + x_3, x_2 + x_4).$$

Then, on the one hand:

$$i_0(\boldsymbol{IC})_0(x', x'') = (x', x'', 0, 0), i_0(\boldsymbol{IC})_1(x) = (x, 0, 0, 0);$$

$$i_1(\boldsymbol{IC})_0(x', x'') = (0, 0, x', x''); \; i_1(\boldsymbol{IC})_1(x) = (0, x, 0, 0).$$

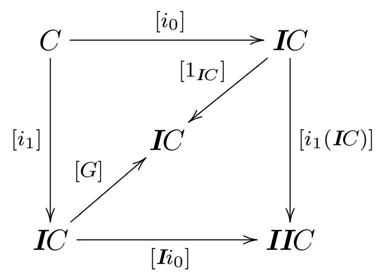
On the other hand:

$$(\boldsymbol{I}i_0)_0(x', x'') = (x', 0, x'', 0), \; (\boldsymbol{I}i_0)_1(x) = (0, 0, x, 0);$$
$$(\boldsymbol{I}i_1)_0(x', x'') = (0, x', 0, x''), \; (\boldsymbol{I}i_1)_1(x) = (0, 0, 0, x).$$

We recall also that the homotopy relation in the I-category $(\mathbf{Chain}_R^+)_c$ can be identified with the chain homotopy relation ([1], Remark 6.9, p. 41).

Now we want to show that $[i_0] : C \to \boldsymbol{IC}$ does not have the HEP in the category $h(\mathbf{Chain}_R^+)_c$.

For this, consider a morphism $G : \boldsymbol{IC} \to \boldsymbol{IC}$ given by $G_1 : (\boldsymbol{IC})_1 = R \to (\boldsymbol{IC})_1 = R, G_1(x) = ax$ and $G_0 : (\boldsymbol{IC})_0 = R \oplus R \to (\boldsymbol{IC})_0 = R \oplus R, G_0(x^1, x^2) = ((a - h + 1)x^1 + (1 - h)x^2, (h - a + 1)x^1 + (h + 1)x^2)$. This is a chain map since $G_0 \circ (\boldsymbol{I}d)_1 = (\boldsymbol{I}d)_1 \circ G_1$. Moreover, the difference $G \circ i_1 - i_0 : C \to \boldsymbol{IC}$ is given by $(G \circ i_1 - i_0)_0(x) = (-hx, hx) = (\boldsymbol{I}d)_1(hx) = ((\boldsymbol{I}d \circ \hbar)_1(x)$, with $\hbar : C_0 = R \to (\boldsymbol{IC})_1 = R$. Thus, we have $G \circ i_1 \simeq i_0$, such that there exists the following commutative diagram in $(\mathbf{Chain}_R^+)_c$.



Now a morphism $F : \boldsymbol{IIC} \to \boldsymbol{IC}$ is given by $F_0 : (\boldsymbol{IIC})_0 = R \oplus R \oplus R \oplus R \to (\boldsymbol{IC})_0 = R \oplus R$, with $F_0(x^1, x^2, x^3, x^4) = (a_1^1 x^1 + a_2^1 x^2 + a_3^1 x^3 + a_4^1 x^4)$ and $F_1 : (\boldsymbol{IIC})_1 = R \oplus R \oplus R \oplus R \to (\boldsymbol{IC})_1 = R$, with $F_1(x^1, x^2, x^3, x^4) = b_1 x^1 + b_2 X^2 + b_3 x^3 + b_4 x^4$. Then, by the relation $F_1 \circ (\boldsymbol{II}d)_2 = 0$, we deduce the relation

$$a_1^1 + a_4^1 = a_2^1 + a_3^1,\ a_1^2 + a_4^2 = a_2^2 + a_3^2.$$

Together with the condition $F_0 \circ (\boldsymbol{II}d)_1 = (\boldsymbol{I}d)_1 \circ F_1$, we obtain

$$b_1 = b_2, b_3 = b_4;\ a_1^1 := \alpha, a_2^1 = \alpha - b_1, a_3^1 = \alpha - b_3, a_4^1 = \alpha - b_1 - b_3;$$
$$a_1^2 := \beta, a_2^2 = \beta + b_1, a_3^2 = \beta + b_3, a_4^2 = \beta + b_1 + b_3.$$

Now the relation $F \circ i_1(IC) \simeq 1_{IC}$ is required, which means that there is a morphism $k : (\boldsymbol{IC})_0 = R \oplus R \to (\boldsymbol{IC})_1 = R$ for which

$$(F \circ i_1(\boldsymbol{IC}))_0 - (1_{\boldsymbol{IC}})_0 = (Id)_1 \circ k_0$$

and $(F \circ i_1(\boldsymbol{IC}))_1 - (1_{\boldsymbol{IC}})_1 = k_0 \circ (\boldsymbol{I}d)_1$. By this we obtain

$$\alpha = 1 + b_3 - k_1, b_1 = 1 - k_1 + k_2, \beta = k_1 - b_3.$$

Now we check to see if there exists a such morphism F such that $F \circ \boldsymbol{I}i_0 \simeq G$.

We have

$$(F \circ \boldsymbol{I}i_0)_0(x^1, x^2) = F_0(x^1, 0, x^2, 0) = (a_1^1 x^1 + a_3^1 x^2, a_1^2 x^1 + a_3^2 x^2) \text{ and}$$
$$(F \circ \boldsymbol{I}i_0)_1(x) = F_1(0, 0, x, 0) = b_3 x.$$

Suppose that a chain homotopy morphism $\gamma : \boldsymbol{IC} \to \boldsymbol{IC}$ is given by $\gamma_0 : (\boldsymbol{IC})_0 = R \oplus R \to (\boldsymbol{IC})_1 = R$, with $\gamma_0(x^1, x^2) = l_1 x^1 + l_2 x^2$. Then the relation $F \circ \boldsymbol{I}i_0 - G = (\boldsymbol{I}d) \circ \gamma + \gamma \circ (\boldsymbol{I}d)$ implies $(F_1 \circ (\boldsymbol{I}i_0)_1 - G_1)(x) = (\gamma_0 \circ (\boldsymbol{I}d)_1))(x) \Rightarrow b_3 - a = l_2 - l_1$ and $(F_0 \circ Ii_0 - G_0)(x^1, x^2) = ((Id)_1 \circ \gamma_0)(x^1, x^2) \Rightarrow$

$$a_1^1 x^1 + a_3^1 x^2 - (a - h + 1)x^1 - (1 - h)x^2 = -l_1 x^1 - l_2 x^2,$$
$$a_1^2 x^1 + a_3^2 x^2 - (h - a + 1)x^1 - (h + 1)x^2 = l_1 x^1 + l_2 x^2.$$

By summing the last relations, we deduce $(a_1^1 + a_1^2)x^1 + (a_3^1 + a_3^2)x^2 - 2x^1 = 0$ $\Rightarrow$

$$a_1^1 + a_1^2 - 2 = 0, a_3^1 + a_3^2 = 0.$$

The first relation implies $\alpha + \beta - 2 = 0$, and then $1 + b_3 - k_1 + k_1 - b_3 - 2 = 0$ $\Rightarrow 1 = 0!$

Thus,we have proved there is $[G] : \boldsymbol{I}C \to \boldsymbol{I}C$ with $[G][i_1] = [i_0]$, but for which $[F] : IIC \to \boldsymbol{I}C$ does not exist, such that $[F][i_1(\boldsymbol{I}C)] = [1_{\boldsymbol{I}C}]$ and $[F][\boldsymbol{I}i_0] = [G]$. Therefore, the morphism $[i_0] : C \to \boldsymbol{I}C$ does not have the HEP in $h(\mathbf{Chain}_R^+)_c$, although i_0 is a cofibration in $(\mathbf{Chain}_R^+)_c$.

Remark 4.4. Taking into consideration examples 3.9 and 4.3, Proposition 4.1 and Theorem 4.2, we formulate the following question: If for an I-category $\mathbf{C}$ the images of cofibrations by the homotopy functor $\hbar$ are morphisms with HEP in $h\mathbf{C}$, or equivalent, if all morphisms in $h\mathbf{C}$ have the HEP, does it follow that $\mathbf{C}$ satisfies the HBH?

Remark 4.5. If A is a cofibrant object in an arbitrary I-category C, then $\emptyset \to A$ has the HEP in $h\mathbf{C}$.

5. The reflection of push-out diagrams

Theorem 5.1. *The homotopy functor $\hbar : \boldsymbol{C} \to h\boldsymbol{C}$ carries a push-out diagram*

$$\begin{array}{ccc} B & \xrightarrow{i} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \overline{f}} \\ X & \xrightarrow[\overline{i}]{} & P \end{array}$$

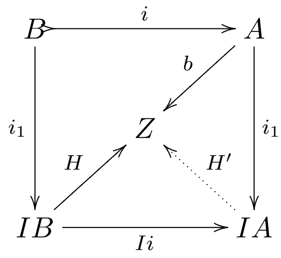
with i a cofibration in $\boldsymbol{C}$, in a weak push-out diagram in the homotopy category $h\boldsymbol{C}$.

Proof. Consider the commutative diagram in $h\mathbf{C}$

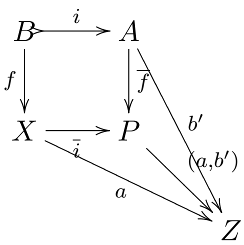
$$\begin{array}{ccc} B & \xrightarrow{[i]} & A \\ {\scriptstyle [f]}\downarrow & & \downarrow{\scriptstyle [\overline{f}]} \\ X & \xrightarrow[{[\overline{i}]}]{} & P \end{array}$$

which we need to verify that it is a weak push-out diagram.

Suppose that $[a] : X \to Z$ and $[b] : A \to Z$, are morphisms in $h\mathbf{C}$ such that $[a][f] = [b][i]$, i.e., $a \circ f \simeq b \circ i$. Let $H : \boldsymbol{I}B \to Z$, be a homotopy morphism in $\mathbf{C}$ with $H \circ i_0 = a \circ f$ and $H \circ i_1 = b \circ i$. Using the HEP for the morphism i, there exists a homotopy morphism $H' : \boldsymbol{I}A \to Z$, satisfying the conditions $H' \circ i_1 = b$ and $H' \circ (\boldsymbol{I}i) = H$.



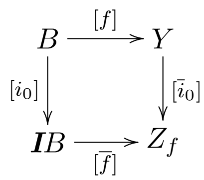
We define $b' = H' \circ i_0 : A \to Z$. Then we have $b' \circ i = (H' \circ i_0) \circ i = H' \circ (i_0 \circ i) = H' \circ ((\boldsymbol{I}i) \circ i_0) = (H' \circ \boldsymbol{I}i) \circ i_0 = H \circ i_0 = a \circ f$. Now using the weak push-out property for the given diagram in $\mathbf{C}$ it follows there exists a morphism $(a, b') : P \to Z$ in $\mathbf{C}$ such that $(a, b') \circ \bar{i} = a$ and $(a, b') \circ \overline{f} = b'$.



Therefore we can write $[(a, b')] \circ [\bar{i}] = [a]$ and $[(a, b')] \circ [\overline{f}] = [b']$. But $[b'] = [b]$, such that $[(a, b')] \circ [\overline{f}] = [b]$.

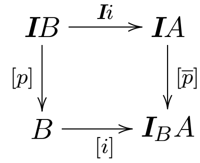
We remember that weak push-out means that the mediating morphism need not be unique. $\square$

Example 5.2. For an arbitrary morphism $f : B \to Y$ in an I-category $\mathbf{C}$, there exists the following weak push-out diagram in the category $h\mathbf{C}$



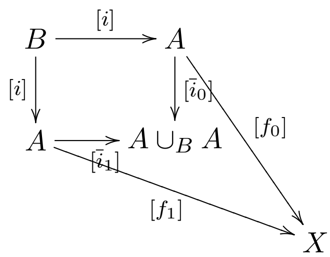
where Z_f is the mapping cylinder of f ([1], p. 31).

Example 5.3. For a cofibration $i : B \rightarrowtail A$, in an I-category $\mathbf{C}$, there exists the following weak push-out diagram in the category $h\mathbf{C}$

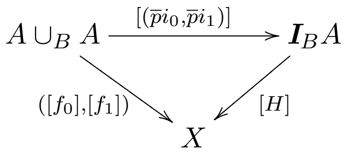


where $\boldsymbol{I}_B A$ is the relative cylinder of A under B ([1], p.28).

Example 5.4. Consider an I-category $\mathbf{C}$ and suppose that $i : B \rightarrowtail A$ is a cofibration. Let $f_0, f_1 : A \to X$ be two morphisms such that $f_0 \circ i \simeq f_1 \circ i$. Then, in $h\mathbf{C}$ there exists the following commutative diagram



and because the square is a weak push-out in $h\mathbf{C}$, there exists a morphism $([f_0],([f_1]) : A \cup_B A \to X$, such that $([f_0],[f_1]) \circ [i_0] = [f_0]$ and $([f_0],[f_1]) \circ [i_1] = [f_1]$. Then, in the category C there is a cofibration $(\overline{p}i_0, \overline{p}i_1) : A \cup_B A \rightarrowtail \boldsymbol{I}_B A$. Then we will say that $[f_0] \simeq [f_1] relB$ in $h\mathbf{C}$ if there is a commutative diagram in $h\mathbf{C}$



If $f_0 \circ i = f_1 \circ i$ and $f_0 \simeq f_1 relB$, then $[f_0] \simeq [f_1] relB$ in $h\mathbf{C}$. But in the general case we can obtain some more general relations of relative homotopy in $\mathbf{C}$.

Example 5.5. If an I-category C satisfies the HBH, then the following diagram in $h\mathbf{C}$

$$\begin{array}{ccc} A & \xrightarrow{[i_1]} & \boldsymbol{I}A \\ {\scriptstyle [i_0]}\downarrow & & \downarrow{\scriptstyle [b_0]} \\ \boldsymbol{I}A & \xrightarrow[{[b_1]}]{} & \boldsymbol{I}A \end{array}$$

is a weak push-out diagram in $h\mathbf{C}$. Thus, if $\mathbf{C}$ satisfies the HBH, then $h\mathbf{C}$ satisfies an 'almost' HBH.

Remark 5.6. There are particular situations in which the image of the diagram from Theorem 5.1 by the functor $\hbar$ is even a push-out diagram. An example in this sense is the sum $A \vee B$ of two objects.

6. Conclusion

Definition 6.1. An almost I-category is a structure $(\mathbf{C}, \mathbf{cof}, \boldsymbol{I}, \emptyset)$ satisfying all axioms of an I-category but we changed the term 'push-out' by the term 'weak push-out'.

Using Theorem 2.1, Proposition 4.1 and Theorem 5.1, and the fact that in the axiom (I4) the morphisms $i \vee i$ and (i_0, i_1) are cofibrations such that the induced square by H is a weak push-out diagram, we can conclude the following theorem.

Theorem 6.2. *Let $(\boldsymbol{C}, \boldsymbol{cof}, \boldsymbol{I}, \emptyset)$ be an I-category satisfying the HBH. Then the homotopy functor $\hbar : \boldsymbol{C} \to h\boldsymbol{C}$ induces on $h\boldsymbol{C}$ a structure of an almost I-category.*

Remark 6.3. It is necessary to specify that, in spite of Theorem 4.2, the cofibrations in the structure of the almost I-category of $h\mathbf{C}$ stated in Theorem 6.2 are only the images by the functor $\hbar$ of the cofibrations of the I-category $\mathbf{C}$ since we need the result of Theorem 5.1 for the axioms (I2) and (I4).

Corollary 6.4. *If the I-category $\boldsymbol{C}$ is one of the categories **Top**, $(\boldsymbol{Top})^*_c$ (of well pointed spaces), **Topp**, or **End**, then $h\boldsymbol{C}$ is an almost I-category for which the cofibrations are the images by the homotopy functor $\hbar$ of the cofibrations of $\boldsymbol{C}$.*

Example 6.5. By Example 4.3 we deduce that the category $h(\mathbf{Chain}^+_R)_c$ does not have a structure of almost I-category induced by the homotopy functor $\hbar$.

By Theorem 6.2 and Proposition 4.18 from [3], p. 35, we obtain the following corollary.

Corollary 6.6. *Let* $(\boldsymbol{C}, \boldsymbol{cof}, \mathbf{I}, \emptyset)$ *be an I-category satisfying the HBH, and let* $Ho(\boldsymbol{C})$ *be the localization of* $\boldsymbol{C}$ *with respect to homotopy equivalences in* $\boldsymbol{C}$ ([2]). *Then* $Ho(\boldsymbol{C})$ *has a natural structure of almost I-category.*

Using Theorem 6.2, Proposition 4. 19 from [3], p. 35, and Example 3.8, we obtain the following corollary.

Corollary 6.7. *Let* $\boldsymbol{C}$ *be an I-category satisfying the HBH and let* B *be an object in* C. *Then* $Ho(\boldsymbol{C}^B)$ *has a natural structure of almost I-category. Particularly,* $Ho(Top^*)$ *is an almost I-category.*

REFERENCES

1. Baues, H.-J. – *Algebraic Homotopy*, Cambridge Studies in Advanced Mathematics, 15, Cambridge University Press, Cambridge, 1989.
2. Baues, H.-J. – *Combinatorial Foundation of Homology and Homotopy,* Applications to spaces, diagrams, transformation groups, compactifications, differential algebras, algebraic theories, simplicial objects, and resolutions, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1999.
3. Baues, H.-J.; Quintero, A. – *Infinite Homotopy Theory,* K-Monographs in Mathematics, 6, Kluwer Academic Publishers, Dordrecht, 2001.

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Faculty of Mathematics,
University "Al.I. Cuza",
11, Bd. Carol I, 700506, Iaşi,
ROMANIA
ioanpop@uaic.ro